Central limit theorem for constrained Poisson systems

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Abstract

We prove that for a class of constrained Poisson white noise fields, the scaling (continuum) limit exists and equals Gaussian white noise, indexed by mean zero test functions. Under natural conditions on the Lévy measure, the (Poisson) moments converge to their Gaussian counterparts.

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1. Introduction

The background to this paper is the possibility to obtain modular forms [6] as moments of random fields on tori evaluated on points of finite order, see [1]. Consider a random field equation

$$LX = \xi$$

where $L$ is a linear partial differential operator (in $\mathbb{R}^d$) with constant coefficients and $\xi$ is a given white noise of Poisson type, say $\xi = \sum \alpha_i \delta_{x_i}$. A formal solution is obtained by defining

$$X = \sum \alpha_i G(\cdot - x_i)$$

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where \( G \) is a fundamental solution to \( L \). In the periodic case, when \( X \) is a random field on a torus, in real dimension two, and \( L \) is the Cauchy–Riemann operator \( \bar{\partial} := \frac{1}{2}(\partial_x + \sqrt{-1}\partial_y) \):

\[
\bar{\partial}X = \sum \alpha_i \delta_{x_i}.
\]

The periodicity of \( X \) requires the constraint, or gauge condition, \( \sum \alpha_i = 0 \) to hold. This means that the \( \alpha_i \) are no more independent. In particular the one-particle space has to be removed.

In a more general setting we study the convergence of constrained massless Poisson (noise) fields to Gaussian white noise indexed by test functions with vanishing mean-value (\( \int f = 0 \)). It is also shown that the moments converge, under appropriate conditions on the associated Lévy measure.

Convergence of the noise implies convergence of \( \sum \alpha_i G(\cdot - x_i) \) to the free Gaussian massless field on the torus. The same goes for convergence of the moments.

2. Convolution semi-groups

For the background in probability and Fourier analysis the reader is referred to [2–5] and [7].

Let \( V \) be a finite dimensional real vector space of dimension \( d \). The canonical pairing \( \text{Hom}(V, \mathbb{R}) \times V \rightarrow \mathbb{R} \) is denoted by \( \langle \xi, v \rangle \). We fix a Euclidean inner product for \( V \).

The corresponding norm is denoted by \( \| \cdot \| \) and the dual norm for \( \text{Hom}(V, \mathbb{R}) \) is also denoted by \( \| \cdot \| \). As for the normalization of Lebesgue measures we shall refer to the norm \( \| \cdot \| \).

2.1. Definition. For a Borel probability measure \( \mu \) on \( V \) its characteristic function (Fourier transform) \( \varphi \) is defined by

\[
\varphi(\xi) := \int_V e^{\sqrt{-1}\langle \xi, v \rangle} \mu(dv), \quad \text{for } \xi \in \text{Hom}(V, \mathbb{R}).
\]

Its \( n \)-fold convolution \( \mu^{*n} \) is defined by

\[
\mu^{*n}(A) := \mu^{\times n}\left(\{(v_1, v_2, \ldots, v_n) \in V^n; v_1 + v_2 + \cdots + v_n \in A\}\right) \quad \text{for } A \in \text{Borel}(V),
\]

where \( \mu^{\times n} \) stands for the \( n \)-fold product of \( \mu \) with itself.

Recall that \( \varphi(\cdot)^n \) is the characteristic function of \( \mu^{*n} \).

We have the following result on the behaviour of \( \varphi \) near the origin.

2.2. Lemma. Suppose that \( \mu \) is a Borel probability measure \( \mu \) on \( V \) with finite second order moment and \( \int v\mu(dv) = 0 \). Then for any \( \varepsilon, 0 < \varepsilon < 1 \), there exists \( \delta > 0 \) such that \( |\varphi(\xi)| \leq 1 - \frac{1}{2}(1 - \varepsilon) \int \langle \xi, v \rangle^2 \mu(dv) \) for all \( \xi \in \text{Hom}(V, \mathbb{R}) \) with \( \int \langle \xi, v \rangle^2 \mu(dv) \leq \delta \).

Proof. Using \( e^x = 1 + x + \left( \frac{1}{2} + \int_0^1 (1 - t)(e^{tx} - 1) dt \right)x^2 \), we see that

\[
\varphi(\xi) = 1 - \int_V \frac{1}{2} \langle \xi, v \rangle^2 \mu(dv) + \int_V \left\{ \int_0^1 (1 - t)\left(1 - e^{\sqrt{-1}t\langle \xi, v \rangle} \right) dt \langle \xi, v \rangle^2 \right\} \mu(dv)
\]

\[
= 1 - \frac{1}{2} \|\xi\|^2 + \left(\xi \otimes \xi, \int_{[0,1] \times V} (1 - t)(1 - e^{\sqrt{-1}t\langle \xi, v \rangle}) v \otimes v dt \mu(dv)\right).
\]
We first suppose that the quadratic form $\text{Hom}(V, \mathbb{R}) \to \mathbb{R}$, $\xi \mapsto \int \langle \xi, v \rangle^2 \mu(dv)$ is non-degenerate. Since $\mu$ has finite second order moment, the dominated convergence theorem leads to

$$\lim_{\xi \to 0} \int_{[0,1] \times V} (1 - t)\left| 1 - e^{\sqrt{-1}t \langle \xi, v \rangle} \right| \|v\|^2 dt \mu(dv) = 0.$$ 

Therefore, given $\epsilon$, $0 < \epsilon < 1$, we can choose $\delta > 0$ so that

$$\int_{[0,1] \times V} (1 - t)\left| 1 - e^{\sqrt{-1}t \langle \xi, v \rangle} \right| \|v\|^2 dt \mu(dv) < \frac{\epsilon}{2}$$

whenever $\|\xi\| \leq \delta$.

In general we write $N := \{\eta \in \text{Hom}(V, \mathbb{R}) : \int \langle \eta, v \rangle^2 \mu(dv) = 0\}$. Then the claim holds on any complementary subspace of $N$. On the other hand $\varphi(\xi + \eta) = \varphi(\xi)$ for all $\eta \in N$. Thus the claim is valid also for the degenerate case.

2.3. Remark. One immediately gets $\lim_{n \to \infty} n(\varphi(\xi/\sqrt{n}) - 1) = -C(\xi)/2$ and the central limit theorem $\lim_{n \to \infty} \varphi(\xi/\sqrt{n})^n = e^{-C(\xi)/2}$ where $C(\xi) := \int \langle \xi, v \rangle^2 \mu(dv)$.

We next discuss a uniform estimate of $\varphi$ away from the origin.

2.4. Lemma. Let $\mu$ be a Borel probability measure on $V$. Suppose $\mu$ is absolutely continuous w.r.t. Lebesgue measure. Then $\sup_{\|\xi\| > \delta} |\varphi(\xi)| < 1$ for all $\delta > 0$.

Proof. It is easy to see that if $|\varphi(\xi)| = 1$ for nonzero $\xi$, then $\mu$ is supported by a countable union of hyperplanes in $V$. In particular, $\mu$ is not absolutely continuous. We know that this is not the case, and thus $|\varphi(\xi)| = 1$ if and only if $\xi = 0$. The result now follows from the Riemann–Lebesgue lemma. □

One observes the following

2.5. Lemma. Let $\mu$ be a Borel probability measure on $V$ with finite second order moment. Suppose $\mu$ is absolutely continuous w.r.t. Lebesgue measure. Then the bilinear form

$$(\xi, \eta) \mapsto \int \langle \xi, v \rangle \langle \eta, v \rangle \mu(dv)$$

defines a Euclidean inner product on $\text{Hom}(V, \mathbb{R})$ and hence, by duality, a Euclidean inner product on $V$.

From now on, provided $\mu$ is absolutely continuous w.r.t. Lebesgue measure, we shall refer to the inner product described in Lemma 2.5. Thus we have

$$\|\xi\|^2 = \int \langle \xi, v \rangle^2 \mu(dv) \quad \text{for } \xi \in \text{Hom}(V, \mathbb{R})$$

and

$$\|v\| = \max \left\{ \langle \xi, v \rangle ; \xi \in \text{Hom}(V, \mathbb{R}), \|\xi\| = 1 \right\} \quad \text{for } v \in V.$$

Combining the lemmas so far we get
2.6. Lemma. Let \( \mu \) be a Borel probability measure on \( V \) with finite second order moment. Suppose that \( \mu \) is absolutely continuous w.r.t. Lebesgue measure and \( \int v \mu (dv) = 0 \). Then there exist \( \delta > 0 \) and \( 0 < c < 1 \) such that

\[
\sum_{n \geq p} \frac{t^n}{n!} |\varphi(\xi)|^n \leq \begin{cases} e^{t \left(1 - \|\xi\|^2/3\right)}, & \|\xi\| \leq \delta, \\ c^{-p} e^{ct |\varphi(\xi)|^p}, & \|\xi\| \geq \delta, \end{cases}
\]

for all \( t \geq 0 \) and \( p \geq 1 \).

The assumption in the next lemma holds if the density \( \rho \) is bounded.

2.7. Lemma. Let \( \mu \) be a Borel probability measure on \( V \) with density \( \rho \) w.r.t. Lebesgue measure. Suppose \( \int \rho(v)^q \, dv < \infty \) for some \( 1 < q \leq 2 \). Then \( \int |\varphi(\xi)|^p \, d\xi < \infty \) for all \( p \geq q/(q-1) \).

Proof. This is an immediate consequence of the Hausdorff–Young inequality. \( \square \)

2.8. Corollary. Let \( \mu \) be a Borel probability measure on \( V \) with finite second order moment and with density \( \rho \) w.r.t. Lebesgue measure. Suppose that \( \int v \mu (dv) = 0 \) and \( \int \rho(v)^q \, dv < \infty \) for some \( 1 < q \leq 2 \). Then \( \mu^n \) has continuous density \( \rho^n \) (n-fold convolution of function \( \rho \)) for \( n \geq q/(q-1) \). Moreover if \( p \geq q/(q-1) \), then

\[
\lim_{t \to \infty} t^{d/2} e^{-t} \sum_{n \geq p} \frac{t^n}{n!} \rho^n(\sqrt{t}u - v) = (2\pi)^{-d/2} e^{-\frac{1}{2} u^2}, \quad \text{for all } u, v \in V,
\]

where the convergence is uniformly bounded in the sense that

\[
\limsup_{t \to \infty} t^{d/2} e^{-t} \sup_{v \in V} \sum_{n \geq p} \frac{t^n}{n!} \rho^n(\sqrt{t}u - v) \leq (2\pi)^{-d/2}, \quad \text{for all } u \in V.
\]

Proof. The characteristic function of \( \mu^n \) equals \( \xi \mapsto \varphi(\xi)^n \), which, by Lemma 2.7, is integrable, provided \( n \geq q/(q-1) \). The (inverse) Fourier transform \( \frac{1}{(2\pi)^d} \int \varphi(\xi)^n e^{-\frac{1}{2} \langle \xi, \cdot \rangle} \, d\xi \) gives the continuous modification \( \rho^n \) for the density function of \( \mu^n \). Due to Lemma 2.6 we have

\[
\lambda_t(v) := \sum_{n \geq p} \frac{t^n \rho^n(v)}{n!} = \frac{1}{(2\pi)^d} \int \sum_{n \geq p} \frac{t^n}{n!} \varphi(\xi)^n e^{-\frac{1}{2} \langle \xi, v \rangle} \, d\xi,
\]

\[
p \geq \frac{q}{q-1}, \quad t \geq 0, \quad v \in V.
\]

After breaking the region of integration into two parts: \( \{\xi: \|\xi\| \leq \delta\} \) and \( \{\xi: \|\xi\| > \delta\} \), with \( \delta > 0 \) is chosen according to Lemma 2.6, we make a change of variable in the first part and get

\[
I^{d/2} e^{-t} \lambda_t(\sqrt{t}u - v) = \frac{1}{(2\pi)^d} \int_{\|\xi\| \leq \sqrt{t}\delta} e^{-t} \sum_{n \geq p} \frac{t^n}{n!} \varphi(\xi/\sqrt{t})^n e^{-\frac{1}{2} \langle \xi/\sqrt{t}, u-v/\sqrt{t} \rangle} \, d\xi
\]

\[
+ \frac{I^{d/2}}{(2\pi)^d} \int_{\|\xi\| > \delta} e^{-t} \sum_{n \geq p} \frac{t^n}{n!} \varphi(\xi)^n e^{-\frac{1}{2} \langle \xi, \sqrt{t}u - v \rangle} \, d\xi =: I + II.
\]
By the central limit theorem (see Remark 2.3), \( \lim_{t \to \infty} e^{-t(e^{\xi/\sqrt{t}} - 1)} = e^{-\|\xi\|^2/2} \) and hence the integrand in \( I \) converges to \( e^{-\|\xi\|^2/2}e^{-\sqrt{-1}(\xi,u)} \) as \( t \to \infty \). By Lemma 2.6 and the dominated convergence theorem, we see that

\[
\lim_{t \to \infty} I = \frac{1}{(2\pi)^d} \int e^{-\|\xi\|^2/2}e^{-\sqrt{-1}(\xi,u)} d\xi = (2\pi)^{-d/2}e^{-\|u\|^2/2}.
\]

It is easily seen that \( \limsup_{t \to \infty} \sup_{v \in V} |I| \leq (2\pi)^{-d/2} \). As for the second term, Lemma 2.6 implies that

\[
|II| \leq (2\pi)^{-d}c^{-p} \int \phi(\xi)|^p d\xi t^{d/2}e^{(c-1)t},
\]

where \( c \) is as in Lemma 2.6. This completes the proof. \( \Box \)

2.9. Remark. One also infers from the discussion above that \( \lim_{n \to \infty} n^{d/2} \rho^n(\sqrt{n}u - v) = (2\pi)^{-d/2}e^{-\|u\|^2/2} \) for all \( u, v \in V \).

3. Conditioned Poisson white noise

Let \( B \) be a standard measurable space and let \( \nu \) be a finite positive measure on \( B \) without point masses. We first introduce a Poisson random measure \( N \) on \( B \) with intensity \( \nu \). \( N \) is a family of nonnegative integer valued random variables indexed by measurable subsets of \( B \). If \( A_1, A_2, \ldots, A_n \) are pairwise disjoint measurable subsets of \( B \), then the joint distribution of \( \{N(A_1), N(A_2), \ldots, N(A_n)\} \) is characterized by

\[
E[z_1^{N(A_1)}z_2^{N(A_2)} \cdots z_n^{N(A_n)}] = e^{(z_1-1)v(A_1)}e^{(z_2-1)v(A_2)} \cdots e^{(z_n-1)v(A_n)}, \quad |z_1| < 1, |z_2| < 1, \ldots, |z_n| < 1. \tag{3.1}
\]

It is known that there exists a unique set valued realization of the Poisson random measure \( N \). To describe this fact precisely (see Lemma 3.3) we introduce the following

3.2. Definition. We use \( \text{Config}(B) \) to denote the space of all finite subsets in \( B \). The measurable structure is the smallest one which makes the mappings \( \text{Config}(B) \to \mathbb{R}, \quad \omega \mapsto \#(\omega \cap A) \) measurable for all measurable subsets \( A \) of \( B \). (\( \#M \) denotes the number of elements in the finite set \( M \).)

3.3. Lemma. There exists a unique random variable \( X \) taking values in \( \text{Config}(B) \) such that \( \#(X \cap A) = N(A) \) a.s. for each measurable subset \( A \) of \( B \).

For a bounded measurable function \( g: B \to \mathbb{C} \) one can deduce from (3.1) that

\[
E[z^\#X \prod_{b \in X} g(b)] = \exp\left\{ \int_B (zg(b) - 1) \nu(db) \right\}, \quad |z| < 1. \tag{3.4}
\]

This implies

\[
E\left[ \prod_{b \in X} g(b); \#X \geq p \right] = e^{-\nu(B)} \sum_{n \geq p} \frac{1}{n!} \left( \int_B g(b) \nu(db) \right)^n, \quad p \geq 1. \tag{3.5}
\]
We now derive some formulae needed to construct “conditioned Poisson white noise”.

3.6. Lemma. Let \( f : B^n \to \mathbb{C} \) respectively \( g : B \to \mathbb{C} \) be bounded measurable functions. Then

\[
E \left[ \sum_{b \in X^n \neq f(b)} f(b) z^{\#X} \prod_{b \in X} g(b) \right] = \int_{B^n} f(b) \prod_{i=1}^{n} g(b_i) v^n(d\mathbf{b}) z^n \exp \left\{ \int_{B} (zg(b) - 1) v(db) \right\}, \quad |z| < 1,
\]

where \( A^n_{\neq} \) stands for \( \{ \mathbf{b} = (b_1, b_2, \ldots, b_n) \in A^n; \ b_i \neq b_j \text{ if } i \neq j \} \) (non-coincident points).

Proof. Let \( A_1, A_2, \ldots, A_n \) be pairwise disjoint measurable subsets of \( B \). Then using the independence and (3.4) we get

\[
E \left[ \#(X \cap A_1) \#(X \cap A_2) \ldots \#(X \cap A_n) z^{\#X} \prod_{b \in X} g(b) \right] = \int_{A_1 \times A_2 \times \cdots \times A_n} \prod_{i=1}^{n} g(b_i) v^n(d\mathbf{b}) z^n \exp \left\{ \int_{B} (zg(b) - 1) v(db) \right\}.
\]

If \( f \) is the indicator function for the product set \( A_1 \times A_2 \times \cdots \times A_n \), then, since the \( A_i \)'s are mutually disjoint, we see that for each \( \omega \in \text{Config}(B) \)

\[
\sum_{b \in \omega^n_{\neq}} f(b) = \#(\{ \mathbf{b} : b_i \in \omega \cap A_i \text{ for all } i \}),
\]

where the right-hand side is equal to \( \#(\omega \cap A_1) \#(\omega \cap A_2) \ldots \#(\omega \cap A_n) \). Therefore the statement holds for such \( f \)'s. Finally exploiting the fact that, \( v \) having no point mass, the product measure \( v^n \) does not have mass on the set of coincident points in \( B^n \), we reach the desired equality. 

We immediately obtain the following

3.7. Corollary. Let \( f \) and \( g \) be as in Lemma 3.6. Then

\[
E \left[ \sum_{b \in X^n \neq f(b)} f(b) \prod_{b \in X} g(b); \#X \geq p \right] = \begin{cases} \int_{B^n} f(b) \prod_{i=1}^{n} g(b_i) v^n(d\mathbf{b}) e^{-v(B)} \sum_{k \geq p-n} \frac{1}{k!} (\int_{B} g(b) v(db))^k, & 1 \leq n < p, \\ \int_{B^n} f(b) \prod_{i=1}^{n} g(b_i) v^n(d\mathbf{b}) \exp \left\{ \int_{B} (g(b) - 1) v(db) \right\}, & p \leq n. \end{cases}
\]

Let \( V \) be a finite dimensional real vector space equipped with a Borel probability measure \( \mu \), as in Section 2, and \( T \) a standard measurable space equipped with a finite measure \( m \). We are interested in the following situation:

3.8. Assumption. The space \( B \) is the product vector bundle \( T \times V \to T \) over base \( T \) with standard fibre \( V \) and the intensity \( v \) is the product measure \( m \times \mu \). The fibre measure \( \mu \) has finite second order moment and bounded continuous density \( \rho \) w.r.t. Lebesgue measure and \( \int v \mu(dv) = 0 \).
Since $\mu$ has no point mass, this holds for $\nu$ as well. We also note that Assumption 3.8 implies all of the conditions in Corollary 2.8.

**3.9. Definition.** $\lambda^p_t(v) := \sum_{n \geq p} t^n \rho^{\ast n}(v)$ for $p \geq 1$, $t > 0$ and $v \in V$.

Here, the convergence is uniform and, by Remark 2.9, $\lambda^p_t(v) > 0$ for all $p \geq 1$, $t > 0$ and $v \in V$.

We see by (3.5) that

$$E\left[ \prod_{(x,v) \in X} e^{\sqrt{-1}(\xi \cdot v)} ; \#X \geq p \right] = e^{-t} \int_V e^{\sqrt{-1}(\xi \cdot v)} \lambda^p_t(v) dv \quad \text{where } t = m(T). \quad (3.10)$$

The conditional expectation $E[\cdot \mid \#X \geq p, \sum_{(x,v) \in X} v = u]$ has a nice continuous version with respect to the parameter $u \in V$, since $\lambda^p_t(\cdot)$ is continuous and strictly positive. To show this precisely, we consider the disintegration of Lebesgue measure on $V^k$ with respect to the family of affine subspaces

$$\left\{ v = (v_1, v_2, \ldots, v_k) : \sum_{i=1}^k v_i = u \right\}, \quad u \in V.$$

**3.11. Definition.** Let $k$ be a positive integer. For each $u \in V$, $\mu^k_u$ denotes the Borel measure on $V^k$ supported by the affine subspace $\{v: \sum_{i=1}^k v_i = u\}$, on which $\mu^k_u$ is $k^{-d/2}$ times the Lebesgue measure. ($\mu^1_u$ is the Dirac measure at $u$.)

The factor $k^{-d/2}$ is obtained from the equality

$$\int_{V^k} f(v) \mu^k_u(dv) = \int_{V^{k-1}} f\left(v_1, \ldots, v_{k-1}, u - \sum_{i=1}^{k-1} v_i\right) dv_1 \cdots dv_{k-1}, \quad k \geq 2.$$  

If $1 \leq n < k$ we have

$$\int_{V^k} f(v_1, v_2, \ldots, v_n) \prod_{i=n+1}^k \rho(v_i) \mu^k_u(dv)$$

$$= \int_{V^n} f(v_1, v_2, \ldots, v_n) \rho^{\ast k-n} \left(u - \sum_{i=1}^n v_i\right) dv_1 dv_2 \cdots dv_n. \quad (3.12)$$

In order to express the disintegration along the fibre $\{\omega: \sum_{(x,v) \in \omega} v = u\}$ in $\text{Config}(B)$, we shall make use of

**3.13. Lemma.** Let $f : V^n \to \mathbb{C}$ be a continuous function with compact support. Then

$$\int_{V^n} f(v) \prod_{i=1}^n e^{\sqrt{-1}(\xi \cdot v_i)} dv = \int_V \int_{V^n} f(v) \mu^n_u(dv) e^{\sqrt{-1}(\xi \cdot u)} du.$$
On the other hand
\[
\int_{V^n} f(v) \prod_{i=1}^{n} e^{\sqrt{-1} \langle \xi, v_i \rangle} \, dv \int_{V} \lambda(v) e^{\sqrt{-1} \langle \xi, v \rangle} \, dv = \int_{V^n} f(v) \lambda \left( u - \sum_{i=1}^{n} v_i \right) \, dv e^{\sqrt{-1} \langle \xi, u \rangle} \, du,
\]
where \( \lambda : V \rightarrow \mathbb{C} \) is integrable.

The straightforward proof is omitted.

To put emphasis on the independence between the \( T - \) and \( V - \) components, we introduce the following notation.

3.14. Definition. \( T^n \times V^n \rightarrow B^n \), \((x, v) \mapsto x \times v\) is the natural mapping, i.e.
\[
x \times v = ( (x_1, v_1), (x_2, v_2), \ldots, (x_n, v_n) ).
\]

With the help of Lemma 3.13, we derive from Corollary 3.7 (where \( g(x, v) = e^{\sqrt{-1} \langle \xi, v \rangle} \)) and (3.10) the following

3.15. Corollary. Let \( p \geq 1 \) and \( t = m(T) \). For each \( u \in V \) there exists a unique probability measure \( P_u \) on \( \text{Config}(B) \) such that
\[
\int_{\text{Config}(B)} \sum_{b \in \omega \cap \mathcal{C}} f(b) P_u(d\omega) = \begin{cases} 
\int_{T^n \times V^n} f(x \times v) m^n(dx) \prod_{i=1}^{n} \rho(v_i) \frac{\lambda^p(u - \sum_{i=1}^{n} v_i)}{\lambda^p(u)} \, dv, & 1 \leq n < p, \\
\int_{T^n \times V^n} f(x \times v) m^n(dx) \prod_{i=1}^{n} \rho(v_i) \frac{\lambda^1(u - \sum_{i=1}^{n} v_i)}{\lambda^1(u)} \, dv + \int_{T^n \times V^n} f(x \times v) m^n(dx) \prod_{i=1}^{n} \rho(v_i) \lambda^p(u) \, dv, & 1 \leq n < p,
\end{cases}
\]
hold for all bounded measurable functions \( f : B^n \rightarrow \mathbb{C} \). Moreover, \( P_u \) is supported by the fibre \( \{ \# \omega \geq p, \sum_{(x, v) \in \omega} v = u \} \) and the system \( \{ P_u : u \in V \} \) is a probability kernel for the conditional expectation \( E[ \cdot | \# X \geq p, \sum_{(x, v) \in X} v] \).

Proof. By virtue of (3.12) we see that (3.16) consistently defines a projective system of measures which extends uniquely to a measure \( P_u \) on \( \text{Config}(B) \) thanks to the Daniell–Kolmogorov theorem. (Note that the measurable structure for \( \text{Config}(B) \) is standard measurable.) The equality \( P_u( \{ \# \omega \geq p, \sum_{(x, v) \in \omega} v = u \}) = 1 \) follows from (3.12). The last statement is a consequence of Corollary 3.7 and (3.10). \( \square \)

3.17. Remark. We evaluate that \( \int \#(\omega \cap (T \times \{0\})) P_u(d\omega) = \rho(0)m(T)\lambda^p(0)^{-1} \) if \( p = 1 \) and \( u = 0 \), and otherwise the integral vanishes. Hence with respect to \( P_u \) the samples \( \omega \) do not meet the zero section of the bundle \( T \times V \rightarrow T \) almost surely unless \( p = 1 \) and \( u = 0 \).

Let \( F : B^n \rightarrow \mathbb{C} \) be a bounded measurable function. We consider the disintegration \( \tilde{F} : V^n \rightarrow \mathbb{C} \) of \( F \) with respect to the product measure \( m \times \mu \), i.e.
\[
\tilde{F}(v) = m(T)^{-n} \int_{T^n} F(x \times v) m^n(dx).
\]
Then we see from (3.16) that
\[ \int_{\text{Config}(B)} \sum_{b \in \omega_n} F(b) P_u(d\omega) = \int_{\text{Config}(B)} \sum_{x \times v \in \omega_n^u} \tilde{F}(v) P_u(d\omega). \]

4. Scaling and central limit theorem

From now on we fix \( p \geq 1 \) and a pair of measures \( m \) and \( \mu \), where the measure \( m \) is normalized to have mass one, \( m(T) = 1 \), and the measure \( \mu \) satisfies the condition in Assumption 3.8. We re-introduce \( t > 0 \) as the scaling parameter. The scaling \( V \ni v \mapsto t^{-1/2} v \in V \) induces the measure \( \mu(\sqrt{t} \cdot) \) on \( V \). At the same time we multiply the measure \( m \) by the factor \( t \). We recall that the probability measure described in Corollary 3.15 is characterized by the intensity \( \nu \) and the conditioning \( u \in V \).

4.1. Definition. We denote by \( P^t_u, u \in V \), the probability measure on \( \text{Config}(B) \) characterized by (3.16) with \( \nu = tm \times \mu(\sqrt{t} \cdot) \), while by \( Q^t_u, u \in V \), the probability measure on \( \text{Config}(B) \) characterized by (3.16) with \( \nu = tm \times \mu \).

The scaling \( V \ni v \mapsto t^{-1/2} v \in V \) of the fibre \( V \) introduces a bundle isomorphism
\[ B^n \to B^n, \quad x \times v \mapsto x \times (t^{-1/2} v) \]
for each positive integer \( n \) and hence \( \text{Config}(B) \to \text{Config}(B), \omega \mapsto t^{-1/2} \omega \). One easily gets from (3.16) that if \( f: \text{Config}(B) \to \mathbb{C} \) is a bounded measurable function, then
\[ \int_{\text{Config}(B)} f(\omega) P^t_u(d\omega) = \int_{\text{Config}(B)} f(t^{-1/2} \omega) Q^t_u(d\omega). \]

In order to describe the behaviour of \( P^t_u \) as \( t \to \infty \), we introduce a generalized random field.

4.3. Definition. For each bounded measurable function \( f: T \to \text{Hom}(V, \mathbb{R}) \), define
\[ Y(f, \omega) := \sum_{(x, v) \in \omega} \langle f(x), v \rangle, \quad \omega \in \text{Config}(B). \]

Using Corollary 3.15 and (3.18) one shows

4.4. Lemma. Let \( \xi \in \text{Hom}(V, \mathbb{R}) \). If \( f(x) = \xi \) for \( m \)-a.e. \( x \), then \( P_u(Y(f) = \langle \xi, u \rangle) = 1 \).

By this observation it suffices to consider \( f: T \to \text{Hom}(V, \mathbb{R}) \) with zero \( m \)-mean, i.e. \( \int_T f(x)m(dx) = 0 \). We easily see that
\[ Y(f, \omega)^2 = \sum_{(x, v) \in \omega} \langle f(x), v \rangle^2 + \sum_{x \times v \in \omega^u_n} \langle f(x_1), v_1 \rangle \langle f(x_2), v_2 \rangle. \]

One finds the expression for \( Y(f, \omega)^k \) with general \( k \) in the proof of Corollary 4.9.

4.5. Lemma. Suppose that the fibre measure \( \mu = \rho dv \) has moments of all orders, i.e. \( \int \|v\|^k \rho(v) dv < \infty \) for all \( k \). Let \( f: T \to \text{Hom}(V, \mathbb{R}) \) be a bounded measurable function with zero \( m \)-mean and let \( k_1, k_2, \ldots, k_n \) be positive integers. Then
\[ t^{\sum_{i=1}^n k_i/2 - n} \sum_{x \times v \in o_n^p} \prod_{i=1}^n (f(x_i), v_i)^{k_i} P_u^i (d\omega) \]

\[
\begin{cases}
 f_{V^n} \left( \prod_{i=1}^n f_T \left( f(x), v_i \right)^{k_i} m(dx) \rho(v_i) \right) \frac{\lambda^{p-n}_i (\sqrt{iu} - \sum_{i=1}^n v_i)}{\lambda^{p}_i (\sqrt{iu})} \, dv, & 1 \leq n < p, \\
 f_{V^n} \left( \prod_{i=1}^n f_T \left( f(x), v_i \right)^{k_i} m(dx) \rho(v_i) \right) \frac{\lambda^{1}_i (\sqrt{iu} - \sum_{i=1}^n v_i)}{\lambda^{1}_i (\sqrt{iu})} \, dv \\
 + f_{V^n} \left( \prod_{i=1}^n f_T \left( f(x), v_i \right)^{k_i} m(dx) \rho(v_i) \right) \lambda^{p}_i \left( \sqrt{iu} \right)^{-1} \mu^p_{\sqrt{iu}} (dv), & p \leq n.
\end{cases}
\] (4.6)

**Proof.** We obtain the result by applying (3.16) to functions on the form

\[ \langle f(x_1), v_1 \rangle^{k_1} \langle f(x_2), v_2 \rangle^{k_2} \cdots \langle f(x_n), v_n \rangle^{k_n} \]

where \( k_1, k_2, \ldots, k_n \) are positive integers, and then using (4.2). \( \square \)

We note that (4.6) vanishes if at least one of the \( k_1, k_2, \ldots, k_n \) is 1 since \( \int_T f(x) m(dx) = 0 \) is assumed in Lemma 4.5 and otherwise \( (k_1 + k_2 + \cdots + k_n)/2 - n \geq 0. \)

**4.7. Lemma.** Let \( n \geq 2 \) and let \( f_j : V \to \mathbb{C}, i = 1, 2, \ldots, n, \) be integrable and square-integrable with respect to Lebesgue measure. Then \( u \mapsto f_{V^n} \prod_{i=1}^n f_i(v_i) \mu^p_{\sqrt{iu}} (dv) \) has a continuous modification vanishing at \( \infty. \)

**Proof.** Calculating the Fourier transform of the function under discussion, then, by virtue of Lemma 3.13, we get

\[ (2\pi)^{d(n-1)/2} \prod_{i=1}^n \hat{f}_i(\cdot), \]

which is integrable, since \( n \geq 2 \) and each of the factors is bounded and square-integrable. Thus the result follows. \( \square \)

Before claiming our main result we recall Assumption 3.8 and that the inner product is the one described in Lemma 2.5.

**4.8. Theorem.** Suppose that the fibre measure \( \mu = \rho dv \) has finite moments of all orders \( \int \|v\|^k \rho(v) \, dv. \) Let \( f : T \to \text{Hom}(V, \mathbb{R}) \) be a bounded measurable function with zero \( m \)-mean and let \( k_1, k_2, \ldots, k_n \) be positive integers. Then

\[
\lim_{t \to \infty} \int \sum_{x \times v \in o_n^p} \prod_{i=1}^n (f(x_i), v_i)^{k_i} P_u^i (d\omega) = \begin{cases} 
(f_T \|f(x)\|^2 m(dx))^n, & \text{if } k_1 = k_2 = \cdots = k_n = 2, \\
0, & \text{otherwise.}
\end{cases}
\]

**Proof.** It follows from Corollary 2.8 that

\[
\lim_{t \to \infty} \frac{\lambda^k_i (\sqrt{iu} - v)}{\lambda^p_i (\sqrt{iu})} = 1 \quad \text{and} \quad \lim_{t \to \infty} \sup_{v \in V} \frac{\lambda^k_i (\sqrt{iu} - v)}{\lambda^p_i (\sqrt{iu})} \leq e^{\frac{1}{2} \|u\|^2},
\]

for all \( 1 \leq k \leq p \) and \( u, v \in V. \) Thus we get the conclusion for the case \( n < p \) from (4.6) and that \( f(\xi, u)^2 \rho(v) \, dv = \|\xi\|^2. \) To handle the term containing the measure \( \mu^p_{\sqrt{iu}} \) we use Lemma 4.7 and Corollary 2.8 for the case \( \max\{p, 2\} \leq n. \) If \( p = n = 1, \) the argument is trivial and hence the proof is complete. \( \square \)
4.9. Corollary. Suppose that the fibre measure \( \mu = \rho \, dv \) has finite moments \( \int \| v \|^k \rho(v) \, dv \) of all orders. Let \( f : T \to \text{Hom}(V, \mathbb{R}) \) be a bounded measurable function. Then
\[
\lim_{t \to \infty} \int e^{\sqrt{-1} Y(f, \omega)} P_u^t(d\omega) = \exp \left\{ \sqrt{-1} \left( \langle \xi, u \rangle - \frac{1}{2} \int_T \| f(x) - \xi \|^2 m(dx) \right) \right\},
\]
where \( \xi = \int_T f(x) m(dx) \).

Proof. Let \( k_1, k_2, \ldots, k_n \) be positive integers and \( k = \sum_{i=1}^n k_i \). For each partition \( \Delta \) of the set \( \{1, 2, \ldots, k\} \) into subsets with \( k_i \) elements, \( i = 1, 2, \ldots, n \), we write
\[
I(f, \Delta, \omega) := \sum_{(x, v) \in \omega} \prod_{i=1}^n \langle f(x_i), v_i \rangle^{k_i}.
\]
(In order to be definite, we may arrange so that \( k_1 \geq k_2 \geq \cdots \geq k_n \).) Then we get
\[
Y(f, \omega)^k = \left( \sum_{(x, v) \in \omega} \langle f(x), v \rangle \right)^k = \sum_{\text{partitions}} I(f, \Delta, \omega),
\]
where \( \Delta \) runs through all partitions of the set \( \{1, 2, \ldots, k\} \). We now see by Theorem 4.8 that if \( \int f(x) m(dx) = 0 \), then
\[
\lim_{t \to \infty} \int Y(f, \omega)^k P_u^t(d\omega) = \begin{cases} 
\frac{k!}{2^{k/2} (k/2)!} \left( \int_T \| f(x) \|^2 m(dx) \right)^{k/2}, & \text{if } k \text{ even} \\
0, & \text{otherwise},
\end{cases}
\]
which characterizes a unique Gaussian distribution. Combining with Lemma 4.4 we reach the statement. \( \square \)

4.10. Example. Let \( Z_t \) be a \( V \)-valued Lévy process with characteristic \( \varphi(\cdot) - 1 \), i.e.
\[
E\left[ e^{\sqrt{-1} \langle \xi, Z_t \rangle} \right] = e^{t(\varphi(\xi) - 1)}, \quad \xi \in \text{Hom}(V, \mathbb{R}), \ t > 0.
\]
Then the scaled process \( (\sqrt{\varepsilon} Z_{t/\varepsilon} : 0 \leq t \leq 1) \) under the condition: \( Z_t \) jumps at least \( p \) times up to time \( t = 1/\varepsilon \) and \( Z_{1/\varepsilon} = \varepsilon^{-1/2} u \) converges in law to a pinned Brownian motion \( B_t \) with mean \( E[B_t] = tu \) and covariance
\[
E[(B_t - tu)(B_s - su)] = s(1-t), \quad 0 \leq s \leq t \leq 1 \text{ as } \varepsilon \to 0.
\]

Proof. One way to realize \( Z_t, \ 0 \leq t \leq 1 \), is to find a measurable mapping \( \psi : T \to [0, 1] \) so that \( m(\{x \in T; \ \psi(x) \leq t\}) = t \) (this is possible because \( T \) is a standard measurable space and \( m \) has no point mass) and then set \( Z_t := \sum_{(x, v) \in X, \psi(x) \leq t} v \). In this picture one may regard \( X \) as describing the jumps of the Lévy process \( Z_t \). With respect to the measure \( P_u \) described in Corollary 3.15 the process \( Z_t \) jumps at least \( p \) times up to time \( t = 1 \) and moreover it is pinned to \( u \in V \) at time \( t = 1 \). \( \square \)

4.11. Remark. The results above can be extended to complex vector spaces \( V \). We start from a (even-dimensional) real vector space with a complex structure \( J \), such that
\[
\int_V \langle \xi, Jv \rangle^2 \mu(dv) = \int_V \langle \xi, v \rangle^2 \mu(dv) \quad \text{for } \xi \in \text{Hom}(V, \mathbb{R}).
\]
(Clearly this holds if $\mu$ is $J$-invariant.) Then the inner product is invariant under $J$ and
\[
\int_{V} \langle \xi, Jv \rangle \langle \eta, v \rangle \mu(dv) + \int_{V} \langle \xi, v \rangle \langle \eta, Jv \rangle \mu(dv) = 0 \quad \text{for } \xi, \eta \in \text{Hom}(V, \mathbb{R}).
\]
We extend the index of the random field $Y$ to measurable functions $f : T \to \text{Hom}(V, \mathbb{C})$:
\[
Y(f, \omega) := \sum_{(x, v) \in \omega} \langle f(x), v \rangle, \quad \omega \in \text{Config}(B).
\]
Here the pairing $\langle \xi, v \rangle$ is considered for $\xi \in \text{Hom}(V, \mathbb{C})$ and $v \in V$. We introduce subspaces:
\[
\text{Hom}^{1,0}(V, \mathbb{C}) := \{ \xi \in \text{Hom}(V, \mathbb{C}) : \xi \circ J = \sqrt{-1} \xi \},
\]
\[
\text{Hom}^{0,1}(V, \mathbb{C}) := \{ \xi \in \text{Hom}(V, \mathbb{C}) : \xi \circ J = -\sqrt{-1} \xi \}.
\]
Then $\text{Hom}(V, \mathbb{C}) = \text{Hom}^{1,0}(V, \mathbb{C}) \oplus \text{Hom}^{0,1}(V, \mathbb{C})$. We have that
\[
\text{if } \xi, \eta \in \text{Hom}^{1,0}(V, \mathbb{C}) \quad \text{(or } \xi, \eta \in \text{Hom}^{0,1}(V, \mathbb{C}) \text{) then } \int_{V} \langle \xi, v \rangle \langle \eta, v \rangle \mu(dv) = 0.
\]
Note that if $\xi \in \text{Hom}^{1,0}(V, \mathbb{C})$ then $\bar{\xi} \in \text{Hom}^{0,1}(V, \mathbb{C})$ and
\[
\int_{V} |\langle \xi, v \rangle|^2 \mu(dv) = 2 \max \{|\langle \xi, v \rangle|^2 ; \ v \in V, \|v\| = 1 \}.
\]
The limit generalized random field, say $Z$, is complex Gaussian and
\[
E[Z(f)^2] = 0, \quad E[|Z(f)|^2] = \int_{T \times V} \left| f(x) - \int_{T} f(m, v) \right|^2 m(dx) \mu(dv)
\]
for bounded measurable functions $f : T \to \text{Hom}^{1,0}(V, \mathbb{C})$.

References