Control solvability of interval systems of max-separable linear equations

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Abstract

For a given matrix interval \( A = \langle A, \overline{A} \rangle \) and a given vector interval \( b = \langle \bar{b}, \overline{b} \rangle \) the notation \( A \otimes x = b \) if \( \oplus = \max \) represents an interval system of linear max-separable equations. Several types of solvability of interval systems are known. We define control solvability, weak control solvability and weak universal solvability and give necessary and sufficient condition for an interval system to be solvable according to each definition.

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1. Introduction

Systems of max-separable linear equations are used in several branches of applied mathematics. Systems of linear equations over max–plus algebra can assist in modelling and analysis of discrete event systems [1,7] and those over max–min algebra in modelling of fuzzy relations [9]. Choosing unsuitable values for the matrix entries and right-hand side can lead to unsolvable systems. Methods of restoring solvability by modifying the input data have been studied in [4,5] and by dropping some equations in [3]. Another possibility is to replace each entry by an interval...
of possible values. Then we talk about an interval system of linear equations. The theory of interval computations and in particular of interval systems in the classical algebra is already quite developed, see e.g. the monograph [10] or [13]. In [8], we studied tolerance and weak tolerance solvability in max–plus and max–min algebra. In this paper, we study other solvability concepts and solutions types and give a necessary and sufficient conditions for them.

2. Preliminaries

Let \((B, \oplus, \otimes)\) be an algebraic structure with two binary operations. \((B, \oplus, \otimes)\) is called max–plus algebra, if \(B = \mathbb{R} \cup \{-\infty\}\), \(a \oplus b = \max\{a, b\}\), \(a \otimes b = a + b\) and it is called max–min algebra, if \(B = \langle 0, 1 \rangle\), \(a \oplus b = \max\{a, b\}\), \(a \otimes b = \min\{a, b\}\).

Let \(m, n\) be given positive integers. Denote by \(M = \{1, 2, \ldots, m\}\), \(N = \{1, 2, \ldots, n\}\). The set of all \(m \times n\) matrices over \(B\) is denoted by \(B(m, n)\) and the set of all column \(n\)-vectors over \(B\) by \(B(n)\).

Then for each matrix \(A \in B(m, n)\) and each vector \(x \in B(n)\) the product \(A \otimes x\) equals:
\[
\left[ A \otimes x \right]_i = \max_{j \in N} \{\min\{a_{ij}, x_j\}\} \text{ for the max–min case and } \left[ A \otimes x \right]_i = \max_{j \in N} \{a_{ij} + x_j\} \text{ for the max–plus case.}
\]
For a given matrix interval \(A = \langle A, A \rangle\) with \(A, A \in B(m, n)\), \(A \leq A\) and a given vector interval \(b = \langle \underline{b}, \overline{b} \rangle\) with \(\underline{b}, \overline{b} \in B(m)\), \(\underline{b} \leq \overline{b}\) the notation
\[
A \otimes x = b
\]
represents an interval system of linear max-separable equations of the form
\[
A \otimes x = b \quad (1)
\]
such that \(A \in A, b \in b\).

Each system of the form (2) is said to be a subsystem of system (1), if \(A \in A, b \in b\).

We say, that an interval system has a constant matrix if \(A = A\) and has a constant right-hand side, if \(b = \overline{b}\).

In what follows we will consider for the max–plus case \(A \in \mathbb{R}(m, n)\).

The principal solution of a system (2) is defined by
\[
x^*_j(A, b) = \min_{i \in M} \{b_i; a_{ij} > b_i\} \quad (3)
\]
(where \(\min \emptyset = 1\) by definition) for the max–min case and
\[
x^*_j(A, b) = \min_{i \in M} \{b_i - a_{ij}\} \quad (4)
\]
for the max–plus case for each \(j \in N\). A general definition of \(x^*(A, b)\) for extremal algebras (where \(x \oplus y = x \lor y\)) is given in [15].

A solvability of an interval system depends on solvability of its subsystems. The following assertions describe the importance of principal solution for solvability of (2).

**Lemma 1** [7,15]. Let \(A \in B(m, n)\) and \(b \in B(m)\) be given. Then

(a) if \(A \otimes x = b\) for some \(x \in B(n)\), then \(x \leq x^*(A, b)\),

(b) \(A \otimes x^*(A, b) \leq b\).
Theorem 1 [6,7]. Let \( A \in B(m, n) \) and \( b \in B(m) \) be given. Then system \( A \otimes x = b \) is solvable if and only if \( x^*(A, b) \) is its solution.

Lemma 2 [2]. A system of inequalities
\[
B \otimes x \leq b, \\
C \otimes x \geq c
\]
has a solution if and only if \( C \otimes x^*(B, b) \geq c \).

3. Control solvability

Definition 1. A vector \( x \) is a control solution of an interval system (1), if for each \( b \in B \) there exists \( A \in A \) such that \( A \otimes x = b \).

Control solutions over classical algebra were introduced by Shary in [14].

Theorem 2. A vector \( x \) is a control solution of an interval system (1) if and only if
\[
A \otimes x \leq b, \\
\overline{A} \otimes x \geq \overline{b}.
\]

Proof. Let us consider the product \([A \otimes x]_i\) as a function (determined by \( x \)) of variables \( a_{i1}, a_{i2}, \ldots, a_{in} \). This is an isotone continuous function and so the image of the \( n \)-dimensional interval \([a_{i1}, \overline{a}_{i1}] \times [a_{i2}, \overline{a}_{i2}] \times \cdots \times [a_{in}, \overline{a}_{in}]\) is the interval \([A \otimes x]_i, [\overline{A} \otimes x]_i\). A vector \( x \) is a control solution if and only if \( b \subseteq \{A \otimes x; A \in A\} \), i.e. \( \{b_i, \overline{b}_i\} \subseteq \{[A \otimes x]_i, [\overline{A} \otimes x]_i\} \) for each \( i \in M \), which is equivalent to the condition stated in the theorem. □

In the following we define types of solvability connected with the existence of control solutions.

Definition 2. We say that an interval system (1) is
(i) control solvable if there exists a vector \( x \in B(n) \) such that for each \( b \in B \) there exists \( A \in A \) such that \( A \otimes x = b \),
(ii) weakly control solvable if for each \( b \in B \) there exist \( x \in B(n) \) and \( A \in A \) such that \( A \otimes x = b \).

It is easy to see that any control solvable interval system is weakly control solvable.

Theorem 3. An interval system (1) is control solvable if and only if
\[
\overline{A} \otimes x^*(A, b) \geq \overline{b}.
\]

Proof. An interval system is control solvable if and only if it has a control solution, which is equivalent to solvability of inequalities (5), (6). By Lemma 2 it is fulfilled if and only if \( \overline{A} \otimes x^*(A, b) \geq \overline{b} \). □
Lemma 3. An interval system (1) is weakly control solvable if and only if
\[ \overline{A} \otimes x^*(\underline{A}, \underline{b}) \geq \underline{b} \] (8)
for each \( \underline{b} \in \underline{b} \).

Proof. Suppose that \( \underline{b} \in \underline{b} \) is fixed. Then the existence of \( x \in B(n) \) and \( A \in A \) such that \( A \otimes x = \underline{b} \) is equivalent to the existence of a control solution of the system with constant right-hand side \( \underline{b} \) which is equivalent to (8) according to Theorem 3. Therefore, an interval system is weakly control solvable if and only if the inequality (8) is fulfilled for each \( \underline{b} \in \underline{b} \). \( \square \)

Lemma 3 gives a necessary and sufficient condition for weak control solvability, but it does not provide an easy method for checking the weak control solvability. Also the following theorem enables to verify a weak control solvability in polynomial time.

For each \( p \in M \) denote by \( c(p) \) the vector with following entries:
\[ c(p)_i = \begin{cases} \bar{b}_i & \text{for } i = p, \\ \underline{b}_i & \text{for } i \neq p, i \in M. \end{cases} \] (9)

Theorem 4. An interval system (1) is weakly control solvable if and only if
\[ \overline{A} \otimes x^*(\underline{A}, c(p)) \geq c(p) \] (10)
for each \( p \in M \).

Proof. The “only if” part is trivial. For the converse implication suppose that there exists a vector \( \underline{b} \in \underline{b} \) such that the inequality (8) is not fulfilled. Then there exists an index \( k \in M \) such that
\[ [\overline{A} \otimes x^*(\underline{A}, \underline{b})]_k < \underline{b}_k. \] (11)
We prove
\[ [\overline{A} \otimes x^*(\underline{A}, c(k))]_k < c(k)_k. \] (12)
Inequality (11) implies that for each \( j \in N \)
\[ \bar{a}_{kj} \otimes x_j^*(\underline{A}, \underline{b}) < \underline{b}_k. \] (13)
We consider two cases:

I. In the max–min algebra the inequality (13) means that at least one of the following cases has occurred: either
\[ \bar{a}_{kj} < \underline{b}_k \quad \text{or} \quad x_j^*(\underline{A}, \underline{b}) < \underline{b}_k. \]
In the first case, we have \( \bar{a}_{kj} < \bar{b}_k = c_k^{(k)} \) which implies \( \bar{a}_{kj} \otimes x_j^*(\underline{A}, c_k^{(k)}) < c_k^{(k)}. \)
In the second case, \( x_j^*(\underline{A}, \underline{b}) = \min_{i \in M} \{ b_i : a_{ij} > b_i \} < \underline{b}_k \) implies the inequality
\[ \min_{i \neq k} \{ b_i : a_{ij} > b_i \} < \bar{b}_k \leq \bar{b}_k. \] (14)
By (9) and by definition of principal solution we have \( x_j^*(\underline{A}, c_k^{(k)}) = \min_{i \in M} \{ c_i^{(k)} : a_{ij} > c_i^{(k)} \} < \min_{i \neq k} \{ c_i^{(k)} : a_{ij} > c_i^{(k)} \} \leq \min_{i \neq k} \{ b_i : a_{ij} > b_i \} \leq \min_{i \neq k} \{ b_i : a_{ij} > b_i \} < \bar{b}_k = c_k^{(k)}. \) (by (14)) Then \( \bar{a}_{kj} \otimes x_j^*(\underline{A}, c_k^{(k)}) < c_k^{(k)} \) which implies (12).
II. In the max–plus algebra the inequality (13) means \( \tilde{a}_{kj} + x_j^*(A, b) < b_k \), or equivalently \( x_j^*(A, b) < b_k - \tilde{a}_{kj} \), i.e. \( \min_{i \in M} \{ b_i - a_{ij} \} < b_k - \tilde{a}_{kj} \), i.e.

\[
\min_{i \neq k} \{ b_i - a_{ij} \}, b_k - a_{kj} < b_k - \tilde{a}_{kj},
\]

then

\[
\min_{i \neq k} \{ b_i - a_{ij} \} < b_k - \tilde{a}_{kj} \leq \tilde{b}_k - \tilde{a}_{kj}.
\]  

(15)

By definition, \( x_j^*(A, c^{(k)}) = \min \{ \min_{i \neq k} \{ b_i - a_{ij} \}, \tilde{b}_k - a_{kj} \} \leq \min \{ \min_{i \neq k} \{ b_i - a_{ij} \}, b_k - a_{kj} \} \) and by (15) we get \( x_j^*(A, c^{(k)}) < \tilde{b}_k - \tilde{a}_{kj} \), i.e. \( \tilde{a}_{kj} + x_j^*(A, c^{(k)}) = \tilde{a}_{kj} \otimes x_j^*(A, c^{(k)}) < \tilde{b}_k = c_k^{(k)} \) for each \( j \in N \) which implies (12). \( \square \)

Relevance of (12) in both cases means that the inequality (10) is not fulfilled for \( p = k \).

Example 1. In the max–min algebra let us take

\[
A = \begin{pmatrix}
0.2, & 0.4 \\
0.4, & 0.5 \\
0.3, & 0.4
\end{pmatrix}, \quad b = \begin{pmatrix}
0.1, & 0.4 \\
0.2, & 0.3 \\
0.5, & 0.6
\end{pmatrix}.
\]

To verify control solvability we compute \( x^*(A, b) = (0.1, 0.1, 0.5)^T, A \otimes x^*(A, b) = (0.5, 0.3, 0.5)^T \), so inequality (7) is not fulfilled and the given interval system is not control solvable. We verify weak control solvability.

By (9) we have

\[
c^{(1)} = \begin{pmatrix}
0.4 \\
0.2 \\
0.5
\end{pmatrix}, \quad c^{(2)} = \begin{pmatrix}
0.1 \\
0.3 \\
0.5
\end{pmatrix}, \quad c^{(3)} = \begin{pmatrix}
0.1 \\
0.2 \\
0.6
\end{pmatrix}
\]

and \( x^*(A, c^{(1)}) = (0.2, 1, 0.5)^T, x^*(A, c^{(2)}) = (0.1, 0.1, 0.5)^T, x^*(A, c^{(3)}) = (0.1, 0.1, 1)^T \).

Since \( A \otimes x^*(A, c^{(1)}) = (0.7, 0.6, 0.5)^T \geq c^{(1)} \), \( A \otimes x^*(A, c^{(2)}) = (0.5, 0.3, 0.5)^T \geq c^{(2)} \), \( A \otimes x^*(A, c^{(3)}) = (0.6, 0.3, 0.8)^T \geq c^{(3)} \), the given interval system is weakly control solvable.

Example 2. In the max–plus algebra let us take

\[
A = \begin{pmatrix}
3, & 10 \\
5, & 7 \\
6, & 8
\end{pmatrix}, \quad b = \begin{pmatrix}
6, & 7 \\
3, & 6 \\
4, & 6
\end{pmatrix}.
\]

To verify control solvability we compute \( x^*(A, b) = (-2, -3, -4)^T \), then \( A \otimes x^*(A, b) = (8, 6, 7)^T \geq \tilde{b} \) which means that the given interval system is control solvable. Then this is weakly control solvable, too.

4. Weak universal solvability

Definition 3. A vector \( x \in B(n) \) is a universal solution of an interval system (1) if \( A \otimes x = b \) for each \( A \in A \) and each \( b \in b \).

Theorem 5 [8]. An interval system (1) with a constant right-hand side \( b = \overline{b} = \tilde{b} \) has a universal solution if and only if

\[
A \otimes x^*(\overline{A}, b) = b.
\]  

(16)
**Definition 4.** An interval system (1) is weakly universally solvable if for each \( b \in \mathbf{b} \) there exists a vector \( x \in \mathcal{B}(n) \) such that \( A \otimes x = b \) for each \( A \in \mathbf{A} \).

**Remark.** If an interval system (1) is weakly universally solvable then it is weakly control solvable.

**Lemma 4.** An interval system (1) is weakly universally solvable if and only if for each \( b \in \mathbf{b} \) the equality \( A \otimes x^*(\overline{A}, b) = b \) is satisfied.

**Proof.** For any fixed \( b \in \mathbf{b} \), weak universal solvability is equivalent to the existence of a universal solution of the system with constant right-hand side \( b \) which is equivalent to (16). Therefore, an interval system is weakly universally solvable if and only if equality (16) is fulfilled for each \( b \in \mathbf{b} \). \(\square\)

Lemma 4 gives a necessary and sufficient condition for weak control solvability, but it is not easy to check it. The following theorem enables to verify a weak control solvability in polynomial time.

**Theorem 6.** An interval system (1) is weakly universally solvable if and only if
\[
A \otimes x^*(\overline{A}, c^{(p)}) = c^{(p)}
\]
for each \( p \in M \).

**Proof.** The “only if” part is trivial. For the converse implication suppose that for some vector \( b \in \mathbf{b} \) equality (16) is broken in the \( k \)th row. Considering Lemma 1(b) and inequality \( \bar{a}_{kj} \leq a_{kj} \) for each \( j \in \mathbb{N} \) we get
\[
\bigoplus_{j \in \mathbb{N}} a_{kj} \otimes x_j^*(\overline{A}, b) < b_k,
\]
i.e., for each \( j \in \mathbb{N} \)
\[
a_{kj} \otimes x_j^*(\overline{A}, b) < b_k. \tag{18}
\]
We consider two cases:

I. In the max–min algebra inequality (18) is fulfilled if \( a_{kj} < b_k \) or \( x_j^*(\overline{A}, b) < b_k \).

In the first case we get \( a_{kj} < b_k \) which implies \( a_{kj} \otimes x_j^*(\overline{A}, c^{(k)}) < b_k = c^{(k)} \). In the second case we have \( \min_{i \in M} \{ b_i : \bar{a}_{ij} > b_i \} < b_k \) which implies
\[
\min_{i \neq k} \{ b_i : \bar{a}_{ij} > b_i \} < b_k \leq b_k. \tag{19}
\]
By (9) and by definition of principal solution we have \( x_j^*(\overline{A}, c^{(k)}) = \min_{i \in M} \{ c_i^{(k)} : \bar{a}_{ij} > c_i^{(k)} \} \leq \min_{i \neq k} \{ c_i^{(k)} : \bar{a}_{ij} > c_i^{(k)} \} \leq \min_{i \neq k} \{ b_i : \bar{a}_{ij} > b_i \} \leq \min_{i \neq k} \{ b_i : \bar{a}_{ij} > b_i \} < \bar{b}_k = c^{(k)} \) (by (19)). We have \( a_{kj} \otimes x_j^*(\overline{A}, c^{(k)}) < c^{(k)} \) for each \( j \in \mathbb{N} \), i.e. \( [A \otimes x^*(\overline{A}, c^{(k)})]_k < c^{(k)}_k \) and so equality (17) does not hold for \( p = k \).
II. For the max–plus case inequality (18) means \( akj + x_j^*(A, b) < b_k, \) or equivalently \( x_j^*(A, b) < b_k - akj, \) i.e. \( \min\{\min_{i \neq k}\{bi - aij\} , b_k - akj\} < b_k - akj. \) This means, that at least one of the following cases has occurred:

\[
\min\{bi - aij\} < b_k - akj \quad \text{or} \quad b_k - akj < b_k - akj.
\]

In the first case \( x_j^*(A, c(k)) = \min\{\min_{i \neq k}\{bi - aij\} , b_k - akj\} \leq \min\{\min_{i \neq k}\{bi - aij\} , b_k - akj\} \leq \min\{\min_{i \neq k}\{bi - aij\} , b_k - akj\}, \) which is equivalent to \( akj + x_j^*(A, c(k)) < b_k. \)

In the second case \( b_k - akj < b_k - akj, \) i.e. \( akj + x_j^*(A, c(k)) < b_k. \) Since in both cases we get \( [A \otimes x^*(A, c(k))]_k < c_k^{(k)}, \) equality (17) does not hold for \( p = k. \)

Example 3. In the max–min algebra let us take

\[
A = \begin{pmatrix}
0.3 & 0.5 \\
0.4 & 0.2 \\
0.2 & 0.7
\end{pmatrix}, \quad b = \begin{pmatrix}
0.4 & 0.5 \\
0.3 & 0.4 \\
0.5 & 0.6
\end{pmatrix}
\]

By (9) we have

\[
c^{(1)} = \begin{pmatrix}
0.5 \\
0.3 \\
0.5
\end{pmatrix}, \quad c^{(2)} = \begin{pmatrix}
0.4 \\
0.4 \\
0.5
\end{pmatrix}, \quad c^{(3)} = \begin{pmatrix}
0.4 \\
0.3 \\
0.6
\end{pmatrix}
\]

and \( x^*(A, c^{(1)}) = (0.3, 0.5, 0.5)^T, x^*(A, c^{(2)}) = (0.4, 0.4, 0.5)^T, x^*(A, c^{(3)}) = (0.3, 0.4, 0.6)^T. \)

Since \( A \otimes x^*(A, c^{(1)}) = c^{(1)}, A \otimes x^*(A, c^{(2)}) = c^{(2)}, A \otimes x^*(A, c^{(3)}) = c^{(3)}, \) the given interval system is weakly universally solvable. Thus it is weakly control solvable, too. To verify control solvability we compute \( x^*(A, b) = (0.3, 0.4, 0.5)^T, \) which by Theorem 3 means that the given interval system is not control solvable.

5. Algebraic solutions

Definition 5. A vector \( x \) is an algebraic solution of an interval system (1) if \( \{ A \otimes x ; A \in A \} = b. \)

Algebraic solutions were first introduced in [11]. Rohn in [12] dealt with algebraic solutions in the classical algebra. This type of solution is easy to characterize:

Theorem 7. A vector \( x \) is an algebraic solution of an interval system (1) if and only if it satisfies

\[
A \otimes x = b, \quad (20)
\]

\[
\overline{A} \otimes x = \overline{b}. \quad (21)
\]

Proof. Using the same consideration as in the proof of Theorem 2 we get \( \{ [A \otimes x]; A \in A \} = \{ [A \otimes x]; [A \otimes x] \}. \) Then the equality of sets \( \{ A \otimes x ; A \in A \} = b \) is equivalent to the system of equalities (20) and (21). \( \square \)

To formulate a necessary and sufficient condition for the existence of an algebraic solution we define a matrix \( A^* \in B(2m, n) \) and a vector \( b^* \in B(2m) \) as follows:
\[
a_{ij}^* = \begin{cases} a_{ij} & \text{for } j \in M, i \in \{1, 2, \ldots, m\}, \\
\bar{a}_{ij} & \text{for } j \in M, i \in \{m+1, m+2, \ldots, 2m\}, \end{cases}
\]

(22)

\[
b_i^* = \begin{cases} b_i & \text{for } i \in \{1, 2, \ldots, m\}, \\
\bar{b}_i & \text{for } i \in \{m+1, m+2, \ldots, 2m\}.
\end{cases}
\]

(23)

**Theorem 8.** An interval system (1) has an algebraic solution if and only if the system

\[
A^* \otimes x = b^*
\]

(24)

is solvable.

**Proof.** Existence of an algebraic solution is equivalent to solvability of the system equations (20) and (21), which is equivalent to solvability of the system (24). □

**Example 4.** In the max–plus algebra let us take

\[
A = \begin{pmatrix}
(1, 2) & (2, 5) & (4, 8) \\
(3, 5) & (1, 7) & (2, 3)
\end{pmatrix}, \quad b = \begin{pmatrix}
(3, 7) \\
(4, 8)
\end{pmatrix}
\]

By (22) and (23) we have

\[
A^* = \begin{pmatrix}
1 & 2 & 4 \\
3 & 1 & 2 \\
2 & 5 & 8 \\
5 & 7 & 3
\end{pmatrix}, \quad b^* = \begin{pmatrix}
3 \\
4 \\
7 \\
8
\end{pmatrix}
\]

Then \(x^*(A^*, b^*) = (1, 1, -1)^T\) and since \(A^* \otimes x^*(A^*, b^*) = b^*\), the given interval system has an algebraic solution \(x = x^*(A^*, b^*)\).

**References**


