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The combinatorics and the homology of the poset of subgroups of p-power index

Michael Weidner

Mathematisches Institut, Universität Freiburg, Albertstr. 23b, D-79104 Freiburg i. Br., Germany

Volkmar Welker *

Institut für Experimentelle Mathematik, Ellernstr. 29, D-45326 Essen, Germany

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Abstract

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For a finite group G and a prime p the poset $S^p(G)$ of all subgroups $H \neq G$ of p-power index is studied. The Möbius number of the poset is given and the homotopy type of the poset is determined as a wedge of spheres. We describe the representation of G on the homology groups of the order complex of $S^p(G)$ and show that this representation can be realized by matrices with entries in the set $\{+1, -1, 0\}$. Finally a CL-shellable subposet of $S^p(G)$ is exhibited for odd primes p.

1. Introduction

For a finite group G and a prime p we study the partially ordered set $S^p(G)$ of all subgroups $H \neq G$ of p-power index (i.e. $[G : H] = p^i$ for some $i \neq 0$). We will write poset for partially ordered set in the sequel. We obtain results on the algebraic combinatorics, the homology and the topology

Correspondence to: M. Weidner, Mathematisches Institut, Universität Freiburg, Albertstr. 23b, D-79104 Freiburg i. Br., Germany. Email: weidner@ibm2.mathematik.uni-freiburg.de.
* Partially supported by the Mittag-Leffler Institute. Email: welker@exp-math.uni-essen.de.

of the poset. In particular, we show that the homotopy type of the geometric realization $|CS^p(G)|$ of the order complex $CS^p(G)$ of $S^p(G)$ is a wedge of spheres of the same dimension. The order complex CP of a finite poset P is the set of all linearly ordered subsets (chains) of the poset P, in particular CP is a simplicial complex. Thereby we can speak of a poset in terms of algebraic topology (e.g. homotopy type) and homology. There are various papers on topological properties of parts of the subgroup lattice (see, e.g., [1,8,10,17]). Recently (see in particular [1]) there have been efforts to prove the conjecture of Quillen which states that the poset $S_p(G) = \{H \mid |H| = p^i \text{ for some } i \neq 0\}$ is contractible if and only if the maximal normal p-subgroup $O_p(G)$ is not 1. We prove a dual of this conjecture (Theorem 2.10) for the poset $S^p(G)$ where the condition $O_p(G) \neq 1$ is replaced by $O^p(G) \neq G$ for the minimal normal subgroup with p-power index. The main source of motivation for the research on the Quillen conjecture is the interest in the representations of the group on the homology groups of the order complex. Here we show that the representation of G on the homology groups of the order complex of $S^p(G)$ reveals a lot about the permutation representations of the group (Theorem 3.4). Thereby we can show that the representation of G on the homology groups can be represented by matrices with entries 1, -1 and 0. We prove in Theorem 2.9 that the poset $S^p(G)$ has the homotopy type of a wedge of spheres by giving a homotopy equivalence to an ordinal sum of antichains. Each antichain corresponds to the maximal subgroups of p-power index which supplement a fixed chief factor. In Section 4 (Theorem 4.9) we determine a CL-shellable (chain lexicographically shellable) subposet of $S^p(G)$ for $p \neq 2$ and for $p = 2$ in the case that the group has no composition factor isomorphic to $\mathrm{PSL}(2,q)$ and $q + 1 = 2^n$ for some n. Since this shellable subposet is G-homotopy equivalent to $S^p(G)$, our observation enables us to retrieve and understand combinatorially the results on the homotopy type for "most" groups. Shellability is a very strong combinatorial tool which has been of use in various branches of group theory (e.g. theory of buildings, Bruhat orders of Coxeter groups). Moreover, this shows that the combinatorial structure of $S^p(G)$ is rather restricted. For general groups it remains open whether there is a CL-shellable subposet of $S^p(G)$ which is G-homotopy equivalent to $S^p(G)$. We refer the reader for an introduction to the theory of shellable posets to [2]. As a general reference for combinatorial tools used in this paper we recommend [12].

Throughout this paper we fix a chief series $\mathcal{R} : 1 = N_0 < \cdots < N_{k_G} = G$ of the group G. In particular, we will write in the sequel k_G for the length of a chief series of the group G. By $\mathcal{I}_p$ we denote the set of indices j for which the set

$$\mathcal{K}_j = \{M \mid MN_j = G \text{ and } N_{j-1} \leq M,\ M \in S^p(G)\}$$

is not empty. We call the elements of $\mathcal{K}_j$ the p-supplements of the chief factor N_j/N_{j-1}. We do not want to exclude the trivial case when $\mathcal{I}_p$ is empty by an additional assumption in the formulation of our statements. Therefore we define the product (resp. the sum) over an empty set to be 1 (resp. 0).

2. The homotopy type of the poset $S^p(G)$

For a finite group G we will determine in this section the homotopy type of $S^p(G)$ and prove a dual of the Quillen Conjecture for $S^p(G)$. Although by far not trivial the poset $S^p(G)$ is more easily tractable than $S_p(G)$ for which a determination of the homotopy type seems to be evasive.

Before we proceed analyzing the poset $S^p(G)$ we give some preparatory definitions and results about the homotopy type of posets. Let P be a poset on which a group G acts order preserving (i.e. for elements $x, y \in P$ for which $x \leq y$ holds and an element $g \in G$ the images x^g and y^g also satisfy $x^g \leq y^g$). We will call a poset with such a G-action a G-poset. Since we are interested in group representations on the homology groups of the order complex of a poset we will analyze the G-homotopy type of a poset P. Of course the G-representations on the homology groups of the order complex of two G-homotopy equivalent posets are isomorphic. We write P^* for the dual poset of P (the poset on the elements of P which is obtained by reversing the order relation of P). Since the order complexes CP and CP^* are G-isomorphic, the posets P and P^* are G-homotopy equivalent (even G-homeomorphic). Therefore, we may interchange "$\leq$" and "$\geq$" in the applications of the following proposition.

Proposition 2.1 (Topological tools). *Let P and Q be G-posets.*

(i) (See [10, Theorem A] and [14, Theorem 1].) *Suppose that $f : Q \to P$ is a G-equivariant poset homomorphism. If for all $p \in P$ the subposet $f^{-1}(p)_{\geq} = \{x \in Q \mid f(x) \geq p\}$ of Q is* $\mathrm{Stab}_G(p)$*-contractible then P and Q are G-homotopy equivalent.*

(ii) (See [3] and [18].) *Let $p \in P$ be an element such that for all $x \in P$ the infimum $p \wedge x$ exists in P then P is* $\mathrm{Stab}_G(p)$*-contractible.*

Proof. The first assertion is the equivariant version [14, Theorem 1] of a result of Quillen [10, Theorem A]. We sketch the proof of the second assertion in order to give the reader a feeling for the topology of order complexes.

We consider the following map:

$$f : P \to Q = \{x \wedge p \mid x \in P\} \subseteq P, \qquad x \mapsto p \wedge x.$$

Obviously f is a $\mathrm{Stab}_G(p)$-equivariant poset morphism. Moreover, f satisfies $f(x) \leq x$ for all $x \in P$. One verifies that in this case f induces a $\mathrm{Stab}_G(p)$-

homotopy equivalence from P to $f(P) = Q$ (This is a special case of an equivariant formulation of the Homotopy Property in [10, (1.3)].) Now p is the greatest element of Q and therefore the order complex of Q is a cone with apex p. This shows that Q is $\mathrm{Stab}_G(p)$ contractible. By composing these homotopy equivalences one shows that P is $\mathrm{Stab}_G(p)$-contractible. $\square$

The argumentation in the proof of Proposition 2.1 (ii) actually shows that f induces an equivariant conical contraction in the sense of [10, (1.5)].

In the next lemma we recall some well-known facts about intersection of supplements of chief factors. Since the results are fundamental for our further investigations we will also provide the proofs.

Lemma 2.2 (Group-theoretical tools). (i) *If U and H are subgroups of G such that $UH = G$, then*

$$[G : (U \cap H)] = [G : U] \cdot [G : H].$$

(ii) *Let $\mathcal{I} \subset \{0, \dots, k_G\}$ be a set of indices of normal subgroups N_i in the fixed chief series. Let K_i for $i \in \mathcal{I}$ be subgroups such that $N_{i-1} \leq K_i$ and $K_i N_i = G$. Then for all indices j*

$$N_j \bigcap_{i \in \mathcal{I}} K_i = \bigcap_{j < i \in \mathcal{I}} K_i \quad \textit{and} \quad [G : \bigcap_{i \in \mathcal{I}} K_i] = \prod_{i \in \mathcal{I}} [G : K_i].$$

Proof. The first assertion is just a special form of Lagrange's lemma since in this case $|G| = |UH| = (|U| \cdot |H|)/|U \cap H|$.

The essential ingredient of the proof of the second assertion is Dedekind's modular law: For $A, B \in G$ and $N \leq B$ the identity $(NA \cap B) = N(A \cap B)$ holds.

Now let j be the least element of the set $\mathcal{I}$. Then for all $j' \in \mathcal{I} - \{j\}$ the subgroup N_j is contained in $K_{j'}$. By assumption $N_j K_j = G$ holds. Hence an application of Dedekind's law proves

$$N_j \bigcap_{i \in \mathcal{I}} K_i = N_j (K_j \cap \bigcap_{j < i \in \mathcal{I}} K_i) = N_j K_j \cap \bigcap_{j < i \in \mathcal{I}} K_i = \bigcap_{j < i \in \mathcal{I}} K_i.$$

Since for arbitrary indices $j \leq j'$ the subgroup N_j is contained in $N_{j'}$, induction on $|\mathcal{I}|$ shows for an arbitrary index j

$$(*) \qquad N_j \bigcap_{i \in \mathcal{I}} K_i = \bigcap_{j < i \in \mathcal{I}} K_i.$$

By definition N_j is a normal subgroup. Therefore, Lagrange's lemma gives

$$(**) \qquad [G : \bigcap_{i \in \mathcal{I}} K_i] = [G : N_j \bigcap_{i \in \mathcal{I}} K_i] \cdot [N_j : (N_j \cap \bigcap_{i \in \mathcal{I}} K_i)].$$

By the first part of the proof and by induction we obtain for the first factor on the right-hand side

$$[G : N_j \bigcap_{i \in \mathcal{I}} K_i] = [G : \bigcap_{j<i \in \mathcal{I}} K_i] = \prod_{j<i \in \mathcal{I}} [G : K_i].$$

Now let j be again the least element of $\mathcal{I}$. Since $N_j \leq K_i$ for all $i \in \mathcal{I} - \{j\}$ and since $N_j K_j = G$ the second factor on the right-hand side of $(**)$ satisfies

$$[N_j : (\bigcap_{i \in \mathcal{I}} K_i \cap N_j)] = [N_j : (K_j \cap N_j)] = [G : K_j].$$

The assertion follows immediately form a combination of the last two identities. □

Now we describe the construction of the ordinal sum of two posets [12, Section 3.2]. This construction will prove to be the key for the investigation of the topological behavior of $S^p(G)$. For two posets P and Q the ordinal sum $P \oplus Q$ is the poset on the disjoint union of P and Q which has the same order relations as P and Q and the additional relations $x \leq y$ for $x \in P$ and $y \in Q$. Obviously the operation $\oplus$ is not commutative in general (i.e. $P \oplus Q$ and $Q \oplus P$ are in general not isomorphic). However, there is an obvious one-to-one correspondence between the chains in $P \oplus Q$ and those in $Q \oplus P$; thus:

Remark 2.3. Let $P = (\cdots(P_1 \oplus P_2) \oplus \cdots \oplus P_{k-1}) \oplus P_k)$ be an ordinal sum of the G-posets $P_1, \ldots, P_k$. Then for every permutation σ in the symmetric group S_k the order complexes of P and $(\cdots(P_{\sigma(1)} \oplus P_{\sigma(2)}) \oplus \cdots \oplus P_{\sigma(k-1)}) \oplus P_{\sigma(k)})$ are G-isomorphic.

The preceding remark shows that for homological and topological purposes we do not have to care about the non-commutativity of $\oplus$. In order to have the poset $\bigoplus_{i=1}^k P_i$ well-defined we set

$$\bigoplus_{i=1}^{k} P_i = (\cdots(P_1 \oplus P_2) \oplus \cdots \oplus P_{k-1}) \oplus P_k).$$

Now we are in position to state the key lemma for the analysis of the G-homotopy type of $S^p(G)$. We have formulated the lemma in very general terms in order to show that our results on $S^p(G)$ generalize easily once sufficient group-theoretical knowledge is provided (see Section 5 (5.4)). For a set X of proper subgroups of a group G we define

$$\mathcal{K}_j^X = \{U \mid UN_j = G \text{ and } N_{j-1} \leq U,\ U \in X\}$$

to be the set of X-supplements of the chief factor N_j/N_{j-1}. By $\mathcal{I}_X$ we denote the set of all j for which $\mathcal{K}_j^X$ is not empty. Of course if $X = S^p(G)$ then

$\mathcal{K}_j^X = \mathcal{K}_j$ and $\mathcal{I}_X = \mathcal{I}_p$ for the sets $\mathcal{K}_j$ and $\mathcal{I}_p$ defined in the Introduction. As already mentioned before we have fixed a chief series throughout this paper.

Lemma 2.4. *Let X be a set of proper subgroups of the group G which is closed under conjugation by elements of G. We regard X as a poset ordered by inclusion. Assume that the following conditions are fulfilled:*

(i) *$U \in X$ implies $N_jU \in X \cup \{G\}$ for all normal subgroups N_j in the fixed chief series.*

(ii) *If $K \in \mathcal{K}_j^X$ and $U \in X$ such that $K \leq UN_{j-1}$ then $U \cap K$ is an element of X.*

Under this hypothesis the following statements hold:

(a) *For each $U \in X$ there exists exactly one index $j(U)$ such that*

$$G = N_{j(U)}U > N_{j(U)-1}U \in \mathcal{K}_{j(U)}^X.$$

(b) *For $K \in \mathcal{K}_j^X$ and $U \in X$ we have $K \leq UN_{j-1}$ if $j(U) < j$ $(= j(K))$ and $K \not\leq UN_{j-1}$ if $j(U) > j$.*

(c) *The poset*

$$X_c = \{\bigcap_{j \in \mathcal{I}} K_j \mid \emptyset \neq \mathcal{I} \subseteq \mathcal{I}_X \text{ and } K_j \in \mathcal{K}_j^X\}$$

is a subposet of X.

(d) *The posets X, X_c and $\bigoplus_{i \in \mathcal{I}_X} \mathcal{K}_i^X$ are G-homotopy equivalent (note that each set $\mathcal{K}_i^X$ can be regarded as a subposet of the poset X).*

Proof. (a) As the normal subgroup N_i is a proper subgroup of N_{i+1} there is exactly one index $j(U)$ such that

$$N_0U = U \leq N_{j(U)-1}U < N_{j(U)}U = G.$$

But in this case by (i) $N_{j(U)-1}U \in X$ and $N_{j(U)}(N_{j(U)-1}U) = G$. Hence $N_{j(U)-1}U$ is an element of $\mathcal{K}_{j(U)}^X$.

(b) Let $K \in \mathcal{K}_j^X$ and $U \in X$. If $j(U) < j$ then $K \leq G = UN_{j(U)} = UN_{j-1}$. If $j(U) > j$ then $G = KN_{j(U)-1} = KN_{j(U)-1}U$. So $K \leq N_{j(U)-1}U$ implies $N_{j(U)-1}U = G$ contradicting the definition of $j(U)$.

(c) Now we prove $X_c \subseteq X$. Let $A \in X_c$ be an arbitrary subgroup in X_c. Thus there is a set of indices $\mathcal{I}$ and $K_i \in \mathcal{K}_i^X$, $i \in \mathcal{I}$, such that $A = \bigcap_{i \in \mathcal{I}} K_i$. Let j be the greatest index in $\mathcal{I}$. By induction we may assume $B = \bigcap_{i \in \mathcal{I}-\{j\}} K_i \in X$. But by Lemma 2.2(ii) we obtain $BN_{j-1} = G > K_j$. Thus $A = B \cap K_j \in X$ by assumption (ii).

(d) In a first step we prove that if X satisfies the conditions (i) and (ii) then X_c satisfies the hypothesis too. From Lemma 2.2(ii) we infer condition (i) for the set X_c. The verification of condition (ii) is a bit more involved. For $U \in X_c$ there is a set $\mathcal{I}$ and $K_i \in \mathcal{K}_i^X$ for $i \in \mathcal{I}$ such that $U = \bigcap_{i \in \mathcal{I}} K_i$. From the assumption and Lemma 2.2(ii) we conclude $\bigcap_{j\leq i\in\mathcal{I}} K_i = UN_{j-1} \geq K$. So $i > j$ implies $i \notin \mathcal{I}$ and if $j \in \mathcal{I}$ then $K_j \geq K$. Thus $U \cap K = \bigcap_{j>i\in\mathcal{I}} K_i \cap K$ is in X_c. This completes the verification of condition (ii). Furthermore, it is easy to conclude from our argumentation that $\mathcal{I}_X = \mathcal{I}_{X_c}$ and $\mathcal{K}_i^X = \mathcal{K}_i^{X_c}$ for $i \in \mathcal{I}_X$.

Thus by the transitivity of G-homotopy equivalence it is enough to show that X and $\bigoplus_{i\in\mathcal{I}_X} \mathcal{K}_i^X$ are G-homotopy equivalent.

At this point it is convenient to have control over the actual ordering in $\bigoplus_{i\in\mathcal{I}_X} \mathcal{K}_i^X$. By definition the ordering is constructed iteratively by $(\cdots(\mathcal{K}_{i_t}^X \oplus \mathcal{K}_{i_{t-1}}^X) \oplus \cdots) \oplus \mathcal{K}_{i_1}^X$ for the ordered index set $\{i_1 < \cdots < i_t\} = \mathcal{I}_X$. In order to distinguish between the order in the ordinal sum and the order in the subgroup lattice we write $\leq_\oplus$ for the order in the ordinal sum. For $K, K' \in \bigoplus_{i\in\mathcal{I}_X} \mathcal{K}_i$ we have defined: $K \leq_\oplus K'$ if either $j(K) > j(K')$ or $j(K) = j(K')$ and $K \leq K'$. So, $K \leq_\oplus K'$ if and only if $K \leq N_{j(K)-1}K'$ (this is a special case of (b)). Now we consider the following map

$$f : X \to \bigoplus_{i\in\mathcal{I}_X} \mathcal{K}_i^X, \qquad U \mapsto N_{j(U)-1}U.$$

Since $N_{j(U)-1}$ is a normal subgroup the mapping f is G-equivariant. Moreover, it is trivial that $U \leq H \in X$ implies $N_jU \leq N_jH$. Thus f is a poset morphism (here the definition of $\leq_\oplus$ is crucial).

Now we want to apply Proposition 2.1(i) to the mapping f. Thus we have to show that for every $K \in \bigoplus_{i\in\mathcal{I}_X} \mathcal{K}_i^X$ the poset $f^{-1}(K)_{\geq_\oplus}$ is $N_G(K)$-contractible. We will prove this by an application of Proposition 2.1(ii) to $f^{-1}(K)_{\geq_\oplus}$ for $p = K$. In order to prove that Proposition 2.1(ii) applies we have to show that $K \cap U$ is in $f^{-1}(K)_{\geq_\oplus}$ for $U \in f^{-1}(K)_{\geq_\oplus}$. Suppose U is a subgroup in $f^{-1}(K)_{\geq_\oplus}$, in particular $N_{j(K)-1}U \geq K$. By assumption (ii) we have $U \cap K \in X$. Furthermore, Dedekind's identity shows that $N_{j(K)-1}(U \cap K) = (N_{j(K)-1}U) \cap K = K$ holds. Thus $j(K \cap U) = j(K)$ and therefore $f(U \cap K) = K \geq_\oplus K$. Finally we conclude $U \cap K \in f^{-1}(K)_{\geq_\oplus}$, which completes the proof. □

Now we prove that the preceding lemma applies to the set of subgroups whose index is a product $\neq 1$ of a fixed set of primes π. In particular we retrieve for $\pi = \{p\}$ our poset $S^p(G)$. Let π be a set of primes and define $S^\pi(G) = \{U \mid [G : U] = \prod_{p\in\pi} p^{n_p} \neq 1\}$.

Proposition 2.5. *Let π be a set of primes. Then the poset $S^\pi(G)$ is G-homotopy equivalent to the poset $\bigoplus_{j\in\mathcal{I}_{S^\pi(G)}} \mathcal{K}_j^{S^\pi(G)}$.*

Proof. By Lemma 2.4(d) it will suffice to verify conditions (i) and (ii) of Lemma 2.4 for the poset $X = S^\pi(G)$. Obviously $UN \in S^{\pi(G)} \cup \{G\}$ for any normal subgroup N and $U \in S^\pi(G)$. Now suppose $U \in S^\pi(G)$ and

$K \in \mathcal{K}_j^{S^\pi(G)}$ are subgroups such that $K \le N_{j-1}U$. By Lemma 2.4(b) we have to treat the following two cases. If $j(U) < j(K) = j$ then $N_{j(U)} \le N_{j-1} \le K$. Thus $UK = G$ and by Lemma 2.2(i) $U \cap K \in S^\pi(G)$. In the second case $j(U) = j(K) = j$ and $K \le UN_{j-1}$. This implies $K = N_{j-1}(U \cap K)$ and therefore

$$\begin{aligned}&[G : (U \cap K)]\\ &\quad = [G : (U \cap K)N_{j-1}] \cdot [(U \cap K)N_{j-1} : (U \cap K)N_{j-1}]\\ &\quad = [G : K] \cdot [N_{j-1} : (U \cap N_{j-1})].\end{aligned}$$

But $[N_{j-1} : (U \cap N_{j-1})] = [UN_{j-1} : U]$ divides $[G : U]$. Hence every prime dividing $[G : (U \cap K)]$ lies in π and therefore $U \cap K \in S^\pi(G)$. $\square$

The following corollary reduces the problem of deciding whether $S^\pi(G)$ is contractible or not to the problem of deciding whether one of the posets $\mathcal{K}_j^{S^\pi(G)}$ is contractible or not.

Corollary 2.6. *The poset $S^\pi(G)$ is contractible if and only if there is a $j \in \mathcal{I}_{S^\pi(G)}$ for which $\mathcal{K}_j^{S^\pi(G)}$ is contractible.*

Proof. This observation follows immediately from the preceding proposition and the fact that the geometric realization of the ordinal sum of posets is the join of the geometric realizations of the summands. Now it is a well-known fact that the join of finitely many topological spaces is contractible if and only if at least one of the spaces is itself contractible. $\square$

In the following we will return to the case $\pi = \{p\}$ for a single prime p. Of course in this case $S^p(G) = S^\pi(G)$ and $\mathcal{K}_i = \mathcal{K}_i^{S^p(G)}$.

Proposition 2.7 (cf. [15, Satz 4.3]). *For all $i \in \mathcal{I}_p$ the elements of $\mathcal{K}_i$ are maximal subgroups of p-power index. Furthermore, every maximal subgroup of p-power index is contained in exactly one of the sets $\mathcal{K}_j$. In particular, each set $\mathcal{K}_j$ regarded as a poset is an antichain. For each $j \in \mathcal{I}_p$ and $K, K' \in \mathcal{K}_j$ we have $[G : K] = [G : K']$. Moreover, $(N_j \cap K)/N_{j-1}$ is a minimal element of $S^p(N_j/N_{j-1})$.* $\square$

We would like to remark that the proof of Proposition 2.7 is based on the classification of the subgroups of prime power index in finite simple groups by Guralnik [7]. Since his proof uses the classification of finite simple groups, our result also depends on this deep theorem.

For a finite poset P we denote by $\mu(P)$ its Möbius number. We refer the reader to the paper of Rota [11] for the definition and the basic properties of

$\mu(P)$. We will use the notation $[H, G]$ to denote the interval $\{U \mid H \leq U \leq G\}$ and we write $[H, G]_c$ for $([H, G] \cap S_c^p(G)) \cup \{G\}$.

In order to clarify the situation we state at this point a result of the first author which shows that $S_c^p(G)$ is independent of the chosen chief series. Moreover, it gives a purely combinatorial characterization of the subgroups $U \in S^p(G)$ which are elements of the subposet $S_c^p(G)$. We do not need this theorem in its full strength but we think it might be helpful for an understanding of the situation.

Proposition 2.8 (cf. [15, Satz 4.8]). *$S_c^p(G)$ is the set of all $U \in S^p(G)$ for which $\mu([U, G]) \neq 0$. Thus $S_c^p(G)$ is independent of the chosen chief series.*

This implies the following: If $U \in S_c^p(G)$ and $U \leq V$, then $V \in S_c^p(G)$ if and only if V is the intersection of all maximal subgroups containing V. □

In the next theorem we determine the homotopy type as well as the Möbius number of $S^p(G)$.

Theorem 2.9. *Let G be a finite group. Then $S^p(G)$ is homotopy equivalent to a wedge of $n = \prod_{i \in \mathcal{I}_p}(|\mathcal{K}_i| - 1)$ spheres of dimension $|\mathcal{I}_p| - 1$. In particular, the Möbius number of $S^p(G)$ is given by*

$$\mu(S^p(G)) = -\prod_{i \in \mathcal{I}_p}(1 - |\mathcal{K}_i|).$$

Proof. By Lemma2.4 we can work in the poset $\bigoplus_{i \in \mathcal{I}_p} \mathcal{K}_i$. Now it is a well-known fact that the ordinal sum of two posets has the same homotopy type as the topological join of their geometric realizations. By Proposition 2.7 the posets $\mathcal{K}_j$ are antichains. Hence they are homotopy equivalent to a wedge of $|\mathcal{K}_j| - 1$ spheres of dimension 0. Now the join of a wedge of i-spheres and a wedge of j-spheres for two non-negative integers i and j is a wedge of $(i + j + 1)$-spheres. The number of $(i + j + 1)$-spheres in the join is the product of the number of i-spheres and the number of j-spheres. By induction this proves that $S^p(G)$ has the homotopy type of a wedge of $\prod_{i \in \mathcal{I}_p}(|\mathcal{K}_i| - 1)$ spheres of dimension $|\mathcal{I}_p| - 1$. The formula for the Möbius number follows combinatorially since the Möbius number of an ordinal sum of two posets is (-1) times the product of the Möbius numbers of the summands. Recall that the Möbius number of an antichain $\mathcal{A}$ is $|\mathcal{A}| - 1$. By topological reasoning we obtain the Möbius number of a poset which is homotopy equivalent to a wedge of i-sphere as $(-1)^i$ times the number of spheres. □

From the preceding theorem one easily derives the following analog of the Quillen Conjecture for $S^p(G)$.

Theorem 2.10. *For a finite group G the following four statements are equivalent.*
(i) $S^p(G)$ *is G-contractible.*
(ii) $S^p(G)$ *is contractible.*
(iii) $\mu(S^p(G)) = 0$.
(iv) $O^p(G) \neq G$.

Proof. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are trivial. Now assume the hypothesis of (iii). From Theorem 2.9 we see that $\mu(S^p(G)) = 0$ implies that there is a chief factor N/M for which a unique subgroup $U \in S^p(G)$ satisfies $UN = G$ and $M \leq U$. But the group U is then necessarily normal in G. Therefore, $O^p(G) \leq U \neq G$ which shows (iv).

Now we assume $O^p(G) \neq G$. Then the element $O^p(G)$ of $S^p(G)$ satisfies the assumptions of Proposition 2.1 (ii) and we are done. □

As an easy consequence of part (iii) of the equivalence we have that the Lefschetz module $\Lambda(S^p(G)) = \sum_{i\geq 0}(-1)^i\widetilde{H}_i(CS^p(G))$ vanishes if and only if $O^p(G) \neq G$. We would like to mention that for a G-poset P the module $\Lambda(P)$ is in general a virtual module. Since in our case only one homology group does not vanish either $\Lambda(S^p(G))$ is a usual module or $-\Lambda(S^p(G))$ is a usual module.

Before we proceed to the determination of the representation of G on the non-vanishing homology group of $S^p(G)$, we will exhibit that the homotopy equivalence between $X = S^p(G)$ and $X_c = S^p_c(G)$, implicitly described in the proof of Lemma 2.4(d), can be chosen to be a deformation retraction. The special structure of $S^p(G)$ comes into play through the following group-theoretical lemma which is of independent interest.

Lemma 2.11. *Let U be an element of $S^p(G)$. Then there is a unique element of $K \in \mathcal{K}_{j(U)}$ which contains U.*

Proof. Suppose that K and K' are two subgroups in $\mathcal{K}_{j(U)}$ which contain H. Hence $N_{j(U)-1} \leq K, K'$ and $N_{j(U)-1}U$ is a subgroup of $K \cap K'$. But $UN_{j(U)-1}$ is also a subgroup in $\mathcal{K}_{j(U)}$. By Proposition 2.7 the set $\mathcal{K}_{j(U)}$ is an antichain. Thus $K = N_{j(U)}U = K'$. □

Lemma 2.12. *The poset $S^p_c(G)$ is a deformation retract of $S^p(G)$. The corresponding homotopy equivalence is G-equivariant.*

Proof. We have to prove that the inclusion $i : S^p_c(G) \hookrightarrow S^p(G)$ induces a G-homotopy equivalence. For an element $U \in S^p(G)$ the preimage $i^{-1}(U)_{\geq} = \{H \mid H \in S^p_c(G), H \geq U\}$ is $\{\bigcap_{i\in\mathcal{I}} K_i \mid U \leq K_i \in \mathcal{K}_i, \emptyset \neq \mathcal{I} \subseteq \mathcal{I}_p\}$.

Let K denote the unique element in $\mathcal{K}_{j(U)}$ containing U (see Lemma 2.11). By definition of $S^p_c(G)$ for each $H \in i^{-1}(U)$ there is a set $\mathcal{I} \subset \mathcal{I}_p$ and $K_i \in \mathcal{K}_i$,

$i \in \mathcal{I}$, such that $H = \bigcap_{i\in\mathcal{I}} K_i$. As $U \leq H$ we conclude $U \leq K_i$. If $j(U) \in \mathcal{I}$ then $K = K_{j(U)}$ and $H \cap K = H$. If $j(U) \notin \mathcal{I}$ then again by the definition of $S_c^p(G)$ we conclude $H \cap K \in S_c^p(G)$ and $U \leq H \cap K$. This proves that $K \cap H$ is in $i^{-1}(U)$ for all $H \in i^{-1}(U)$. Now Proposition 2.1(ii) shows that $i^{-1}(U)$ is $N_G(K)$-contractible. But Lemma 2.11 implies that every element of G which normalizes U also normalizes K. Therefore, $i^{-1}(U)$ is also $N_G(U)$-contractible. So i is a G-homotopy equivalence by Proposition 2.1(i). □

So far we have related the poset $\bigoplus_{i\in\mathcal{I}_p} \mathcal{K}_i$ to the poset $S^p(G)$. Our next project is to determine the representation of the group G on the homology of $\bigoplus_{i\in\mathcal{I}_p} \mathcal{K}_i$. Before we can do that we will state some general results on the homology of ordinal sum of antichains.

3. Homology representations of automorphism groups of ordinal sums of antichains

In this section we will determine the representation of a group G on the reduced simplicial homology of the order complex of an ordinal sum of antichains on which the group G acts.

The results will remind the reader about some facts about semimodular lattices [4,13]. Indeed, ordinal sums of antichains belong to the class of totally semimodular posets which generalize semimodularity to non-lattices. Since our intention is mainly the determination of the representation of G on $\widetilde{H}_i(S^p(G))$ we will confine ourselves to the particular situation (ordinal sum of antichains) occurring here.

For a simplicial complex Δ we denote by $\widetilde{H}_i(\Delta)$ the reduced simplicial homology of Δ. In general we assume that the coefficient ring is the ring of integers $\mathbb{Z}$.

So for the rest of this section let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be antichains which are G-sets for a finite group G. The main application of this situation will of course be the case $\mathcal{A}_i = \mathcal{K}_i$. The first part of the following remark is a well-known fact. The second part follows immediately from the first together with Lemma 2.4 (see also Theorem 2.9).

Remark 3.1. Let $P = \bigoplus_{i=1}^k \mathcal{A}_i$ be an ordinal sum of k antichains. Then $\widetilde{H}_j(CP) = 0$ if $i \neq k-1$ and $\widetilde{H}_{k-1}(CP)$ is a free $\mathbb{Z}$-module of rank $(-1)^{k-1} \cdot \mu(P) = \prod_{i=1}^k (|\mathcal{A}_i| - 1)$. In particular, $\widetilde{H}_j(CS^p(G)) = 0$ if $i \neq |\mathcal{I}_p| - 1$ and $\widetilde{H}_{|\mathcal{I}_p|-1}(CS^p(G))$ is a free $\mathbb{Z}$-module of rank $(-1)^{|\mathcal{I}_p|-1} \cdot \mu(CS^p(G)) = \prod_{i=1}^{|\mathcal{I}_p|} (|\mathcal{K}_i| - 1)$.

In the following theorem the representation of G on the reduced homology $\widetilde{H}_i(CP)$ for $P = \bigoplus_{i=1}^k \mathcal{A}_i$ is determined.

We denote for a G-set X the permutation module over $\mathbb{Z}$ with basis X by $\mathbb{Z}X$. The augmentation map $\varepsilon : \mathbb{Z}X \to \mathbb{Z}$ is the unique $\mathbb{Z}$-module homomorphism, that sends x to 1 for all $x \in X$. Let $\mathbf{I}(X)$ denote the kernel of ε. We would like to thank the referee for the permission to include his proof and his formulation of the following theorem. His results simplify our previous approach which was inspired by results given in [5].

Theorem 3.2. *Let $P = \bigoplus_{i=1}^{k} \mathcal{A}_i$ be an ordinal sum of antichains $\mathcal{A}_i$. Assume that the antichains $\mathcal{A}_i$ are G-sets for a group G. Then*

$$\widetilde{H}_{k-1}(CP) \cong \mathbf{I}(\mathcal{A}_1) \otimes \mathbf{I}(\mathcal{A}_2) \otimes \cdots \otimes \mathbf{I}(\mathcal{A}_k).$$

as G-modules.

Proof. We verify the claim by induction on k. Clearly $\widetilde{H}_0(C\mathcal{A}_1) \cong \mathbf{I}(\mathcal{A}_1)$. Let $Q = \bigoplus_{i=1}^{k-1} \mathcal{A}_i$. By $C_*(P)$ we denote the augmented chain complex of P, so that $C_{-1}(P) = \mathbb{Z}$ and we write $\partial_P : C_*(P) \to C_*(P)$ for the boundary operator.

Now the mapping $a_1 < a_2 \ldots < a_{k-1} < a_k \mapsto a_1 < a_2 \ldots < a_{k-1} \otimes a_k$ for $a_i \in \mathcal{A}_i$ yields an isomorphism:

$$C_{k-1}(P) \cong C_{k-2}(Q) \otimes C_0(\mathcal{A}_k).$$

Similarly we obtain:

$$C_{k-2}(P) \cong (C_{k-3}(Q) \otimes C_0(\mathcal{A}_k)) \oplus (C_{k-2}(Q) \otimes C_{-1}(\mathcal{A}_k)).$$

Via these isomorphisms we can represent the boundary operator $\partial_P : C_{k-1}(P) \to C_{k-2}(P)$ by the homomorphism $(\delta_Q \otimes \mathrm{id}) \oplus ((-1)^k \mathrm{id} \otimes \partial_{\mathcal{A}_k})$ (note that $\partial_{\mathcal{A}_k} : C_0(\mathcal{A}_k) \to C_{-1}(\mathcal{A}_k)$ is the augmentation map). As $C_k(P) = 0$ we get

$$\begin{aligned}
\widetilde{H}_{k-1}(CP) &= \mathrm{Ker}(\partial_P : C_{k-1}(P) \to C_{k-2}(P)) \\
&\cong \mathrm{Ker}((\partial_Q \otimes \mathrm{id}) \oplus ((-1)^k \mathrm{id} \otimes \partial_{\mathcal{A}_k})) \\
&= \mathrm{Ker}(\partial_Q \otimes \mathrm{id}) \cap \mathrm{Ker}((-1)^k \mathrm{id} \otimes \partial_{\mathcal{A}_k}) \\
&= \mathrm{Ker}(\partial_Q \otimes \mathrm{id}) \cap \mathrm{Ker}(\mathrm{id} \otimes \partial_{\mathcal{A}_k}).
\end{aligned}$$

The short exact sequence $0 \to \mathbf{I}(\mathcal{A}_k) \to C_0(\mathcal{A}_k) \to C_{-1}(\mathcal{A}_k) \to 0$ splits over $\mathbb{Z}$ and therefore remains exact after tensoring with an arbitrary module. Moreover, the exact sequence $0 \to \widetilde{H}_{k-2}(Q) \to C_{k-2}(Q) \to C_{k-3}(Q)$ (as $C_{k-1}(Q) = 0$) remains exact after tensoring with a free $\mathbb{Z}$-module (in particular with $C_i(\mathcal{A}_k)$). The following diagram is therefore a commutative diagram of exact sequences.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \widetilde{H}_{k-2}(Q)\otimes \mathbf{I}(\mathcal{A}_k) \arrow[r] \arrow[d] & \widetilde{H}_{k-2}(Q)\otimes C_0(\mathcal{A}_k) \arrow[r] \arrow[d] & \widetilde{H}_{k-2}(Q)\otimes C_{-1}(\mathcal{A}_k) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{k-2}(Q)\otimes \mathbf{I}(\mathcal{A}_k) \arrow[r] & C_{k-2}(Q)\otimes C_0(\mathcal{A}_k) \arrow[r, "\mathrm{id}\otimes\partial_{\mathcal{A}_k}"'] \arrow[d, "\partial_Q\otimes \mathrm{id}"] & C_{k-2}(Q)\otimes C_{-1}(\mathcal{A}_k) \arrow[r] & 0 \\
 & & C_{k-3}(Q)\otimes C_0(\mathcal{A}_k) & &
\end{tikzcd}
$$

Now

$$
\begin{aligned}
&\widetilde{H}_{k-2}(Q)\otimes \mathbf{I}(\mathcal{A}_k)\\
&\quad\cong \operatorname{Ker}(\widetilde{H}_{k-2}(Q)\otimes C_0(\mathcal{A}_k)\to C_{k-2}(Q)\otimes C_{-1}(\mathcal{A}_k))\\
&\quad\cong \operatorname{Ker}(\partial_Q\otimes \mathrm{id})\cap \operatorname{Ker}(\mathrm{id}\otimes\partial_{\mathcal{A}_k})\cong \widetilde{H}_{k-1}(CP). \qquad \square
\end{aligned}
$$

If we fix some $x_0 \in X$, then $\{x - x_0 \mid x_0 \neq x \in X\}$ is a $\mathbb{Z}$-basis of $\mathbf{I}(X)$. Thus $\mathbf{I}(X)$ is free as a $\mathbb{Z}$-module and the matrix representation of G with respect to this basis has entries $1, -1$ or 0. This implies the following corollary.

Corollary 3.3. *The representation of G on $\widetilde{H}_{k-1}(CP)$ for $P = \bigoplus_{i=1}^{k} \mathcal{A}_i$ can be realized by matrices with entries $+1$, -1 and 0.* $\square$

By Theorem 2.9 and Lemma2.4 we can apply the results of this section to $CS^p(G)$. This gives:

Theorem 3.4. *Let G be a group.*
Then $\widetilde{H}_k(CS^p(G))$ is zero for $k \neq |\mathcal{I}_p| - 1$ and

$$\widetilde{H}_{|\mathcal{I}_p|-1}(CS^p(G)) \cong \mathbf{I}(\mathcal{K}_1)\otimes\mathbf{I}(\mathcal{K}_2)\otimes\cdots\otimes\mathbf{I}(\mathcal{K}_{|\mathcal{I}_p|})$$

as G-modules.
Moreover, the representation of G on $\widetilde{H}(CS^p(G))$ can be realized by matrices with entries $+1$, -1 and 0. $\square$

Note that G acts by conjugation on $\mathcal{K}_i$. Thus the orbits of G on $\mathcal{K}_i$ are just the conjugacy classes of subgroups in $\mathcal{K}_i$.

For $K \in \mathcal{K}_i$ the action of G on the orbit of K is isomorphic to the action of G on the cosets of (the normalizer) $N_G(K)$ in G. As K is a maximal subgroup, we conclude that either $N_G(K) = K$ or that K is normal (and then $\widetilde{H}_k(CS^p(G)) = 0$ for all k).

4. Combinatorial properties of a subposet of $S^p(G)$

Having seen the nice topological behavior of $S_c^p(G)$ it is natural to ask whether this is caused by more combinatorial structure than revealed so far. This is actually true. In this section we will prove that $S_c^p(G)$ is a CL-shellable poset if either $p \neq 2$ or $p = 2$ and G has no composition factor isomorphic to some $\mathrm{PSL}(2,q)$ for $q + 1$ a 2-power, in other words q is a Mersenne prime. In the sequel we abbreviate the second condition by (notPSL). As one consequence of the fact that a poset is shellable we would like to mention that the geometric realization of its order complex is a Cohen–Macaulay space [2]. As every CL-shellable poset is homotopy equivalent to a wedge of spheres of the same dimension, this improves the results of Section 2. More combinatorial, topological and algebraic (in particular the connections to Cohen–Macaulay rings) implications can be found in [2] and [12].

The shellability of parts of the subgroup lattice has been investigated by the second author in [17]. Applied to our situation this result implies that the intervals in $S_c^p(G)$ are EL-shellable if G is a solvable group. It will turn out that we do not need to employ the methods developed in [17] in our particular situation since proper upper intervals $[U, G]_c$ in $S_c^p(G)$ are geometric lattices for groups satisfying (notPSL) (see Proposition 4.7). It is well known [2] that geometric lattices are shellable posets. On the other hand the global structure of $S_c^p(G)$ is far less trivial and for general groups G also the structure of the intervals in $S_c^p(G)$ is not known at all.

Recall that a poset P is bounded if there is a unique minimal element O_P and a unique maximal element 1_P in P (i.e. $P = [0_P, 1_P]$). If P is a bounded poset, we call $a \in P$ an atom if $[0_P, a]$ has exactly two elements. A poset P is graded if it is bounded and all maximal chains have the same length.

Definition 4.1. A recursive atom ordering of a bounded poset P is a linear ordering $<_{\mathrm{atom}}$ of the atoms of P which satisfies:

(C_1) Let x be an element of P such that for the atoms y and z the relations $y < x$ and $z < x$ hold. If $y <_{\mathrm{atom}} z$ then there is an atom $w <_{\mathrm{atom}} z$ of P and an atom z_1 of the interval $[z, x]$ such that z_1 covers w.

(C_2) Let y be an atom of P. Then there is a recursive atom ordering of $[y, 1]$ such that the atoms of $[y, 1]$ which cover some atom z of P preceding y in $<_{\mathrm{atom}}$ come first.

Proposition 4.2 (Björn and Wachs [2, Theorem 3.2]). *A graded poset is CL-shellable if and only if it admits a recursive atom ordering.* □

Define $\overline{S_c^p(G)} = S_c^p(G) \cup \{G, 1\}$ (here 1 is the trivial subgroup). We will prove that $\overline{S_c^p(G)}$ is a graded poset which (if $p \neq 2$ or (notPSL) holds) admits a recursive atom ordering. It is well known that adding (resp. removing) a least or greatest element from a CL-shellable poset leaves a CL-shellable poset. Therefore, $S_c^p(G)$ is CL-shellable if and only if $\overline{S_c^p(G)}$ is and the exhibition of a recursive atom ordering for $\overline{S_c^p(G)}$ satisfies in order to prove CL-shellability of $S_c^p(G)$.

Proposition 4.3. *Let $\mathcal{I}, \mathcal{I}' \subset \mathcal{I}_p$ be two index sets and let $K_i \in \mathcal{K}_i$ for $i \in \mathcal{I}$ and $K_i' \in \mathcal{K}_i$ for $i \in \mathcal{I}'$ be p-supplements. Suppose $\bigcap_{i\in\mathcal{I}} K_i$ is a subgroup of $\bigcap_{i\in\mathcal{I}'} K_i'$. Then the set $\mathcal{I}'$ is a subset of $\mathcal{I}$. If in addition $\mathcal{I} = \mathcal{I}'$ then $\bigcap_{i\in\mathcal{I}} K_i = \bigcap_{i\in\mathcal{I}} K_i'$.*

Suppose $1 < U_l < \cdots < U_1 < G$ is a maximal chain in $\overline{S_c^p(G)}$. Then there is an enumeration $\{i_1, i_2, \dots, i_{|\mathcal{I}_p|}\} = \mathcal{I}_p$ and $K_{i_j} \in \mathcal{K}_{i_j}$ such that $U_k = \bigcap_{k\leq j} K_{i_j}$. Furthermore, $\overline{S_c^p(G)}$ is a graded lattice.

Proof. Set $K_i = G$ for $i \notin \mathcal{I}$ and $K_i' = G$ for $i \notin \mathcal{I}'$. Then we infer from Dedekind's law $\bigcap_{i\in\mathcal{I}} K_i N_{j-1} = \bigcap_{j\leq i\in\mathcal{I}} K_i \leq \bigcap_{j\leq i\in\mathcal{I}'} K_i'$ for all $j \in \mathcal{I}_p$. Thus

$$(\bigcap_{i\in\mathcal{I}} K_i) N_{j-1} \cap N_j = N_j \cap K_j \leq N_j \cap K_j'.$$

Hence $K_j = G$ implies $K_j' = G$. This shows the inclusion $\mathcal{I}' \subseteq \mathcal{I}$. If $\mathcal{I} = \mathcal{I}'$ then $\bigcap_{i\in\mathcal{I}} K_i = \bigcap_{i\in\mathcal{I}} K_i'$ since Proposition 2.7 gives $|\bigcap_{i\in\mathcal{I}} K_i| = |\bigcap_{i\in\mathcal{I}} K_i'|$.

Now we fix a maximal chain $1 < U_l < \cdots < U_1 < G$ in $\overline{S_c^p(G)}$. As $U_k \in S_c^p(G)$ we have $U_k = \bigcap_{i\in\mathcal{I}_k} K_{k,i}$ for some $\mathcal{I}_k \subseteq \mathcal{I}_p$ and some $K_{k,i} \in \mathcal{K}_i$, $i \in \mathcal{I}_k$. As $U_k \leq U_{k-1}$ we get (as just proved) $\mathcal{I}_{k-1} \subset \mathcal{I}_k$. Fix some $i_k \in \mathcal{I}_k \setminus \mathcal{I}_{k-1}$. Thus $U_k \leq U_{k-1} \cap K_{k,i_k} \in S_c^p$. But $U_{k-1} \cap K_{k,i_k} < U_{k-1}$ (see Proposition 2.7), and so the maximality of our chain gives $U_k = U_{k-1} \cap K_{k,i_k}$.

By setting $K_{i_k} = K_{k,i_k}$ the preceding reasoning inductively proves $U_k = \bigcap_{k\leq j} K_{i_j}$.

Now it is obvious that $\overline{S_c^p(G)}$ is a graded poset.

If A and B are in $\overline{S_c^p(G)} \setminus \{1\}$ then the intersection of all maximal subgroups containing both A and B is the join of A and B in $\overline{S_c^p(G)}$ (see the second part of Proposition 2.8). As 1 is contained in A and B, the join of all elements contained in A and B is the meet of A and B. Hence $\overline{S_c^p(G)}$ is a lattice. □

Now we introduce the crucial ordering $\leq_{\text{atom}}$ on the atoms of $\overline{S_c^p(G)}$. For this purpose we fix for each set $\mathcal{K}_j \cup \{G\}$, $j \leq k_G$, a linear ordering $\leq_i$ such that G is the greatest element. For $H \in S_c^p$ we define H^j to be the least element (with respect to $\leq_j$) of $\mathcal{K}_j \cup \{G\}$ which contains H as a subgroup.

From Proposition 4.3 we deduce that $H = \bigcap_{j\in\mathcal{I}_p} H^j = \bigcap_{j\leq k_G} H^j$, and that $H \in S_c^p(G) \cup G$ is an atom in $\overline{S_c^p(G)}$ if and only if $H^j \neq G$ for all $j \in \mathcal{I}_p$.

Definition 4.4. For two different atoms $U, H \in \overline{S_c^p(G)}$ let $j_0 \in \mathcal{I}_p$ be the minimal index, such that $H^j = U^j$ for all $j_0 < j \leq k_G$. We set $H <_{\text{atom}} U$ if $H^{j_0} <_{j_0} U^{j_0}$.

Obviously $<_{\text{atom}}$ is a linear ordering of the atoms of $\overline{S_c^p(G)}$.

Lemma 4.5. *Let $U_1 <_{\text{atom}} U_2$ be atoms of $\overline{S_c^p(G)}$. Let $j_0 \leq k_G$ be the index such that $U_1^j = U_2^j$ for $j_0 < j \leq k_G$ and $U_1^{j_0} \neq U_2^{j_0}$. Then there is no $K \in \mathcal{K}_{j_0}$ such that $K \geq U_1$ and $K \geq U_2$.*

Proof. Assume such a K exists. Thus this K satisfies $U_2 \leq_{j_0} K$ and $N_{j_0-1} \leq K$. From the assumption we infer $N_{j_0}U_1 = \bigcap_{j_0<j} U_1^j = N_{j_0}U_2$.

Furthermore, $U_1N_{j_0} \cap K \geq U_1N_{j_0-1} = \bigcap_{j_0 \leq j} U_1{}^j$ and equality holds by Proposition 4.3. But $U_2 \leq U_2N_{j_0} = U_1N_{j_0}$, so that $U_2 \leq U_1N_{j_0} \cap K \leq (U_1)^{j_0}$. Therefore, $(U_2)^{j_0} \leq_{j_0} (U_1)^{j_0}$, contradicting $U_1 <_{\text{atom}} U_2$. □

Proposition 4.6. *The poset $\overline{S_c^p(G)}$ with the atom ordering $<_{\text{atom}}$ satisfies condition* (C_1) *of Definition* 4.1.

Proof. Assume the situation of condition (C_1). Hence let $U_1 <_{\text{atom}} U_2$ be two atoms of $\overline{S_c^p(G)}$ and let H be another subgroup in $\overline{S_c^p(G)}$ which contains U_1 and U_2. Let $j_0 \in \mathcal{I}_p$ be the index such that $U_1^j = U_2^j$ for $j_0 < j \leq k_G$ and $U_1^{j_0} \neq U_2^{j_0}$. Lemma 4.5 gives $H^{j_0} = G$.

Now we define subgroups $W_j \in \mathcal{K}_j$ for $j \in \mathcal{I}_p$ as follows:

$$W_j = \begin{cases} H^j & \text{if } j < j_0 \text{ and } H^j \neq G, \\ U_2^j & \text{if } j < j_0 \text{ and } H^j = G, \\ U_1^j & \text{if } j = j_0, \\ U_2^j & \text{if } j > j_0. \end{cases}$$

Set $W = \bigcap_{j \in \mathcal{I}_p} W_j$. As U_1 and U_2 are atoms of $\overline{S_c^p(G)}$ we deduce that $U_1^j \neq G$ and $U_2^j \neq G$ for all $j \in \mathcal{I}_p$. Thus $W_j \neq G$ for all $j \in \mathcal{I}_p$. By definition W is a subgroup in $S_c^p(G) \cup \{G\}$. Hence we conclude from Proposition 4.3 that W is an atom of $\overline{S_c^p(G)}$. By definition $W^j \leq_j W_j \leq_j U_2^j$ for $j \geq j_0$. From this observations we infer $W \leq_{\text{atom}} U_2$. Now we set $V = \bigcap_{j_0 \neq j \in \mathcal{I}_p} W_j$. Here Proposition 4.3 implies that V is an atom of the interval $[W, G]_c$. Analogously we deduce from $W_j \geq U_2$ for $j_0 \neq j \leq k_G$ that V is an atom of the interval $[U_2, G]_c$. So far we have shown that V covers W and U_2 in the lattice $\overline{S_c^p(G)}$. It remains to prove that V is a subgroup of H.

For the verification of this claim we will use Lemma 2.2(ii):

As $U_2 \leq H$ we conclude $N_{j_0}H \geq N_{j_0}U_2 = N_{j_0}V$ by definition of V. For $j < j_0$ we have $W_j \leq H^j$ and as $V_{j_0} = G = H^{j_0}$ we get:

$$V = N_{j_0}V \bigcap_{j<j_0} W_j \leq N_{j_0}H \bigcap_{j<j_0} H^j = H. \qquad \square$$

In the sequel we denote by $\mathbf{1}_U \uparrow^G$ the $\mathbb{F}_pG$-module induced from the trivial U-module $\mathbf{1}_U$ of a subgroup U of G. The definition of the prime p will always be obvious from the context. Furthermore, we denote by $\mathbf{J}$ the radical of $\mathbb{F}_pG$.

Proposition 4.7 (cf. [15, Satz 5.8]). *Suppose $p \neq 2$ or G fulfills* (notPSL). *Let $[\mathbf{1}_U \uparrow^G \mathbf{J}/\mathbf{J}^2]$ be the lattice of all submodules of $\mathbf{1}_U \uparrow^G \mathbf{J}/\mathbf{J}^2$. Then for $U \in S_c^p(G)$ the lattices $[U,G]_c$ and $[\mathbf{1}_U \uparrow^G \mathbf{J}/\mathbf{J}^2]$ are isomorphic.*

Since $\mathbf{1}_U \uparrow^G \mathbf{J}/\mathbf{J}^2$ is semisimple, the lattices $[U,G]_c$ and $[\mathbf{1}_U \uparrow^G \mathbf{J}/\mathbf{J}^2]$ are geometric lattice. $\square$

As geometric lattices are upper semimodular, the next lemma applies to the interval $[U,G]_c$ under the assumptions of the preceding proposition:

Lemma 4.8 (Björn and Wachs [2, Theorem 5.1]). *Suppose P is upper semimodular. Then every linear ordering of the atoms of P is a recursive atom ordering of P.* $\square$

Theorem 4.9. *Let G be a group. Then $\overline{S_c^p(G)}$ is a graded lattice. If p is an odd prime or if G satisfies* (notPSL), *then $\leq_{\text{atom}}$ is a recursive atom ordering of $\overline{S_c^p(G)}$. In particular, the poset $S_c^p(G)$ is CL-shellable.*

Proof. By Proposition 4.3 the poset $\overline{S_c^p(G)}$ is graded. Proposition 4.6 tells us that $\leq_{\text{atom}}$ satisfies condition (C_1) of Definition 4.1.

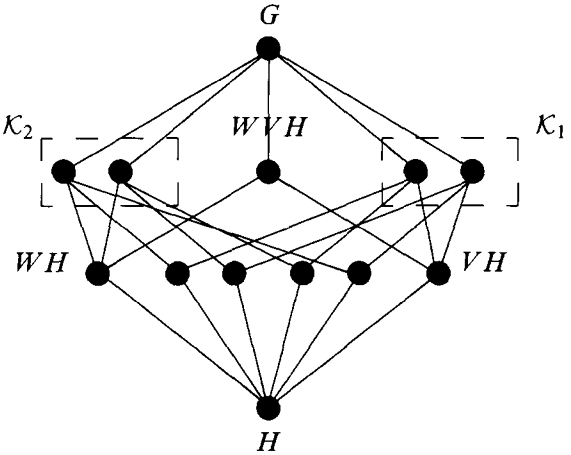
To verify condition (C_2) assume that y is an atom of $\overline{S_c^p(G)}$ and choose a linear ordering $\leq_{\text{atom}'}$ of the atoms of $[y,G]_c$ such that the atoms that cover some atom of $\overline{S_c^p(G)}$ preceding y in $\leq_{\text{atom}}$ come first.

As $[y,G]_c$ is a geometric lattice (Proposition 4.7) the ordering $\leq_{\text{atom}'}$ is a recursive atom ordering (Lemma 4.8). Thus condition (C_2) holds. So far we have shown that $\leq_{\text{atom}}$ is a recursive atom ordering. Now by Proposition 4.2 the poset $\overline{S_c^p(G)}$ is CL-shellable, and so is $S_c^p(G)$ (see the remark after Proposition 4.2). $\square$

However, Proposition 4.7 is not true in every group, as the following counterexample shows:

We define G as the semidirect product of $\mathrm{PSL}(3,2) \cong \mathrm{PSL}(2,7)$ (so, as expected G does not fulfill (notPSL)) and the direct sum $V \oplus W$ of two non-isomorphic irreducible 3-dimensional $\mathbb{F}_2$-modules V and W. Since V

and W are irreducible, every chief series of G has three chief factors whose order is divisible by 2. We may assume that the chief series has the form $1 \leq N_1 = V \leq N_2 = V \oplus W \leq N_3 = G$. Some computations show that there are two conjugacy classes of complements of the first and of the second chief factor. Since there is only one conjugacy class of subgroups of 2-power index in $\mathrm{PSL}(2,7)$, the third chief factor has exactly one conjugacy class of supplements. Fix a 2-complement H in G. Then in every conjugacy class of maximal subgroups of p-power index there is exactly one subgroup containing H. The following figure depicts the interval $[H,G]_c$ described above:



We see that one pair of atoms generates G and the other pairs do not. But the poset $[H,G]_c$ has rank 3. Since in an upper semimodular lattice of rank 3 two atoms generate a rank-2 element, the depicted poset (indeed a lattice) is not upper semimodular and hence not geometric.

Indeed we have provided an atom $H \in \overline{S_c^p(G)}$ such that the interval $[H,G]_c$ is not upper semimodular.

Finally we will give the explicit structure of $S_c^p(G)$ for a special class of groups. The intention is to give the reader a feeling for this poset in particular cases.

Proposition 4.10. *For a group G let $\mathbf{P}_1(G)$ be the projective cover in $\mathbb{F}_pG$ of the trivial G-module. Assume that two different irreducible submodules of $\mathbf{P}_1(G)\mathbf{J}/\mathbf{J}^2$ are not isomorphic G-modules. Then for a subgroup $U \in S_c^p(G)$ and for an index $i \in \mathcal{I}_p$ there is at most one $K_i \in \mathcal{K}_i$ such that $U \leq K_i$.*

Especially if $K_1, \dots K_l, K_1', \dots, K_{l'}'$ are maximal subgroups of p-power index and $\bigcap_{i \leq l} K_i = \bigcap_{i \leq l'} K_i'$, then $l = l'$ and there is a permutation π such that $K_i = K_{\pi(i)}'$.

Proof. Let N denote a minimal normal subgroup of G and $K, K' \in S^p(G)$ such that $NK = NK' = G$ and $K \cap K' \in S^p(G)$. Using induction, it is enough to prove $K = K'$.

As $[N : (K \cap K' \cap N)]$ divides $[G : K \cap K']$ we have $N \cap K = N \cap K'$ by Proposition 2.7. Now $\langle K, K' \rangle \leq N_G(K \cap K' \cap N)$ (the normalizer of $K \cap K' \cap N$ in G) so either $K = K'$ or $K \cap K' \cap N = 1$ and so $|N| = [G : K]$. Thus we may assume that N is an elementary abelian p-group. (Now the assumptions imply that K and K' are conjugate, but we do not need this.)

Now we use the notation of Proposition 4.7 and use the arguments established in [15].

Note that $K \cap K' \in S^p(G)$ implies that $\mathbf{1}_{K \cap K'} \uparrow^G$ is an epimorphic image of $\mathbf{1}_{K \cap K'} \uparrow^G$. Let MK (resp. MK') denote the kernel of the epimorphisms (induced from the embedding $K \cap K' \subset K$) from $\mathbf{1}_{K \cap K'} \uparrow^G$ onto $\mathbf{1}_K \uparrow^G / \mathbf{1}_K \uparrow^G \mathbf{J}^2$. As $\mathbf{1}_K \uparrow^G \mathbf{J}/\mathbf{J}^2 \cong N \cong \mathbf{1}_{K'} \uparrow^G \mathbf{J}/\mathbf{J}^2$ we have $\mathbf{1}_{K \cap K'} \uparrow^G \mathbf{J}/MK \cong N$ and the same for K'. Thus, by our assumption on $\mathbf{P}_1\mathbf{J}/\mathbf{J}^2$ we get $MK = MK'$. So $MG \neq \langle MK, MK' \rangle$ and $G \neq \langle K, K' \rangle$. In particular this implies $K = K'$. □

We would like to mention that for a solvable group the assumptions of the preceding proposition are satisfied if and only if for all $i, j \in \mathcal{I}_p$ the G-modules N_j/N_{j-1} and N_i/N_{i-1} are not isomorphic for $i \neq j$.

Corollary 4.11. *If two different irreducible submodules of* $\mathbf{P}_1(G)\mathbf{J}/\mathbf{J}^2$ *are not isomorphic G-modules then the dual poset (i.e. the poset with the reversed order relation) of* $S^p_c(G) \cup \{G\}$ *and the order complex* CP *of the ordinal sum* $P = \bigoplus_{i \in \mathcal{I}_p} \mathcal{K}_i$ *(regarded as a poset ordered by set-inclusion) are isomorphic posets. In particular,* $S^p_c(G)$ *is a shellable poset.*

Proof. By Proposition 4.10 we can establish a bijection ϕ from $S^p_c(G) \cup \{G\}$ onto the order complex of $\bigoplus_{i \in \mathcal{I}_p} \mathcal{K}_i$. The group G itself corresponds to the empty set. For a group H in $S^p_c(G)$ the identity $H = \bigcap_{K \in \phi(H)} K$ holds and therefore $\phi(H) \leq \phi(U)$ for subgroups $U \leq H$. This shows that ϕ is a monotone mapping between the dual of $S^p_c(G) \cup \{G\}$ and the order complex of $\bigoplus_{i \in \mathcal{I}_p} \mathcal{K}_i$. An analogous reasoning shows that ϕ^{-1} is monotone too. It remains to show that the order complex of $\bigoplus_{i \in \mathcal{I}_p} \mathcal{K}_i$ is CL-shellable. Here we regard a simplicial complex as a poset with inclusion of sets as the order relation. Now a set A is an element of the order complex of $\bigoplus_{i \in \mathcal{I}_p} \mathcal{K}_i$ if and only if $|A \cap \mathcal{K}_i| = 0, 1$ for all $i \in \mathcal{I}_p$. This proves that the order complex of $\bigoplus_{i \in \mathcal{I}_p} \mathcal{K}_i$ is the set of independent sets of the transversal matroid on the sets $\mathcal{K}_i$, $i \in \mathcal{I}_p$. From this observation the shellability follows by [4]. □

5. Concluding remarks

5.1. If G is a solvable group and π is a set of primes then the subgroups in $S_c^\pi(G)$ generalize the concept of Prefrattinisubgroups introduced by Gaschütz [6] (see [9] and [18]). In this case, $S_c^\pi(G) = \{\bigcap_{p\in\pi} U_p < G \mid \text{ for some } U_p \in S_c^p(G) \cup \{G\}\}$. Thus, studying $S_c^p(G)$ is the first step in the analysis of $S_c^\pi(G)$ (of course only for solvable groups).

Anyway, if π is an arbitrary set of primes, Proposition 2.5 tells us that $S^\pi(G)$ is contractible if and only if one of the posets $\mathcal{K}_i^{S^\pi(G)}$ is. Thus $S^\pi(G)$ is contractible if $O^\pi(G) \neq G$, but it remains open whether the converse is also true. Moreover, we regard the investigation of the posets $\mathcal{K}_i^{S^\pi(G)}$ as a very challenging and interesting problem.

5.2. In this remark we would like to stress the relation of the structure of $S^p(G)$ to the representation theory of G. Let $S^{p+}(G)$ denote the poset of all $U < G$ for which $\mathbf{1}_U\uparrow^G$ is a factormodule of $\mathbf{P}_1(G)$ (notation as in Proposition 4.10). Then $S^p(G) \subseteq S^{p+}(G)$ (equality holds for p-solvable groups). Thus $S^p(G)$ gives information about $S^{p+}(G)$ and $\mathbf{P}_1(G)$. For example, we can show the following:

$\mathbf{1}$ is a factormodule of $\mathbf{P}_1(G)\mathbf{J}$ if and only if $S^p(G)$ is contractible. Moreover, $S^{p+}(G)$ is contractible if $S^p(G)$ is.

5.3. Besides $S^p(G)$ and $S_c^p(G)$, there are several other interesting posets associated to a finite group:

(i) The poset of all submodules of $\mathbf{P}_1(G)\mathbf{J}/\mathbf{J}^2$ (this is a projective geometry).
(ii) The G-invariant subgroups of $\bigoplus_{i\in\mathcal{I}_p} N_i/N_{i+1}$.
(iii) The orbit poset of $S_c^p(G)\cup\{G\}$ (i.e. the poset on the conjugacy classes $[H] = \{H^g \mid g \in G\}$ of subgroups in $S_c^p(G)$ ordered by $[H] \le [U] \Leftrightarrow \exists g \in G\colon H^g \le U$).
(iv) Maximal intervals in $S_c^p(G)\cup\{G\}$.

For p-solvable groups all these posets are isomorphic (the equivalence of the first two posets is a famous theorem of Gaschütz, whereas the other equivalences follow from results in [15,16,19]). On the other hand there exist groups, such that no two of these posets are isomorphic. Let for example be $p = 2$ and G be defined as $G = G_1 \oplus G_2$ where G_1 is the group constructed in Section 4. Now a simple count of poset elements shows that only the posets in (iii) and (iv) can be isomorphic for G_1. If we choose G_2 to be a group with two conjugacy classes of 2-complements the last two posets are not isomorphic for G_2.

5.4. Another interesting topic is the dependence of the sets $\mathcal{K}_i$ on the chosen chief series. Obviously the sets $\mathcal{K}_i$ and $\mathcal{I}_p$ depend on the chief series. For the rest of this remark we write $\mathcal{K}_i^{\mathcal{R}}$ for $\mathcal{K}_i$ and $\mathcal{I}_p^{\mathcal{R}}$ for $\mathcal{I}_p$ in order to stress the dependence of the sets on the chosen chief series $\mathcal{R}$. It is easily seen that the set $\mathcal{K} = \bigcup_{i\in\mathcal{I}_p}\mathcal{K}_i^{\mathcal{R}}$ of p-supplements and the cardinality $|\mathcal{I}_p^{\mathcal{R}}|$ are independent of the choice of the chief series. In order to get an idea of how the single sets $\mathcal{K}_i^{\mathcal{R}}$ change, when the chief series changes, we define $\mathrm{Trans}^{\mathcal{R}}(G)$ to be the set of all subsets $\mathcal{U}$ of $\bigcap_{i\in\mathcal{I}_p}\mathcal{K}_i^{\mathcal{R}}$ such that $|\mathcal{U}\cap\mathcal{K}_i^{\mathcal{R}}| \leq 1$ for all i (see also the proof of Corollary 4.11). Note that there is a natural map φ from $\mathrm{Trans}^{\mathcal{R}}(G)$ onto $S_c^p(G)\cup\{G\}$ sending $\mathcal{U}$ to $\bigcap_{U\in\mathcal{U}} U$.

Now $\mathrm{Trans}^{\mathcal{R}}(G)$ depends on the fixed chief series of G, and there are two ways to get rid of this dependence:

(i) Let $\mathrm{Trans}(G)$ denote the union of all sets $\mathrm{Trans}^{\mathcal{R}}(G)$ where $\mathcal{R}$ varies over all chief series.

(ii) Let $\mathrm{Indep}(G)$ denote the set of all subsets $\mathcal{U} \subseteq \mathcal{K}$ such that $[G : \bigcap_{U\in\mathcal{U}}U] = \prod_{U\in\mathcal{U}}[G : U]$ (the definition might remind the reader to probability-theoretic independence).

If G is solvable we have $\mathrm{Trans}(G) = \mathrm{Indep}(G)$ (see [16]). Moreover, the structure of $\mathrm{Trans}(G)$ is well known in this case. The set $\mathrm{Trans}(G)$ is the set of independent sets of a matroid, whose lattice of flats is isomorphic to the poset of conjugacy classes of subgroups in $\overline{S_c^p(G)}$ (see 5.3). Therefore, $\mathrm{Trans}(G)$ is the set of independent sets of a direct product of projective geometries [19].

Note that under the assumptions of Proposition 4.10 the mapping φ (see the proof of Corollary 4.11) is a poset isomorphism and $\mathrm{Trans}(G) = \mathrm{Trans}^{\mathcal{R}}(G)$. Thus studying $\mathrm{Trans}(G)$ might as well be useful in the analysis of $S_c^p(G)$.

5.5. We conjecture that $S_c^p(G)$ is CL-shellable for every group.

Note that $S_c^p(G)$ is CL-shellable for the group defined in Section 4. Furthermore, we can deduce from [17] that the intervals $[H,G]_c \subseteq S_c^p(G)$ are CL-shellable for all finite groups.

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