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Obtaining simultaneous solutions of linear subsystems of inequalities and duals

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Abstract

Given a set S of linear relations (equations and/or inequalities) among n variables, the problem of solving the systems resulting after selecting any subsets of S is dealt with. An algorithm that obtains all the necessary information to solve this problem, even if the operator in each linear relation is chosen, at wish, as an equality or inequality $\leqslant$, $<$, $\geqslant$, or $>$ is given. In addition, this algorithm simultaneously obtains the orthogonal set (dual cone) of a linear space (cone) generated by any subset of a given set of vectors (including sign selection), and allows simplifying the representation of the resulting linear spaces and cones to their minimal representations. The proposed methods are illustrated with several examples. © 2002 Elsevier Science Inc. All rights reserved.

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1. Introduction and motivation

Systems of linear equations have played an important role in the history of Mathematics, Physics and Engineering. The problems of finding methods for a computationally efficient and precise search of the general solutions have been the occupation

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of many mathematicians and scientists for many years (see, for example, [1,3,4,6,13], etc.). However, in practice, linear inequalities arise more frequently than equations, and applied scientists are unavoidably faced with them.

While the problem of finding the general solution of systems of equations implies dealing with linear spaces, and is simple, finding the general solution of systems of inequalities implies working with cones and dual cones, and is not an easy task (see [5,8–12,14,15], etc.).

The key step in solving systems of inequalities is the generation or identification of the dual cone of a given cone. Only after 1953, when Motzkin et al. [8] published the method of double description, important advances were made. Castillo et al. [2] have given an algorithm that leads to a minimal representation of the dual cone.

To clarify the problem we deal with, consider the Euclidean space E^n and the set of linear relations

$$\begin{array}{rl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \geqslant = \leqslant b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & \geqslant = \leqslant b_2, \\ & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & \geqslant = \leqslant b_m, \end{array} \tag{1}$$

where $\geqslant = \leqslant$ must be understood as any one of the three relations. System (1), using the extra variable x_{n+1}, i.e., moving from E^n to E^{n+1}, can be written as

$$\begin{array}{rl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n - b_1x_{n+1} & \geqslant = \leqslant 0, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n - b_2x_{n+1} & \geqslant = \leqslant 0, \\ & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n - b_mx_{n+1} & \geqslant = \leqslant 0, \\ x_{n+1} & = \quad 1. \end{array} \tag{2}$$

This homogenization technique for going from a polyhedron in E^n to a polyhedral cone in E^{n+1} is well known in the polyhedral theory (see [15]).

The main advantage of this new statement is that we first solve the homogeneous system

$$(\mathbf{A}| - \mathbf{b}) \begin{pmatrix} \mathbf{x} \\ x_{n+1} \end{pmatrix} \geqslant = \leqslant \mathbf{0}, \tag{3}$$

and then, force the extra condition $x_{n+1} = 1$.

The set of solutions of (3) is the dual of the cone generated by the rows of $(\mathbf{A}| - \mathbf{b})$ with the corresponding sign. Thus, if we have a method for obtaining the generators of the dual cone, we also have a method for solving homogeneous systems of inequalities. With this, it becomes clear that solving homogeneous systems of inequalities is closely related to finding dual cones. Thus, the importance of dual cones and the need to deal with them.

In this paper, we are interested not in a single system of inequalities, but in obtaining the general solution of all possible subsets of linear systems that can be generated by selecting a subset of relations in (1) and freely selecting one of the relations of the form $\geqslant, =, \leqslant, <$, or $>$ in each row. Our aim consists of obtaining all the necessary information to solve this problem without the need for starting from scratch each time we select a subset of constraints and relations. Since we look for the information to solve any set of constraints, we call this problem the simultaneous solution of linear subsystems of inequalities. So, our problem requires obtaining simultaneously many dual cones.

One interesting practical application of this is the case of overconstrained systems (i.e., systems with no feasible solution), where we are interested first in determining which subset of constraints must be removed to get a feasible subsystem, and next in finding the associated general solutions.

In addition, once we have the representation of dual cones, it is convenient to obtain their minimal representations, that is, obtaining minimal sets of generators, to have the general solutions in their simplest forms. We shall show that the information above also allows obtaining these minimal representations.

The paper is structured as follows. In Section 2, we introduce some basic concepts, as polyhedral convex cones, dual cones and minimal representations, that are needed in the following sections. In Section 3, we describe how simultaneous dual cones can be obtained. In Section 4, we give an algorithm for obtaining the corresponding minimal representations. In Section 5, we return to our initial problem of simultaneously solving systems of inequalities. In Section 6, we discuss the special case of strict inequalities. Finally, in Section 7, we end with some conclusions.

2. Some necessary background on cone structures

In this section we remind the reader about some elementary concepts about cones, and the basic notation is introduced.

2.1. Polyhedral convex cones

Polyhedral convex cones play an important role in dealing with systems of inequalities. Its definition, standard form, and minimal representation are discussed below.

Definition 1 (*Polyhedral convex cone*). Let $\mathbf{A}$ be a matrix, and $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ be the set of its columns. The set

$$\mathbf{A}_\pi \equiv \left\{\mathbf{x} \in E^n \mid \mathbf{x} = \pi_1\mathbf{a}_1 + \cdots + \pi_m\mathbf{a}_m \text{ with } \pi_i \geqslant 0;\ i = 1, \ldots, m\right\}$$

of all nonnegative linear combinations of the column vectors of $\mathbf{A}$ is known as the finitely generated cone, the polyhedral convex cone, or simply the cone generated by $\mathbf{A}$. The vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m$ are called the cone generators.

Similarly, we shall denote $\mathbf{A}_\rho$ as the linear space generated by the columns of $\mathbf{A}$. In this paper the Greek letters π and ρ are used to refer to nonnegative real and unrestricted real numbers, respectively.

Definition 2 (*Standard form of a cone*). If we have a cone $\mathbf{A}_\pi$, we can classify its generators $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ into two groups:
1. The generators whose opposite vectors belong to the cone, that is, $\mathbf{B} \equiv \{\mathbf{a}_i \mid -\mathbf{a}_i \in \mathbf{A}_\pi\}$.
2. The generators whose opposite vectors do not belong to the cone, that is, $\mathbf{C} \equiv \{\mathbf{a}_i \mid -\mathbf{a}_i \notin \mathbf{A}_\pi\}$.

Thus, we can express the cone in the following form: $\mathbf{A}_\pi \equiv (\mathbf{B}| -\mathbf{B}|\mathbf{C})_\pi \equiv \mathbf{B}_\rho + \mathbf{C}_\pi$, which is known as the standard form of a cone. The standard form of a cone distinguishes between its linear space part $\mathbf{B}_\rho$ of the cone, and its proper cone part $\mathbf{C}_\pi$.

Definition 3 (*Minimal representation of a cone*). A standard form of a cone $\mathbf{B}_\rho + \mathbf{C}_\pi$ is said to be a minimal representation if the set of generators of the linear space $\mathbf{B}_\rho$ and the proper cone $\mathbf{C}_\pi$ are minimal sets.

2.2. Dual cones

Definition 4 (*Nonpositive dual or polar cone*). Let $\mathbf{A}_\pi$ be a cone in E^n, with generators $\mathbf{a}_1, \ldots, \mathbf{a}_m$. The nonpositive dual or polar $\mathbf{A}_\pi^p$ (denoted with a p superindex) of $\mathbf{A}_\pi$ is defined as the set

$$\mathbf{A}_\pi^p \equiv \left\{\mathbf{v} \in E^n \mid \mathbf{A}^{\mathrm{T}}\mathbf{v} \leqslant \mathbf{0}\right\} \equiv \left\{\mathbf{v} \in E^n \mid \mathbf{a}_i^{\mathrm{T}}\mathbf{v} \leqslant 0;\ i = 1, \ldots, m\right\}.$$

Note that the dual of a cone is the set of vectors such that their dot products by those of the cone are nonpositive. Minkowski [7] proved that $\mathbf{A}_\pi^p$ is a cone too.

Remark 1. The nonpositive dual of a linear space coincides with its orthogonal:

$$\begin{aligned}\mathbf{A}_\rho^p &\equiv (\mathbf{A}, -\mathbf{A})_\pi^p \\ &\equiv \left\{\mathbf{v} \in E^n \mid \mathbf{a}_i^{\mathrm{T}}\mathbf{v} \leqslant 0 \text{ and } -\mathbf{a}_i^{\mathrm{T}}\mathbf{v} \leqslant 0; i = 1, \ldots, m\right\} \\ &\equiv \left\{\mathbf{v} \in E^n \mid \mathbf{a}_i^{\mathrm{T}}\mathbf{v} = 0; i = 1, \ldots, m\right\}.\end{aligned}$$

Pillers [12] gives a method for obtaining $\mathbf{A}_\pi^p$, and Castillo et al. [2] give an algorithm, the Γ-algorithm, for obtaining $\mathbf{A}_\pi^p$ in its simplified standard form $\mathbf{B}_\rho + \mathbf{C}_\pi$.

3. Obtaining simultaneous dual cones

The Γ-algorithm sequentially obtains the duals of the cones generated by $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_h\}$, the vectors associated with the columns of $\mathbf{A} = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_h, \dots, \mathbf{a}_m)$ up to a given index h. It starts with the dual E^n of the empty set cone and, in iteration h, a new vector $\mathbf{a}_h$ is incorporated to get the generators of the dual cone $(\mathbf{a}_1, \dots, \mathbf{a}_h)_\pi^p$. All vectors that are not minimal generators are removed from the tableau. The result is that we start with the unit matrix that generates E^n, the dual of the empty set, and end with a minimal set of generators for the dual $(\mathbf{a}_1, \dots, \mathbf{a}_m)_\pi^p$.

In the algorithm given below we have modified the Γ-algorithm taking into account that:

1. *Any generator can be selected with any sign.* Since our aim is to get the generators of any subset of vectors in $\mathbf{A}$ or $-\mathbf{A}$, in each iteration h we must incorporate the vector $\mathbf{a}_h$ and the vector $-\mathbf{a}_h$, because we are allowed to select the sign, and keep, in the transformed tableau of generators, all possible generators, that is, the required set of generators in case we select $\mathbf{a}_h$, and the required set of generators in case we select $-\mathbf{a}_h$.
2. *Each tableau must contain the dual generators that arise in previous tableaux.* In addition, we must ensure that the dual cones, that can be minimally generated with the generators in the h-iteration tableau, can also be generated minimally with the generators in the $h+1$-iteration tableau.
3. *The final tableau must contain the generators of all possible duals.* If one tableau contains the minimal generators of the duals of the cones $(\pm\mathbf{a}_j)_\pi$; $j = 1, 2, \dots, m$ generated by any single generator, in the initial set and with any sign, it also contains the minimal set of generators of the dual of any cone generated by a subset of generators of $(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m)$.
4. *We need information to select the generators of any desired polars.* Once we have obtained the final tableau, that contains all possible generators, we need a decision rule to decide which of these possible generators are required to generate the dual of a given initial cone, defined by the subset of generators of $(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m)$. To this end, a careful bookkeeping of indices (the sets $I_{A^0}(\mathbf{v})$, $I_{A^-}(\mathbf{v})$ and $I_{A^+}(\mathbf{v})$ below) is required.

The details of how this is done are given in the algorithm and the illustrative examples below. Considering Remark 1, this algorithm is also valid for obtaining orthogonals.

The standard Γ-algorithm (see [2]) works with two different situations:

1. Γ_{I} in which we still have a linear space component in the dual to be used as pivot. This allows performing linear space operations with linear space generators, and cone operations (nonnegative linear combinations of vectors) with cone generators.
2. Γ_{II}, essentially the double description procedure (see [8]), in which we need to perform cone operations only (i.e., only nonnegative linear combinations of vectors are allowed).

Algorithm 1 (*Simultaneous orthogonals and dual cones of cones generated by subsets of a set of vectors*).

- *Input*: A cone defined by the set of its generators $\mathbf{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ in the Euclidean space E^n.
- *Output*: All the information required to readily write the sets orthogonal to the linear spaces generated by any subset of columns of $\mathbf{A}$ (with any sign), and the dual cones of the cones generated by the same subset, i.e., a matrix $\mathbf{V}^{(m)}$ containing as columns $\mathbf{v}_j$ all the possible generators of these linear spaces and cones, and two sets, per each $\mathbf{v}$-vector, containing the indices k such that: (a) $\mathbf{v}^{\mathrm{T}}\mathbf{a}_k = 0$ (the $I_{A^0}(\mathbf{v})$ set), and (b) $\mathbf{v}^{\mathrm{T}}\mathbf{a}_k \leqslant 0$ (the $I_{A^-}(\mathbf{v})$ set), that is,

  $$I_{A^0}(\mathbf{v}) \equiv \left\{k \mid \mathbf{v}^{\mathrm{T}}\mathbf{a}_k = 0\right\};\ I_{A^-}(\mathbf{v}) \equiv \left\{k \mid \mathbf{v}^{\mathrm{T}}\mathbf{a}_k < 0\right\}.$$

  The matrix $\mathbf{V}^{(m)}$ is partitioned into two submatrices $\mathbf{V}$ and $\bar{\mathbf{V}}$, that refer to vectors that have and have not been used for pivoting, respectively. The $\bar{\mathbf{v}}$-vectors in matrix $\mathbf{V}^{(m)}$ must be used as linear space generators, and the $\mathbf{v}$-vectors, as cone generators, but with any sign (for the sake of simplicity we only work with half of these vectors). When considering vectors $-\mathbf{v}$, i.e., with its sign changed, we must change the corresponding $I_{A^-}(\mathbf{v})$ set, that is,

  $$I_{A^-}(-\mathbf{v}) \equiv \left\{1, 2, \ldots, m\right\} - I_{A^0}(\mathbf{v}) - I_{A^-}(\mathbf{v}).$$

Initialization:

- Let $\mathbf{V}^{(0)} = \mathbf{I}_n$, where $\mathbf{I}_n$ is the identity matrix of dimension n, and let $I_{\bar{V}} \equiv \{1, \ldots, n\}$.
- Initialize the $I_{A^0}(\mathbf{v}_j)$ and $I_{A^-}(\mathbf{v}_j)$ sets to empty sets, for $j = 1, 2, \ldots, n$, and let $h = 1$.

Step 1. *Calculate the dot products*. Calculate $\mathbf{t} = \mathbf{a}_h^{\mathrm{T}}\mathbf{V}^{(h-1)}$. Append to $I_{A^0}(\mathbf{v}_j)$ the index h for all j such that $t_j = 0$.

Step 2. *Look for the pivot*. Find a pivot column ($t_{\text{pivot}} \neq 0$) among those columns with indices in $I_{\bar{V}}$.

Step 3. *Test for Γ_{I} or Γ_{II} processes*. If no pivot has been found, go to *Process* II (Step 5). Otherwise go to *Process* I (Step 4).

Step 4. *Process* I (*updating of* $\mathbf{V}$, $I_{A^0}(\mathbf{v})$ *and* $I_{A^-}(\mathbf{v})$). If $t_{\text{pivot}} > 0$ change the sign of the pivot column and then perform the pivotal process by letting $v_{ij} = \operatorname{sign}(t_{\text{pivot}})$ $t_{\text{pivot}}v_{ij} + t_j v_{i\ \text{pivot}}$ for all $j \neq$ pivot such that $t_j \neq 0$ and all $i = 1, 2, \ldots, n$. Append the index h to the set $I_{A^-}(\mathbf{v}_{\text{pivot}})$ and to the $I_{A^0}(\mathbf{v}_j)$ sets for all $j \neq$ pivot. Remove index pivot from $I_{\bar{V}}$. If desired, simplify the column vectors of $\mathbf{V}$ by dividing each of them by the greater common divisor of the absolute values of all its components. Then, go to Step 6.

Step 5. *Process* II (*updating of* $\mathbf{V}$, $I_{A^0}(\mathbf{v})$ *and* $I_{A^-}(\mathbf{v})$). Append the index h to $I_{A^-}(\mathbf{v}_j)$ for all j such that $t_j < 0$. Determine the set

$$\mathbf{I}^{+-} \equiv \left\{i \mid \mathbf{v}_i^{\mathrm{T}}\mathbf{a}_h \neq 0;\ i \in I_V\right\}. \tag{4}$$

If $\mathbf{I}^{+-} \equiv \emptyset$, then go to Step 6. Otherwise, for each part (i, j) of distinct $(i \neq j)$ indices in $\mathbf{I}^{+-}$:
1. if $(I_{A^0}(\mathbf{v}_i) \cap I_{A^0}(\mathbf{v}_j)) \cup \{h\} \not\subset I_{A^0}(\mathbf{v}_s)\ \forall s \notin I_V$, append to $\mathbf{V}$ the vector

$$\mathbf{v}^* = t_j \mathbf{v}_i - t_i \mathbf{v}_j,$$

2. let $I_{A^0}(\mathbf{v}^*) \equiv (I_{A^0}(\mathbf{v}_i) \cap I_{A^0}(\mathbf{v}_j)) \cup \{h\}$, and
3. let $I_{A^-}(\mathbf{v}^*) \equiv \{k \in \{1, 2, \dots, h\} \mid \mathbf{v}^{*\mathrm{T}}\mathbf{a}_k < 0\}$.

Step 6. If $h < m$, let $h = h + 1$ and go to Step 1; otherwise, return matrix $\mathbf{V}^{(m)}$, and sets $I_{A^0}(\mathbf{v})$ and $I_{A^-}(\mathbf{v})$; $\forall \mathbf{v} \in \mathbf{V}^{(m)}$, and exit.

It is worthwhile mentioning that the set $I_{A^+}(\mathbf{v}_h)$, which is defined as

$$I_{A^+}(\mathbf{v}_h) \equiv \left\{i \mid \mathbf{a}_i^{\mathrm{T}} \mathbf{v}_h > 0\right\} \equiv \left\{1, 2, \dots, m\right\} - I_{A^0}(\mathbf{v}_h) - I_{A^-}(\mathbf{v}_h),$$

can be directly obtained from the output of Algorithm 1. This set will be useful for solving inequalities of the form $\geqslant$.

Next, we show that Algorithm 1 satisfies the required conditions.
1. *Any generator can be selected with any sign.* We consider two cases:
   (a) Γ_{I} process. The transformation in the Γ_{I} process is:

$$\mathbf{v}_j^h = \begin{cases} -\mathrm{sign}\big(t_{\mathrm{pivot}}^{h-1}\big)\mathbf{v}_{\mathrm{pivot}}^{h-1} & \text{if } j = \mathrm{pivot} \in J, \\ \mathrm{sign}\big(t_{\mathrm{pivot}}^{h-1}\big)t_{\mathrm{pivot}}^{h-1}\mathbf{v}_j^{h-1} - \mathrm{sign}\big(t_{\mathrm{pivot}}^{h-1}\big)t_j^{h-1}\mathbf{v}_{\mathrm{pivot}}^{h-1} & \text{if pivot} \neq j \in J. \end{cases}$$

   Since a sign change of $\mathbf{a}_h$ implies a sign change in t_j^{h-1}, t_{pivot}^{h-1} and $\mathrm{sign}(t_{\mathrm{pivot}}^{h-1})$, then $\mathbf{v}_{\mathrm{pivot}}^h$ also changes sign, but $\mathbf{v}_j^h$; $j \neq$ pivot do not. This means that we need $\mathbf{v}_{\mathrm{pivot}}^h$ and $-\mathbf{v}_{\mathrm{pivot}}^h$ in the transformed tableau of the new generators. Instead, we write only the vector $\mathbf{v}_{\mathrm{pivot}}^h$, but we interpret it with both signs.
   (b) Γ_{II} process. The transformation in the Γ_{II} process adds vectors of the form:

$$\mathbf{v}^* = t_j \mathbf{v}_i - t_i \mathbf{v}_j. \tag{5}$$

   Since a sign change of $\mathbf{a}_h$ also implies a sign change in t_i and t_j, and these, a sign change of the new vector $\mathbf{v}^*$, we must consider $\mathbf{v}^*$ with both signs, though we write only one of them.
2. *Each tableau must contain the dual generators that arise in previous tableaux.* Assume that the dual, $\mathbf{B}_\rho^{h-1} + \mathbf{C}_\pi^{h-1}$, of the cone generated by a given subset of vectors of $\{\mathbf{a}_1, \dots, \mathbf{a}_{h-2}\}$ has minimal generators $\{\mathbf{v}_k \mid k \in K\}$ for $\mathbf{B}^{h-1}$ and $\{\mathbf{v}_j \mid j \in J\}$ for $\mathbf{C}^{h-1}$.
   To analyze whether or not the transformed tableau contains a minimal set of generators for the same dual, we consider two cases:
   (a) *Γ_{I} process.* If we are in the Γ_{I} process, the transformed generators are $\mathbf{B}^h = \{\mathbf{v}_k^h \mid k \in K\}$ and $\mathbf{C}^h = \{\mathbf{v}_j^h \mid j \in J\}$, where

$$\mathbf{v}_k^h = \begin{cases} -\operatorname{sign}\big(t_{\text{pivot}}^{h-1}\big)\mathbf{v}_{\text{pivot}}^{h-1} & \text{if } k = \text{pivot} \in K, \\ \operatorname{sign}\big(t_{\text{pivot}}^{h-1}\big)t_{\text{pivot}}^{h-1}\mathbf{v}_k^{h-1} - \operatorname{sign}\big(t_{\text{pivot}}^{h-1}\big)t_k^{h-1}\mathbf{v}_{\text{pivot}}^{h-1} & \text{if pivot} \neq k \in K. \end{cases}$$

and

$$\mathbf{v}_j^h = \operatorname{sign}\big(t_{\text{pivot}}^{h-1}\big)t_{\text{pivot}}^{h-1}\mathbf{v}_j^{h-1} - \operatorname{sign}\big(t_{\text{pivot}}^{h-1}\big)t_j^{h-1}\mathbf{v}_{\text{pivot}}^{h-1} \quad \text{if pivot} \neq j \in J.$$

Thus,

$$\mathbf{v}_k^h = \alpha\mathbf{v}_k^{h-1} + \beta_k\mathbf{v}_{\text{pivot}}^{h-1} \text{ with } \alpha = \operatorname{sign}\big(t_{\text{pivot}}^{h-1}\big)t_{\text{pivot}}^{h-1} > 0 \text{ and } \beta_k \in \mathbb{R} \ \forall k \in K,$$

$$\mathbf{v}_j^h = \alpha\mathbf{v}_j^{h-1} + \beta_j\mathbf{v}_{\text{pivot}}^{h-1} \text{ with } \alpha = \operatorname{sign}\big(t_{\text{pivot}}^{h-1}\big)t_{\text{pivot}}^{h-1} > 0 \text{ and } \beta_j \in \mathbb{R} \ \forall j \in J$$

and then $\mathbf{B}_\rho^{h-1} \equiv \mathbf{B}_\rho^h$ and $\mathbf{C}_\rho^{h-1} \equiv \mathbf{C}_\rho^h$, which implies that the new representations are also minimal.

(b) Γ_{II} *process*. In the Γ_{II} process all the vectors in the previous tableau are kept (note that in the Γ-process we only add vectors but we remove none).

Thus, the final conclusion is that any dual, that can be written in a minimal representation using vectors in the tableau associated with iteration h, can also be written in a minimal representation using vectors in the tableau associated with any iteration h' such that $h' > h$.

3. *The final tableau must contain the generators of all possible duals.* This can be shown using Algorithm 1 starting with the final tableau instead of the unit matrix. If we do this, since the generators of the duals of cones generated by any single vector $\mathbf{a}_j$ or $-\mathbf{a}_j$ are in the final tableau, the transformed tableaux remain the same after introducing each of the desired $\mathbf{a}_h$ vectors in any order. Thus, we get as the final tableau of this process the initial one, showing that it contains the generators of all possible duals.
4. *We need information to select the generators of any desired polars.* Since in the final tableau we have all possible minimal generators, we need a rule to decide which ones are in the dual and which ones are not. This decision rule is very simple and requires only the definition of the dual, i.e., knowing whether the dot products of the generators with the candidate to generator vectors are negative, null or positive, i.e., knowing the sets $I_{A^0}(\mathbf{v})$, $I_{A^-}(\mathbf{v})$ and $I_{A^+}(\mathbf{v})$.

Unfortunately, in the worst case, the running time increases exponentially with the number of generators. However, this problem is not due to the algorithm, but to the complexity of the problem itself.

Example 1 (*Simultaneous dual cones*). Consider the cone generated by the columns of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 & 3 & 0 & 0 \\ 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & -2 & 0 & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{pmatrix}. \tag{6}$$

Our aim is to obtain enough information for obtaining the duals of the cone generated by any subset of columns of $\mathbf{A}$ with any sign. To this end we apply Algorithm 1. In Table 1 the different steps of the process are shown. In each iteration the new generator $\mathbf{a}_h$ appears in the first column, and the dot products in the $\mathbf{t}$ row. Since we can find pivot columns in the first three iterations, we use the Γ_{I} process, and use the pivotal transformation to update the matrix $\mathbf{V}$, where the $\mathbf{v}$ and $\bar{\mathbf{v}}$ vectors refer to vectors that have and have not been used for pivoting, respectively. In the fourth iteration we are unable to find a pivot column, because all the dot products associated with the $\bar{\mathbf{v}}$-vectors are null; thus, we use the Γ_{II}-process to update the $\mathbf{V}$ matrix, and finally get the final tableau, which contains all the information for obtaining the dual of the cone generated by any subset of columns of $\mathbf{A}$ with any sign. For the sake of completeness, we have also included the set $I_{A^+}(\mathbf{v})$ in the final tableau.

Some of the vectors in the $\mathbf{V}$ matrix will be linear space generators and some, cone generators of the corresponding duals. Which of these two cases apply depend on the $I_{A^0}(\mathbf{v})$, $I_{A^-}(\mathbf{v})$ and $I_{A^+}(\mathbf{v})$ sets, as we will see in the following sections.

Once the final matrix $\mathbf{V}^{(m)}$ and the sets $I_{A^0}(\mathbf{v})$ and $I_{A^-}(\mathbf{v})$; $\forall \mathbf{v} \in \mathbf{V}^{(m)}$ are available, the dual of a cone can be easily obtained using one of the two algorithms below that are justified because of the following theorem.

Theorem 1. *The cone* $(\mathbf{v}_1)_\rho + (\mathbf{w}_1)_\pi$ *is contained in the dual of the cone* $(\mathbf{v}_2)_\rho + (\mathbf{w}_2)_\pi$ *iff*

$$\mathbf{v}_1^{\mathrm{T}}\mathbf{v}_2 = 0; \quad \mathbf{w}_1^{\mathrm{T}}\mathbf{w}_2 \leqslant 0; \quad \mathbf{w}_1^{\mathrm{T}}\mathbf{v}_2 = 0; \quad \mathbf{v}_1^{\mathrm{T}}\mathbf{w}_2 = 0. \tag{7}$$

Proof. It is obvious because

$$\begin{aligned}&(\rho_1\mathbf{v}_1 + \pi_1\mathbf{w}_1)^{\mathrm{T}}(\rho_2\mathbf{v}_2 + \pi_2\mathbf{w}_2)\\ &\quad = \rho_1\rho_2\mathbf{v}_1^{\mathrm{T}}\mathbf{v}_2 + \rho_1\pi_2\mathbf{v}_1^{\mathrm{T}}\mathbf{w}_2 + \pi_1\rho_2\mathbf{w}_1^{\mathrm{T}}\mathbf{v}_2 + \pi_1\pi_2\mathbf{w}_1^{\mathrm{T}}\mathbf{w}_2 \leqslant 0\\ &\qquad \forall \rho_1, \rho_2, \pi_1 \text{ and } \pi_2,\end{aligned}$$

is equivalent to (7). □

Algorithm 2 (*Dual of a cone*).

- *Input*: A cone $\mathbf{B}_\rho + \mathbf{C}_\pi$, where the columns of matrices $\mathbf{B}$ and $\mathbf{C}$ are disjoint subsets of the columns of matrices $\mathbf{A}$ or $-\mathbf{A}$.
- *Output*: Its dual cone $(\mathbf{B}_\rho + \mathbf{C}_\pi)^p$ in standard form $\mathbf{M}_\rho + \mathbf{N}_\pi$.

Step 1. Let I_B, I_C^+ and I_C^- be the sets of indices of the columns of $\mathbf{A}$ corresponding to the columns of $\mathbf{B}$ and $\mathbf{C}$ with signs plus or minus, respectively.

Step 2. Let $\mathbf{M}$ be the matrix containing the column vectors $\mathbf{v}$ in the final tableau of Algorithm 1 such that $I_{A^0}(\mathbf{v})$ contains $I_B \cup I_C^+ \cup I_C^-$.

Step 3. Let $\mathbf{N}$ be the matrix containing:

Table 1
Iterations for calculating the dual cone in Example 1 using Algorithm 1

Iteration 1 $\mathbf{V}^{(0)}$						Iteration 2 $\mathbf{V}^{(1)}$						Iteration 3 $\mathbf{V}^{(2)}$						Iteration 4 $\mathbf{V}^{(3)}$					
$\mathbf{a}_1$	$\bar{\mathbf{v}}_1$	$\bar{\mathbf{v}}_2$	$\bar{\mathbf{v}}_3$	$\bar{\mathbf{v}}_4$	$\bar{\mathbf{v}}_5$	$\mathbf{a}_2$	$\mathbf{v}_1$	$\bar{\mathbf{v}}_2$	$\bar{\mathbf{v}}_3$	$\bar{\mathbf{v}}_4$	$\bar{\mathbf{v}}_5$	$\mathbf{a}_3$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\bar{\mathbf{v}}_3$	$\bar{\mathbf{v}}_4$	$\bar{\mathbf{v}}_5$	$\mathbf{a}_4$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\bar{\mathbf{v}}_4$	$\bar{\mathbf{v}}_5$
1	**1**	0	0	0	0	2	−1	**−2**	0	−1	0	0	1	−2	**2**	1	0	3	3	−2	−2	3	−2
2	**0**	1	0	0	0	1	0	**1**	0	0	0	2	−2	1	**−1**	−2	0	1	−2	1	1	−2	1
0	**0**	0	1	0	0	−1	0	**0**	1	0	0	1	0	0	**3**	0	0	−2	4	−2	−3	4	−3
1	**0**	0	0	1	0	0	0	**0**	0	1	0	0	0	0	**0**	3	0	1	0	0	0	1	0
0	**0**	0	0	0	1	0	0	**0**	0	0	1	1	0	0	**0**	0	1	−1	0	0	0	0	1
$\mathbf{t}$	**1**	2	0	1	0		−2	**−3**	−1	−2	0		−4	2	**1**	−4	1		−1	−1	1	0	0
$I_{A^0}(\mathbf{v})$								1	1	1	1			1	1	1	1			1	1	1	1
													2		2	2	2		2		2	2	2
																			3	3		3	3
$I_{A^-}(\mathbf{v})$							1						1	2					1	2	3		

Table 1 *(Continued)*

Iteration 5
$\mathbf{V}^{(4)}$

$\mathbf{a}_5$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v_4}$	$\mathbf{v}_5$	$\mathbf{v}_6$	$\mathbf{v}_7$	$\mathbf{v}_8$
0	3	−2	−2	**3**	−2	−5	1	−4
0	−2	1	1	**−2**	1	3	−1	2
0	4	−2	−3	**4**	−3	−6	1	−5
−1	0	0	0	**1**	0	0	0	0
0	0	0	0	**0**	1	0	0	0
t	0	0	0	**−1**	0	0	0	0
$I_{A^0}(\mathbf{v})$	2	1	1	1	1	3	2	1
	3	3	2	2	2	4	4	4
				3	3			
				4	4			
$I_{A^-}(\mathbf{v})$	1	2	3			2	1	2
	4	4					3	3

Iteration 6
$\mathbf{V}^{(5)}$

$\mathbf{a}_6$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v_4}$	$\mathbf{v}_5$	$\mathbf{v}_6$	$\mathbf{v}_7$	$\mathbf{v}_8$
0	3	−2	−2	3	**−2**	−5	1	−4
0	−2	1	1	−2	**1**	3	−1	2
0	4	−2	−3	4	**−3**	−6	1	−5
0	0	0	0	1	**0**	0	0	0
−1	0	0	0	0	**1**	0	0	0
t	0	0	0	0	**−1**	0	0	0
$I_{A^0}(\mathbf{v})$	2	1	1	1	1	3	2	1
	3	3	2	2	2	4	4	4
	5	5	5	3	3	5	5	5
				4	4			
					5			
$I_{A^-}(\mathbf{v})$	1	2	3	5		2	1	2
	4	4					3	3

Final tableau
$\mathbf{V}^{(6)}$

	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v_4}$	$\mathbf{v}_5$	$\mathbf{v}_6$	$\mathbf{v}_7$	$\mathbf{v}_8$
	3	−2	−2	3	−2	−5	1	−4
	−2	1	1	−2	1	3	−1	2
	4	−2	−3	4	−3	−6	1	−5
	0	0	0	1	0	0	0	0
	0	0	0	0	1	0	0	0
$I_{A^0}(\mathbf{v})$	2	1	1	1	1	3	2	1
	3	3	2	2	2	4	4	4
	5	5	5	3	3	5	5	5
	6	6	6	4	4	6	6	6
				6	5			
$I_{A^-}(\mathbf{v})$	1	2	3	5	6	2	1	2
	4	4					3	3
$I_{A^+}(\mathbf{v})$			4			1		

Pivot columns are boldfaced.

1. the column vectors $\mathbf{v}$ not in $\mathbf{M}$ such that $I_{A^0}(\mathbf{v})$ contains I_B, $I_{A^0}(\mathbf{v}) \cup I_{A^-}(\mathbf{v})$ contains I_C^+, and $I_{A^0}(\mathbf{v}) \cup I_{A^+}(\mathbf{v})$ contains I_C^-, and
2. the column vectors $-\mathbf{v}$ such that $\mathbf{v}$ is not in $\mathbf{M}$, $I_{A^0}(\mathbf{v})$ contains I_B, $I_{A^0}(\mathbf{v}) \cup I_{A^+}(\mathbf{v})$ contains I_C^+, and $I_{A^0}(\mathbf{v}) \cup I_{A^-}(\mathbf{v})$ contains I_C^-.

Step 4. The cone $\mathbf{M}_\rho + \mathbf{N}_\pi$ is the desired dual.

Example 2 (*Dual of a pure cone*). The dual of the cone $\mathbf{S}_\pi$ generated by columns $\mathbf{a}_1, \mathbf{a}_3$ and $-\mathbf{a}_4$ of $\mathbf{A}$, can be obtained using Algorithm 2. In this case we have $I_B \equiv \emptyset$, $I_C^+ \equiv \{1, 3\}$ and $I_C^- \equiv \{4\}$. The set $\mathbf{M}$ of column vectors $\mathbf{v}$ in Table 1 such that $I_{A^0}(\mathbf{v})$ contains $I_B \cup I_C^+ \cup I_C^- \equiv \{1, 3, 4\}$ is the set $\{\mathbf{v}_4, \mathbf{v}_5\}$. Thus $\mathbf{M} \equiv \{\mathbf{v}_4, \mathbf{v}_5\}$. The column vectors $\mathbf{v}$ not in $\mathbf{M}$ such that $I_{A^0}(\mathbf{v})$ contains $I_B \equiv \emptyset$, $I_{A^0}(\mathbf{v}) \cup I_{A^-}(\mathbf{v})$ contains $I_C^+ \equiv \{1, 3\}$, and $I_{A^0}(\mathbf{v}) \cup I_{A^+}(\mathbf{v})$ contains $I_C^- \equiv \{4\}$ is the set $\{\mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8\}$ and the column vectors $-\mathbf{v}$ such that $\mathbf{v}$ is not in $\mathbf{M}$, $I_{A^0}(\mathbf{v})$ contains $I_B = \emptyset$, $I_{A^0}(\mathbf{v}) \cup I_{A^+}(\mathbf{v})$ contains $I_C^+ \equiv \{1, 3\}$, and $I_{A^0}(\mathbf{v}) \cup I_{A^-}(\mathbf{v})$ contains $I_C^- \equiv \{4\}$ is the set $\{-\mathbf{v}_2, -\mathbf{v}_6\}$. Thus, $\mathbf{N} \equiv \{\mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8, -\mathbf{v}_2, -\mathbf{v}_6\}$, and its dual has parametric equations

$$
\begin{aligned}
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} &= \rho_1 \mathbf{v}_4 + \rho_2 \mathbf{v}_5 + \pi_1 \mathbf{v}_3 + \pi_2 \mathbf{v}_7 + \pi_3 \mathbf{v}_8 - \pi_4 \mathbf{v}_2 - \pi_5 \mathbf{v}_6 \\
&= \rho_1 \begin{pmatrix} 3 \\ -2 \\ 4 \\ 1 \\ 0 \end{pmatrix} + \rho_2 \begin{pmatrix} -2 \\ 1 \\ -3 \\ 0 \\ 1 \end{pmatrix} + \pi_1 \begin{pmatrix} -2 \\ 1 \\ -3 \\ 0 \\ 0 \end{pmatrix} + \pi_2 \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \pi_3 \begin{pmatrix} -4 \\ 2 \\ -5 \\ 0 \\ 0 \end{pmatrix} \\
&\quad - \pi_4 \begin{pmatrix} -2 \\ 1 \\ -2 \\ 0 \\ 0 \end{pmatrix} - \pi_5 \begin{pmatrix} -5 \\ 3 \\ -6 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}
\tag{8}
$$

and the corresponding orthogonal set $\mathbf{S}_\pi^\perp$ has parametric equations

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \rho_1 \begin{pmatrix} 3 \\ -2 \\ 4 \\ 1 \\ 0 \end{pmatrix} + \rho_2 \begin{pmatrix} -2 \\ 1 \\ -3 \\ 0 \\ 1 \end{pmatrix}. \tag{9}
$$

Example 3 (*Dual of a general cone*). The dual of the cone with parametric equations

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \rho_1 \mathbf{a}_1 + \rho_2 \mathbf{a}_2 + \pi_1 \mathbf{a}_3 - \pi_2 \mathbf{a}_4
= \rho_1 \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \rho_2 \begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \pi_1 \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \pi_2 \begin{pmatrix} 3 \\ 1 \\ -2 \\ 1 \\ -1 \end{pmatrix}
$$

can be obtained as follows. Since the linear space generators of $\mathbf{A}_\pi$, the columns of $\mathbf{B}$, are columns $\mathbf{a}_1$ and $\mathbf{a}_2$ of $\mathbf{A}$, and the cone generators of $\mathbf{A}_\pi$, the columns of $\mathbf{C}$, are columns $\mathbf{a}_3$ and $-\mathbf{a}_4$ of $\mathbf{A}$, we have $I_B \equiv \{1, 2\}$, $I_C^+ \equiv \{3\}$, and $I_C^- \equiv \{4\}$. Thus, we have:

1. The set of vectors $\mathbf{v}$ in the final tableau of Table 1 such that their $I_{A^0}(\mathbf{v})$ contains the set $I_B \cup I_C^+ \cup I_C^- \equiv \{1, 2, 3, 4\}$ is the set $\{\mathbf{v}_4, \mathbf{v}_5\}$. So, $M \equiv \{\mathbf{v}_4, \mathbf{v}_5\}$.
2. The column vectors $\mathbf{v}$ not in $\mathbf{M}$ such that $I_{A^0}(\mathbf{v})$ contains $I_B \equiv \{1, 2\}$, $I_{A^0}(\mathbf{v}) \cup I_{A^-}(\mathbf{v})$ contains $I_C^+ \equiv \{3\}$, and $I_{A^0}(\mathbf{v}) \cup I_{A^+}(\mathbf{v})$ contains $I_C^- \equiv \{4\}$ is $\{\mathbf{v}_3\}$.
3. The column vectors $-\mathbf{v}$ such that $\mathbf{v}$ is not in $\mathbf{M}$, $I_{A^0}(\mathbf{v})$ contains $I_B \equiv \{1, 2\}$, $I_{A^0}(\mathbf{v}) \cup I_{A^+}(\mathbf{v})$ contains $I_C^+ \equiv \{3\}$, and $I_{A^0}(\mathbf{v}) \cup I_{A^-}(\mathbf{v})$ contains $I_C^- \equiv \{4\}$ is empty.

   Thus, $N \equiv \{\mathbf{v}_3\}$, and we get the cone with parametric equations:

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \rho_1 \mathbf{v}_4 + \rho_2 \mathbf{v}_5 + \pi_1 \mathbf{v}_3 = \rho_1 \begin{pmatrix} 3 \\ -2 \\ 4 \\ 1 \\ 0 \end{pmatrix} + \rho_2 \begin{pmatrix} -2 \\ 1 \\ -3 \\ 0 \\ 1 \end{pmatrix} + \pi_1 \begin{pmatrix} -2 \\ 1 \\ -3 \\ 0 \\ 0 \end{pmatrix}. \tag{10}
$$

Remark 2. Algorithm 1 works in the most general case including unbounded polyhedra with rays. The Γ_{I} and Γ_{II} procedures are prepared for such a case. So, the original polyhedron can contain rays, though this implies a reduction in the dual dimension.

4. Minimal representations of dual cones and solutions

As it has been shown, Algorithm 1 is useful for obtaining dual cones associated with subsets of generators. However, the representations obtained from Algorithm 2 are not necessarily minimal, i.e., they do not need to contain minimal sets of generators. To simplify the resulting representations of cones and linear spaces, some extra work is needed. However, all the required information is already in the final tableau of Algorithm 1. In this section we give one algorithm for obtaining these minimal representations. This implies that the algorithm is valid for obtaining extremes and extreme directions of cones, polytopes and polyhedra.

Using Algorithm 1 we get a set of generators of a cone, that can be written in standard form as $\mathbf{V}_\rho + \mathbf{W}_\pi$, i.e., as a minimal sum of a linear space and a cone. Algorithm 3 below obtains this standard form.

The following lemma and theorem give theoretical support to Algorithm 3. The reader not interested in theoretical results can skip them.

Lemma 1 (Dual after removing some generators). *Let* $\mathbf{V}_\pi = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s)_\pi$ *be a cone, and* $\mathbf{A}_\pi \equiv \mathbf{V}_\pi^p \equiv \mathbf{B}_\rho + \mathbf{C}_\pi$ *its dual, where* $\mathbf{C} = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p)$. *Let* $\mathbf{C}_1$ *be a matrix that contains some columns of* $\mathbf{C}$ *and* $V^0(\mathbf{C}_1)$ *be the set of vectors in* $\mathbf{V}$ *that are orthogonal to* $\mathbf{C}_1$, *that is*:

$$V^0(\mathbf{C}_1) = \left\{\mathbf{v}_i \mid \mathbf{v}_i^{\mathrm{T}}\mathbf{C}_1 = \mathbf{0}\right\}.$$

Then, the dual of the cone $\mathbf{S}_\pi = (V^0(\mathbf{C}_1))_\pi$ *is the cone*

$$\mathbf{S}_\pi^p = (\mathbf{B}, \mathbf{C}_1)_\rho + (\mathbf{C}, \sim \mathbf{C}_1)_\pi,$$

where the symbol $\sim$ *is used to indicate the removed columns.*

Proof. The proof is based on the Γ-algorithm (see [2]). Assume that we use the Γ-algorithm to obtain the dual of $\mathbf{S}_\pi^p$. Since we already know the dual of $\mathbf{V}_\pi^p$ and $\mathbf{S}_\pi^p = \{\mathbf{s} \in \mathbf{V}_\pi^p \mid -\mathbf{s}^{\mathrm{T}}\mathbf{C}_1 \leqslant \mathbf{0}^{\mathrm{T}}\}$, we need to perform only as many new iterations in the Γ-algorithm as the number of columns of $\mathbf{C}_1$, by introducing the vectors in $-\mathbf{C}_1$, one by one. However, when calculating the dot products by the $\mathbf{a}_i$ vectors, we obtain nonnegative values, because $\mathbf{v}_i^{\mathrm{T}}\mathbf{C}_1 \leqslant \mathbf{0}^{\mathrm{T}}$ (note that $\mathbf{v}_i \in \mathbf{V}_\pi^{pp} = \mathbf{V}_\pi$ implies $-\mathbf{v}_i^{\mathrm{T}}\mathbf{C}_1 \geqslant \mathbf{0}^{\mathrm{T}}$). According to the Γ-algorithm, we must keep only those $\mathbf{v}_i \in \mathbf{V}$ such that $\mathbf{v}_i \in V^0(\mathbf{C}_1)$, which proves the theorem. $\square$

Theorem 2 (Simplifying generators of a cone). *Consider the cone* $\mathbf{V}_\pi$, *and let* $\mathbf{A}_\pi = \mathbf{V}_\pi^p = \mathbf{B}_\rho + \mathbf{C}_\pi$. *If*

$$I_{A^0}(\mathbf{v}_{i_1}) \subseteq I_{A^0}(\mathbf{v}_{i_2}), \quad i_1 \neq i_2, \tag{11}$$

then $\mathbf{V}_\pi = (\mathbf{V}, \sim \mathbf{v}_{i_1})_\pi$. *In other words, we can remove the vector* $\mathbf{v}_{i_1}$ *from the set of generators* $\mathbf{V}$ *of* $\mathbf{V}_\pi$.

Proof. Let $\mathbf{C}_1 \equiv \{\mathbf{a}_j \in \mathbf{C} \mid \mathbf{a}_j^{\mathrm{T}}\mathbf{v}_{i_2} < 0\}$. Relation (11) implies that $\mathbf{a}_j^{\mathrm{T}}\mathbf{v}_{i_1} < 0$ if $\mathbf{a}_j^{\mathrm{T}}\mathbf{v}_{i_2} < 0$. Thus, $\mathbf{a}_j^{\mathrm{T}}\mathbf{v}_{i_1} < 0$; $\forall \mathbf{a}_j \in \mathbf{C}_1$.

According to Lemma 1, the dual cone of the cone

$$(\mathbf{B}, \mathbf{C}_1)_\rho + (\mathbf{C}, \sim \mathbf{C}_1)_\pi$$

is the cone $(V^0(\mathbf{C}_1))_\pi$ that does not contain neither the vector $\mathbf{v}_{i_1}$ nor the vector $\mathbf{v}_{i_2}$.

To obtain the dual of the cone $(V^0(\mathbf{C}_1), \mathbf{v}_{i_2})_\pi$ we use the Γ-algorithm and introduce the vector $\mathbf{v}_{i_2}$ in a new iteration. Since $\mathbf{a}_j^{\mathrm{T}}\mathbf{v}_{i_2} < 0\ \forall \mathbf{a}_j$ and $\mathbf{a}_k^{\mathrm{T}}\mathbf{v}_{i_2} < 0\ \forall \mathbf{a}_k \notin \mathbf{C}_1$, all dot products are nonpositive and then all vectors in $\mathbf{C}_1$ remain. Thus, the dual of the cone $(V^0(\mathbf{C}_1), \mathbf{v}_{i_2})_\pi$ is the initial cone $\mathbf{V}_\pi$. Since the dual of the cone $(V^0(\mathbf{C}_1), \mathbf{v}_{i_2})_\pi$, that is generated without the vector $\mathbf{v}_{i_1}$ coincides with the initial cone, generated with it, then $\mathbf{v}_{i_1}$ can be eliminated from the set of generators without any alteration. $\square$

Algorithm 3 (*Minimal set of generators*).
- *Input*: A nonnecessarily minimal representation $\mathbf{B}_\rho^* + \mathbf{C}_\pi^*$ of the dual cone $\mathbf{A}_\pi^p$ of a cone generated by a subset of vectors in $\mathbf{A}$ or $-\mathbf{A}$.
- *Output*: A minimal representation $\mathbf{B}_\rho + \mathbf{C}_\pi$ of the cone $\mathbf{B}_\rho^* + \mathbf{C}_\pi^*$.

Step 1. Let $J_C \equiv \{h|\mathbf{c}_s = \mathbf{a}_h \text{ or } \mathbf{c}_s = -\mathbf{a}_h\}$, i.e., the set of indices corresponding to the $\mathbf{a}$ or $-\mathbf{a}$ vectors from $\mathbf{A}$ contained in $\mathbf{C}$, and let $i = 1$, and $\mathbf{B}^* = \mathbf{C}^* = \emptyset$.

Step 2. If $\mathbf{B}^* \equiv \emptyset$, then go to Step 5. Otherwise, let $\mathbf{B} = \{\mathbf{v}_1^*\}$, the first vector in $\mathbf{B}^*$, $K = I_{A^0}(\mathbf{v}_1^*)$, and $s = 1$.

Step 3. If $s < |\mathbf{B}^*|$ let $s = s + 1$ and go to Step 4. Otherwise go to Step 5.

Step 4. Let $K_1 = K \cap I_{A^0}(\mathbf{v}_s^*)$ and if $K_1 \neq K$, then let $K = K_1$ and add $\mathbf{v}_s^*$ to $\mathbf{B}$. Go to Step 3.

Step 5. If $\mathbf{C}^* = \emptyset$, then go to Step 7. Otherwise, for $i = 1$ to $|\mathbf{C}^*|$ and $\mathbf{v}_i \in \mathbf{C}^*$ do $I_{\bar{A}^0}(\mathbf{v}_i) = J_C \cap I_{A^0}(\mathbf{v}_i)$.

Step 6. Based on Theorem 2, take as $\mathbf{C}$ one of the largest subsets of $\mathbf{C}^*$ such that $I_{\bar{A}^0}(\mathbf{v}_i^*) \not\subset I_{\bar{A}^0}(\mathbf{v}_j^*)$ for all $i \neq j$.

Step 7. Return $\mathbf{B}$ and $\mathbf{C}$ and exit.

The following two examples illustrate this algorithm.

Example 4 (*Minimal representations*). Consider the dual cone $\mathbf{B}_\rho^* + \mathbf{C}_\pi^*$ of the cone $\mathbf{S}_\pi$ generated by columns $\mathbf{a}_1, \mathbf{a}_3$ and $-\mathbf{a}_4$ of $\mathbf{A}$, that was obtained in (8), i.e.,

$$\mathbf{B}^* = (\mathbf{v}_4, \mathbf{v}_5), \quad \mathbf{C}^* = (\mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8, -\mathbf{v}_2, -\mathbf{v}_6).$$

To obtain its dual cone in standard form, we use Algorithm 3, as follows:

Step 1. Let $J_C = \{1, 3, 4\}$, and make $i = 1$.

Step 2. Let $\mathbf{B} = \{\mathbf{v}_4\}$, $K = I_{A^0}(\mathbf{v}_4) = \{1, 2, 3, 4, 6\}$ (see Table 1), and $s = 1$.

Steps 3 and 4. Let $s = 2$ and $K_1 = \{1, 2, 3, 4, 6\} \cap I_{A_0}(\mathbf{v}_5) = \{1, 2, 3, 4, 6\} \cap \{1, 2, 3, 4, 5\} = \{1, 2, 3, 4\}$ (see Table 1). Since $K_1 \neq K$ we let $K = \{1, 2, 3, 4\}$, add $\mathbf{v}_5$ to $\mathbf{B}$, to get $\mathbf{B} = \{\mathbf{v}_4, \mathbf{v}_5\}$, and go to Step 3.

Step 3. Since $s = |\mathbf{B}^*|$ we go to Step 5.

Step 5. We obtain

$$
\begin{aligned}
I_{\bar{A}^0}(\mathbf{v}_3) &= \{1,3,4\} \cap \{1,2,5,6\} = \{1\},\\
I_{\bar{A}^0}(\mathbf{v}_7) &= \{1,3,4\} \cap \{2,4,5,6\} = \{4\},\\
I_{\bar{A}^0}(\mathbf{v}_8) &= \{1,3,4\} \cap \{1,4,5,6\} = \{1,4\},\\
I_{\bar{A}^0}(-\mathbf{v}_2) &= \{1,3,4\} \cap \{1,3,5,6\} = \{1,3\},\\
I_{\bar{A}^0}(-\mathbf{v}_6) &= \{1,3,4\} \cap \{3,4,5,6\} = \{3,4\}.
\end{aligned}
$$

Step 8. Since $I_{\bar{A}^0}(\mathbf{v}_3) \subseteq I_{\bar{A}^0}(\mathbf{v}_8)$, $I_{\bar{A}^0}(\mathbf{v}_7) \subseteq I_{\bar{A}^0}(\mathbf{v}_8)$, we select $\mathbf{C} = \{\mathbf{v}_8, -\mathbf{v}_2, -\mathbf{v}_6\}$.

Step 9. We return $\mathbf{B} = \{\mathbf{v}_4, \mathbf{v}_5\}$ and $\mathbf{C} = \{\mathbf{v}_8, -\mathbf{v}_2, -\mathbf{v}_6\}$ and exit.
Thus, the given cone is simplified.

Example 5 (*Minimal representations*). Consider the dual cone $\mathbf{B}^*_\rho + \mathbf{C}^*_\pi$ of the cone $\mathbf{S}_\pi$ generated by columns $\mathbf{a}_1, \mathbf{a}_3$ and $\mathbf{a}_5$ of $\mathbf{A}$. With Algorithm 1 and Table 1, we get

$$\mathbf{B}^* = (\mathbf{v}_2, \mathbf{v}_5), \quad \mathbf{C}^* = (\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_7, \mathbf{v}_8, -\mathbf{v}_6).$$

To obtain its dual cone in standard form, we use Algorithm 3, as follows:

Step 1. Let $J_C = \{1, 3, 5\}$, and let $i = 1$.

Step 2. Let $\mathbf{B} = \{\mathbf{v}_2\}$, $K = I_{A^0}(\mathbf{v}_2) = \{1, 3, 5, 6\}$ (see Table 1), and $s = 1$.

Steps 3 and 4. Let $s = 2$ and $K_1 = \{1, 3, 5, 6\} \cap I_{A_0}(\mathbf{v}_5) = \{1, 3, 5, 6\} \cap \{1, 2, 3, 4, 5\} = \{1, 3, 5\}$ (see Table 1). Since $K_1 \neq K$ we let $K = \{1, 3, 5\}$, add $\mathbf{v}_5$ to $\mathbf{B}$, to get $\mathbf{B} = \{\mathbf{v}_2, \mathbf{v}_5\}$, and go to Step 3.

Step 3. Since $s = |\mathbf{B}^*|$ we go to Step 5.

Step 5. We obtain

$$
\begin{aligned}
I_{\bar{A}^0}(\mathbf{v}_1) &= \{1,3,5\} \cap \{2,3,5,6\} = \{3,5\},\\
I_{\bar{A}^0}(\mathbf{v}_3) &= \{1,3,5\} \cap \{1,2,5,6\} = \{1,5\},\\
I_{\bar{A}^0}(\mathbf{v}_4) &= \{1,3,5\} \cap \{1,2,3,4,6\} = \{1,3\},\\
I_{\bar{A}^0}(\mathbf{v}_7) &= \{1,3,5\} \cap \{2,4,5,6\} = \{5\},\\
I_{\bar{A}^0}(\mathbf{v}_8) &= \{1,3,5\} \cap \{1,4,5,6\} = \{1,5\},\\
I_{\bar{A}^0}(-\mathbf{v}_6) &= \{1,3,5\} \cap \{3,4,5,6\} = \{3,5\}.
\end{aligned}
$$

Step 8. Since $I_{\bar{A}^0}(\mathbf{v}_7) \subseteq I_{\bar{A}^0}(\mathbf{v}_3)$, $I_{\bar{A}^0}(\mathbf{v}_8) \subseteq I_{\bar{A}^0}(\mathbf{v}_3)$ and $I_{\bar{A}^0}(-\mathbf{v}_6) \subseteq I_{\bar{A}^0}(\mathbf{v}_1)$ we select $\mathbf{C} = \{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\}$.

Step 9. We return $\mathbf{B} = \{\mathbf{v}_2, \mathbf{v}_5\}$ and $\mathbf{C} = \{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\}$ and exit.

Thus, the representation of the given cone is simplified.

Table 2 shows the minimal representations of the duals $\mathbf{V}_\rho + \mathbf{W}_\pi$ of the cones generated by all possible subsets of $\mathbf{A}$ (for the sake of simplicity, changes in sign are not considered).

Table 2
Cone and dual cones in standard form

Initial cone (**a** indices)	Dual generators **V**	**W**	Initial cone (**a** indices)	Dual generators **V**	**W**
{1}	{4, 5, 3, 2}	{1}	{2, 3, 5}	{5, 1}	{4, 3, 2}
{2}	{4, 5, 3, 1}	{2}	{2, 3, 6}	{4, 1}	{5, 3, 2}
{3}	{4, 5, 1, 2}	{3}	{2, 4, 5}	{5, 7}	{4, 1, 8}
{4}	{4, 5, 7, 6}	{1}	{2, 4, 6}	{4, 7}	{5, 1, 8}
{5}	{5, 3, 1, 2}	{4}	{2, 5, 6}	{3, 1}	{4, 5, 2}
{6}	{4, 3, 1, 2}	{5}	{3, 4, 5}	{5, 6}	{4, 1, 7}
{1, 2}	{4, 5, 3}	{1, 2}	{3, 4, 6}	{4, 6}	{5, 1, 7}
{1, 3}	{4, 5, 2}	{3, 1}	{3, 5, 6}	{1, 2}	{4, 5, 3}
{1, 4}	{4, 5, 8}	{2, 7}	{4, 5, 6}	{7, 6}	{4, 5, 1}
{1, 5}	{5, 3, 2}	{4, 1}	{1, 2, 3, 4}	{4, 5}	{1, 2, 7, 8}
{1, 6}	{4, 3, −2}	{5, 1}	{1, 2, 3, 5}	{5}	{4, 3, 1, 2}
{2, 3}	{4, 5, 1}	{3, 2}	{1, 2, 3, 6}	{4}	{5, 3, 1, 2}
{2, 4}	{4, 5, 7}	{1, 8}	{1, 2, 4, 5}	{5}	{4, −3, 7, 8}
{2, 5}	{5, 3, 1}	{4, 2}	{1, 2, 4, 6}	{4}	{5, −3, 7, 8}
{2, 6}	{4, 3, 1}	{5, 2}	{1, 2, 5, 6}	{3}	{4, 5, 1, 2}
{3, 4}	{4, 5, 6}	{1, 7}	{1, 3, 4, 5}	{5}	{4, 2, −6, 8}
{3, 5}	{5, 1, 2}	{4, 3}	{1, 3, 4, 6}	{4}	{5, 2, −6, 8}
{3, 6}	{4, 1, 2}	{5, 3}	{1, 3, 5, 6}	{2}	{4, 5, 3, 1}
{4, 5}	{5, 7, 6}	{4, 1}	{1, 4, 5, 6}	{8}	{4, 5, 2, 7}
{4, 6}	{4, 7, 6}	{5, 1}	{2, 3, 4, 5}	{5}	{4, 1, 7, 6}
{5, 6}	{3, 1, 2}	{4, 5}	{2, 3, 4, 6}	{4}	{5, 1, 7, 6}
{1, 2, 3}	{4, 5}	{3, 1, 2}	{2, 3, 5, 6}	{1}	{4, 5, 3, 2}
{1, 2, 4}	{4, 5}	{−3, 7, 8}	{2, 4, 5, 6}	{7}	{4, 5, 1, 8}
{1, 2, 5}	{5, 3}	{4, 1, 2}	{3, 4, 5, 6}	{6}	{4, 5, 1, 7}
{1, 2, 6}	{4, 3}	{5, 1, 2}	{1, 2, 3, 4, 5}	{5}	{4, 1, 2, 7, 8}
{1, 3, 4}	{4, 5}	{2, −6, 8}	{1, 2, 3, 4, 6}	{4}	{5, 1, 2, 7, 8}
{1, 3, 5}	{5, 2}	{4, 3, 1}	{1, 2, 3, 5, 6}	{ }	{4, 5, 3, 1, 2}
{1, 3, 6}	{4, 2}	{5, 3, 1}	{1, 2, 4, 5, 6}	{ }	{4, 5, −3, 7, 8}
{1, 4, 5}	{5, 8}	{4, 2, 7}	{1, 3, 4, 5, 6}	{ }	{4, 5, 2, −6, 8}
{1, 4, 6}	{4, 8}	{5, 2, 7}	{2, 3, 4, 5, 6}	{ }	{4, 5, 1, 7, 6}
{1, 5, 6}	{3, 2}	{4, 5, 1}	{1, 2, 3, 4, 5, 6}	{ }	{4, 5, 1, 2, 7, 8}
{2, 3, 4}	{4, 5}	{1, 7, 6}			

Indices refer to vectors in **A** of (6) and **v** in the final tableau of Table 1. Positive and negative indices refer to **a** and −**a** generators, respectively.

Remark 3. Since the representation of the dual cone is minimal, then, if there are no linear space generators each polytope generator must be identified with an extreme point, and each pure cone generator can be identified with an extreme direction. Thus, the proposed algorithm enumerates the extreme points and directions of any of the original polyhedra.

5. Simultaneous solutions of subsystems of equations and inequalities

Once we have solved the problem of simultaneous generation of duals, the problem of simultaneous solutions of systems of inequalities is straightforward.

It is well known that the general solution of a system of equations and inequalities is a polyhedron, that can be written, in its minimal form, as the sum of a linear space, a polyhedral convex cone and a polytope, i.e.,

$$\mathbf{x} = \sum_i \rho_i \mathbf{v}_i + \sum_j \pi_j \mathbf{w}_j + \sum_k \lambda_k \mathbf{q}_k, \quad \rho_i \in \mathbb{R}, \ \ \pi_j, \lambda_k \geqslant 0, \ \ \sum_k \lambda_k = 1,$$

where $\mathbf{v}_i, \mathbf{w}_j, \mathbf{q}_k \in E^n$. Our aim is to obtain a minimal set of generators for this polyhedron.

In this paper the Greek letter λ is used to refer to nonnegative reals adding up to 1, i.e., for linear convex combinations, and the polytope generated by the columns of matrix $\mathbf{A}$ is denoted by $\mathbf{A}_\lambda$.

5.1. Solving systems of inequalities using dual cones

In this section we first show how a system of inequalities can be solved by transforming the polyhedron of solutions in E^n into a dual cone in E^{n+1}. Next, we give a theorem that allows representing the solution polyhedron based on the dual cone.

Consider the system of inequalities of the form

$$\mathbf{Ax} \leqslant \mathbf{b}, \tag{12}$$

where $\mathbf{A}$ is a matrix $m \times n$ of real numbers, and $\mathbf{x}$ and $\mathbf{b}$ are column matrices of dimensions n and m, respectively, that represent vectors in E^n and E^m. It is worth noting that it is always possible to write a given system of linear equations and inequalities in this form.

One way of solving system (12) consists of writing it in the more convenient form of a homogeneous system with an extra constraint

$$\begin{aligned} &\mathbf{Cx}^* = \begin{bmatrix} \mathbf{A} & | & -\mathbf{b} \\ -- & + & -- \\ 0 & | & -1 \end{bmatrix} \begin{pmatrix} \mathbf{x} \\ x_{n+1} \end{pmatrix} \leqslant \mathbf{0}, \\ &x_{n+1} = 1, \end{aligned} \tag{13}$$

where

$$\mathbf{x}^* = \begin{pmatrix} \mathbf{x} \\ x_{n+1} \end{pmatrix} \in E^{n+1} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} \mathbf{A} & | & -\mathbf{b} \\ -- & + & -- \\ 0 & | & -1 \end{bmatrix}.$$

The redundant (because $x_{n+1} = 1$) constraint $-x_{n+1} \leqslant 0$ is introduced for convenience, because it facilitates the generation of the solution later.

If $\mathbf{X}$ is the set of solutions of (13), the definition of polar cone leads to

$$\mathbf{X} \equiv \left\{ \mathbf{x}^* \in \mathbf{C}_\pi^{\mathrm{T}p} \mid x_{n+1} = 1 \right\}, \tag{14}$$

that is, the solution of system (13) is the subset of the dual cone of the cone $\mathbf{C}_\pi^{\mathrm{T}}$, such that $x_{n+1} = 1$. This implies that we can find first the polar cone $\mathbf{C}_\pi^{\mathrm{T}p}$, associated with the first two sets of constraints of (13), and then impose the extra constraint $x_{n+1} = 1$.

This method has a geometric interpretation, which is illustrated in Fig. 1. To obtain the solution polyhedron (shadowed and bounded by bold lines pentagon in Fig. 1)

$$S = \left\{ \mathbf{x} \mid \mathbf{A}\mathbf{x} \leqslant \mathbf{b} \right\},$$

we first build the cone (dashed in Fig. 1)

$$C_S = \mathbf{C}_\pi^{\mathrm{T}p} = \left\{ \begin{pmatrix} \mathbf{x} \\ x_{n+1} \end{pmatrix} \middle| \, \mathbf{A}\mathbf{x} - x_{n+1}\mathbf{b} \leqslant \mathbf{0};\ -x_{n+1} \leqslant 0 \right\},$$

and then we obtain its intersection with the hyperplane $x_{n+1} = 1$.

The following theorem gives a minimal representation of the polyhedron in terms of a minimal representation of the dual cone.

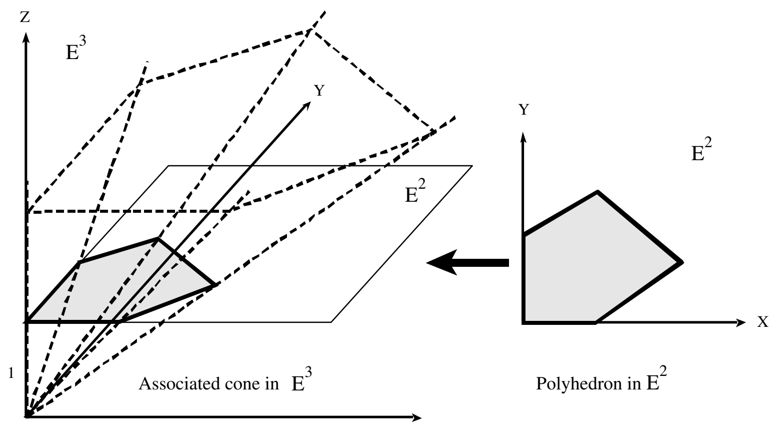

Fig. 1. Illustration of how the cone associated with a polyhedron is obtained.

Theorem 3 (Polyhedron and associated cone generators). *Let* $\mathbf{C}$ *be a matrix and consider the cone* $\mathbf{C}_{\pi}^{p}$, *i.e., the cone* $\mathbf{C}^{\mathrm{T}}\mathbf{x} \leqslant \mathbf{0}$, *where we assume that the constraint* $-x_{n+1} \leqslant 0$ *is in* $\mathbf{C}^{\mathrm{T}}$. *If* $\mathbf{C}_{\pi}^{p} = \mathbf{V}_{\rho} + \mathbf{Z}_{\pi}$ *is the general form of* $\mathbf{C}_{\pi}^{p}$, *then the general form of the polyhedron* $\mathbf{C}^{\mathrm{T}}\mathbf{x} \leqslant \mathbf{0}$; $x_{n+1} = 1$ *is*

$$C_S = \mathbf{V}_{\rho} + \mathbf{W}_{\pi} + \mathbf{Q}_{\lambda},$$

where $\mathbf{W}$ *and* $\mathbf{Q}$ *are the matrices containing all column vectors in* $\mathbf{Z}$ *such that their last components are null and nonnull*, *respectively*, *and* $\mathbf{Q}$ *has been suitably normalized to have unit last components. When the dual cone is in its minimal representation*, *then the representation of the polyhedral based on it is also minimal.*

Proof. Assume that $\mathbf{C}_{\pi}^{p} \equiv \mathbf{V}_{\rho} + \mathbf{Z}_{\pi}$. Then, all the vectors in $\mathbf{V}$ must have a null last component, because of the redundant constraint $-x_{n+1} \leqslant 0$. In addition, and without loss of generality, we can assume that z_{n+1k}, the last component of any vectors $\mathbf{z}_k$ in $\mathbf{Z}$, is 0 or 1, because if it is not, we can divide the whole vector by z_{n+1k}.

Finally, condition $x_{n+1} = 1$ leads to

$$\sum_{k \in K_Z} \pi_k = 1, \tag{15}$$

where

$$K_Z \equiv \left\{k | z_{n+1k} \neq 0\right\} \tag{16}$$

is the set of all column vectors in $\mathbf{Z}$ with nonnull last component.

We have the following two cases:

1. $K_Z \equiv \emptyset$: in this case problem (13) has no solution, since this implies $x_{n+1} = 0 \neq 1$.
2. $K_Z \neq \emptyset$: letting $\lambda_k = \pi_k$, (15) becomes

   $$\sum_{k \in K_Z} \lambda_k = 1, \tag{17}$$

   and then the cone $\mathbf{V}_{\rho} + \mathbf{Z}_{\pi}$ transforms to the polyhedron $\mathbf{V}_{\rho} + \mathbf{W}_{\pi} + \mathbf{Q}_{\lambda}$, where $\mathbf{W}$ and $\mathbf{Q}$ are the matrices whose columns have null and unit last component in $\mathbf{Z}$, respectively. $\square$

Thus, if we have a procedure to identify the cone generators, we immediately have a procedure to identify the polyhedron generators.

In summary, the system of linear inequalities (12) can be solved using the following steps:

1. Obtaining the dual cone $\mathbf{C}_{\pi}^{\mathrm{T}P} = \mathbf{V}_{\rho} + \mathbf{Z}_{\pi}$.
2. Normalizing the vectors of $\mathbf{Z}$ with nonnull last component z_{n+1} by dividing them by z_{n+1}.
3. Writing $\mathbf{C}_{\pi}^{\mathrm{T}P}$ as $\mathbf{V}_{\rho} + \mathbf{W}_{\pi} + \mathbf{Q}_{\lambda}$, where $\mathbf{W}$ and $\mathbf{Q}$ are the vectors in $\mathbf{Z}$ with null and unit last component, respectively.

4. Removing the $n+1$ component of all vectors implied, i.e., returning from E^{n+1} to E^n.

Finally, using the methods developed in Sections 3 and 4 we obtain a minimal representation of the dual cone $C_S = \mathbf{V}_\rho + \mathbf{Z}_\pi$. Next, we identify those generators in $\mathbf{Z}$ with last null component $\mathbf{W}$, and the rest $\mathbf{Q}$. Finally, Theorem 3 allows us to write

$$S = \mathbf{V}_\rho + \mathbf{W}_\pi + \mathbf{Q}_\lambda,$$

that is, the solution of the selected subsystem of $\mathbf{Ax} \leqslant \mathbf{b}$, in its simplest (minimal) form.

5.2. *Choosing inequality relations*

As indicated, Algorithm 1 also allows obtaining the general solution of any subsystem of a given linear system of inequalities, after choosing the relation $\leqslant, \geqslant, =, >$ or $<$.

To this end, the system

$$\begin{aligned} &\mathbf{B}^{\mathrm{T}}\mathbf{x} = \mathbf{0}, \\ &\mathbf{C}^{\mathrm{T}}\mathbf{x} \leqslant \mathbf{0}, \\ &\mathbf{D}^{\mathrm{T}}\mathbf{x} \geqslant \mathbf{0}, \\ &x_{n+1} = 1 \end{aligned} \tag{18}$$

can be solved using Algorithm 1 if the rows of $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are disjoint subsets of the columns of $\mathbf{A}$, as follows:

1. The solution of

   $$\begin{aligned} &\mathbf{B}^{\mathrm{T}}\mathbf{x} = \mathbf{0}, \\ &\mathbf{C}^{\mathrm{T}}\mathbf{x} \leqslant \mathbf{0}, \\ &\mathbf{D}^{\mathrm{T}}\mathbf{x} \geqslant \mathbf{0} \end{aligned}$$

   is the dual cone of the cone $\mathbf{B}_\rho + (\mathbf{C}| - \mathbf{D})_\pi$, that can be obtained using Algorithm 2, after changing sign the $\mathbf{v}$ vectors in $\mathbf{D}$ and exchanging the corresponding sets $I_{A^-}(\mathbf{v})$ and $I_{A^+}(\mathbf{v})$.
2. Use the methods, developed in Section 3, to obtain the corresponding minimal representation.
3. Impose the constraint x_{n+1} to obtain the solution of (18).

Example 6 (*General system*). Consider the system

$$\begin{aligned} &2x_1 + x_2 - x_3 \geqslant 0, \\ &2x_2 + x_3 \leqslant -1, \\ &3x_1 + x_2 - 2x_3 + x_4 \geqslant 1, \end{aligned} \tag{19}$$

which, using the artificial variable $x_5 = 1$, can be written as

$$\begin{aligned} &2x_1 + x_2 - x_3 \geqslant 0,\\ &2x_2 + x_3 + x_5 \leqslant 0,\\ &3x_1 + x_2 - 2x_3 + x_4 - x_5 \geqslant 0,\\ &x_5 = 1. \end{aligned} \tag{20}$$

Since the inequalities correspond to columns $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{a}_4$ of matrix $\mathbf{A}$ in (6), we build the sets

$$I_B = \{\ \}, \quad I_C = \{3\}, \quad I_D = \{2, 4\}.$$

Then, we need to move indices 2 and 4 from $I_{A^-}(\mathbf{v}_j)$ to $I_{A^+}(\mathbf{v}_j)$ and vice versa, for all j.

Applying now Algorithm 2, with $I_C \cup I_D = \{2, 3, 4\}$ we get $\mathbf{N} = \{\mathbf{v}_3, \mathbf{v}_7, -\mathbf{v}_1, -\mathbf{v}_2, -\mathbf{v}_6\}$ and $\mathbf{M} = \{\mathbf{v}_4, \mathbf{v}_5\}$, and with Algorithm 3, we obtain $\mathbf{B} \equiv \{\mathbf{v}_4, \mathbf{v}_5\}$, and $\mathbf{C} \equiv \{-\mathbf{v}_1, -\mathbf{v}_6, \mathbf{v}_7\}$.

Thus, the solution of system (20) without the last equality is the cone with parametric equations

$$\begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5 \end{pmatrix} = \rho_1 \begin{pmatrix} 3\\ -2\\ 4\\ 1\\ 0 \end{pmatrix} + \rho_2 \begin{pmatrix} -2\\ 1\\ -3\\ 0\\ 1 \end{pmatrix} + \pi_1 \begin{pmatrix} -3\\ 2\\ -4\\ 0\\ 0 \end{pmatrix} + \pi_2 \begin{pmatrix} 5\\ -3\\ 6\\ 0\\ 0 \end{pmatrix} + \pi_3 \begin{pmatrix} 1\\ -1\\ 1\\ 0\\ 0 \end{pmatrix}$$

and imposing the last constraint $x_5 = 1$ we get $\rho_2 = 1$ and the general solution of the initial system (the last component removed)

$$\begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4 \end{pmatrix} = \begin{pmatrix} -2\\ 1\\ -3\\ 0 \end{pmatrix} + \rho_1 \begin{pmatrix} 3\\ -2\\ 4\\ 1 \end{pmatrix} + \pi_1 \begin{pmatrix} -3\\ 2\\ -4\\ 0 \end{pmatrix} + \pi_2 \begin{pmatrix} 5\\ -3\\ 6\\ 0 \end{pmatrix} + \pi_3 \begin{pmatrix} 1\\ -1\\ 1\\ 0 \end{pmatrix}. \tag{21}$$

6. Strict inequalities

In this section we deal with the case of strict inequalities. Assume that you have a system of equations and inequalities, some of which are strict. To obtain the solution of such a system, we first solve the problem considering linear inequalities with the corresponding nonstrict inequalities, and then we restrict the solutions as follows.

The general solution of a linear system of equations and inequalities has the form

$$\begin{aligned} &\mathbf{x} = \sum_{i\in I} \rho_i \mathbf{v}_i + \sum_{j\in J} \pi_j \mathbf{w}_j + \sum_{k\in K} \lambda_k \mathbf{q}_k,\\ &\quad \rho_i \in \mathbb{R},\ \ \pi_j \geqslant 0,\ \ \lambda_k \geqslant 0,\ \ \sum_{k\in K} \lambda_k = 1. \end{aligned}$$

For each strict inequality of the form $<$ ($>$), at least one of the vectors $\mathbf{v}$ ($\mathbf{w}_j$ or $\mathbf{q}_k$) such that its $I_{A^-}(\mathbf{v})$ ($I_{A^+}(\mathbf{v})$) set contains the inequality index, must have its coefficients π_j or λ_k positive.

Example 7 (*Strict inequalities*). Consider the following system, which has been obtained by modifying some inequality relations in the system in Example 6

$$\begin{aligned} &2x_1 + x_2 - x_3 > 0, \\ &2x_2 + x_3 < -1, \\ &3x_1 + x_2 - 2x_3 + x_4 \geqslant 1. \end{aligned} \tag{22}$$

To solve this system we start from solution (21) of system (19). Since the strict inequalities correspond to columns 2 and 3 of matrix $\mathbf{A}$ in (6), we look, in Table 1 (final tableau), for indices 2 and 3 in $I_{A^+}(\mathbf{v})$ and $I_{A^-}(\mathbf{v})$, respectively, and find that the index 2 belongs to $I_{A^-}(\mathbf{v}_6)$, which is associated with coefficients π_2 and that the index 3 belongs to $I_{A^+}(\mathbf{v}_7)$, which is associated with coefficients π_3. Thus, the final conclusion is that the solution of (22) is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -3 \\ 0 \end{pmatrix} + \rho_1 \begin{pmatrix} 3 \\ -2 \\ 4 \\ 1 \end{pmatrix} + \pi_1 \begin{pmatrix} -3 \\ 2 \\ -4 \\ 0 \end{pmatrix} + \pi_2 \begin{pmatrix} 5 \\ -3 \\ 6 \\ 0 \end{pmatrix} + \pi_3 \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \tag{23}$$

where $\pi_2 > 0$ or $\pi_3 > 0$.

A simpler way of solving this system consists of substituting solution (23) in system (22) and get

$$\pi_2 > 0, \quad \pi_3 > 0, \quad \pi_1 \geqslant 0,$$

which is the condition above, because the third constraint ($\pi_1 \geqslant 0$) is always satisfied.

7. Conclusions

The simultaneous generation of duals and the solution of subsystems of equations and inequalities can be easily handled by the proposed algorithm. The final tableau of the proposed method contains all the information needed to find the dual cone of the cone generated by any possible subset of vectors of the given set, with any sign. Similarly, the solution of any possible subset of the initially given system of equations and inequalities including any change in the relations $\leqslant, <, =, >$ or $\geqslant$ can be obtained. In addition, the orthogonal sets of the linear spaces generated by any subset of vectors of the initially given set can also be obtained from the same tableau. An additional algorithm allows simplifying the representations of the resulting cones and linear spaces.

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