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Coxeter transformation and inverses of Cartan matrices for coalgebras

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ABSTRACT

Let C be a coalgebra and let $\mathbb{Z}_{\blacktriangleright}^{I_C}, \mathbb{Z}_{\blacktriangleleft}^{I_C} \subseteq \mathbb{Z}^{I_C}$ be the Grothendieck groups of the category C^{op}-inj and C-inj of the socle-finite injective right and left C-comodules, respectively. One of the main aims of the paper is to study the Coxeter transformation $\boldsymbol{\Phi}_C : \mathbb{Z}_{\blacktriangleright}^{I_C} \longrightarrow \mathbb{Z}_{\blacktriangleleft}^{I_C}$ and its dual $\boldsymbol{\Phi}_C^- : \mathbb{Z}_{\blacktriangleleft}^{I_C} \longrightarrow \mathbb{Z}_{\blacktriangleright}^{I_C}$ of a pointed sharp Euler coalgebra C, and to relate the action of $\boldsymbol{\Phi}_C$ and $\boldsymbol{\Phi}_C^-$ on a class of indecomposable finitely cogenerated C-comodules N with the ends of almost split sequences starting with N or ending at N. By applying Chin, Kleiner, and Quinn (2002) [5], we also show that if C is a pointed K-coalgebra such that the every vertex of the left Gabriel quiver ${}_CQ$ of C has only finitely many neighbours then for any indecomposable non-projective left C-comodule N of finite K-dimension, there exists a unique almost split sequence $0 \longrightarrow \tau_C N \longrightarrow N' \longrightarrow N \longrightarrow 0$ in the category $C\text{-Comod}_{fc}$ of finitely cogenerated left C-comodules, with an indecomposable comodule $\tau_C N$. We show that $\mathbf{dim}\,\boldsymbol{\tau}_C N = \boldsymbol{\Phi}_C(\mathbf{dim}\,N)$, if C is hereditary, or more generally, if $\text{inj.dim}\,DN = 1$ and $\text{Hom}_C(C, DN) = 0$.

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Throughout we fix an arbitrary field K and $D(-) = (-)^* = \text{Hom}_K(-, K)$ is the ordinary K-linear duality functor. We recall that a K-coalgebra C is said to be pointed if all simple C-comodules are one-dimensional. Let C be a pointed K-coalgebra and $C^* = \text{Hom}_K(C, K)$ the K-dual (pseudocompact [17]) K-algebra with respect to the convolution product, see [9,14]. We denote by C-Comod and C-comod the category of left C-comodules and finite-dimensional left C-comodules, respectively. The

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corresponding categories of right C-comodules are denoted by C^{op}-Comod and C^{op}-comod. The socle of a comodule M in C-Comod is denoted by $\operatorname{soc} M$.

We recall from [5] and [6] that, for a class of coalgebras C (including left semiperfect ones), given an indecomposable non-injective C-comodule M in C-comod and an indecomposable non-projective C-comodule N in C-comod there exist almost split sequences

$$0 \longrightarrow M \longrightarrow M' \longrightarrow \tau_C^- M \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \tau_C N \longrightarrow N' \longrightarrow N \longrightarrow 0 \tag{$*$}$$

in C-Comod, where $C\text{-Comod}^{\bullet}_{fc} \underset{\tau_C}{\overset{\tau_C^-}{\rightleftarrows}} C\text{-comod}_{f\mathcal{P}}$ are the Auslander–Reiten translate operators (1.17). On the other hand, for a class of computable coalgebras C, a Cartan matrix $\mathfrak{c}_C \in \mathbb{M}_{I_C}(\mathbb{Z})$, its inverse $\mathfrak{c}_C^{-1}$, and a corresponding Coxeter transformation $\boldsymbol{\Phi}_C : K_0(C) \longrightarrow K_0(C)$ is defined and studied in [19] (see also [4] and [11]), where $K_0(C) = K_0(C\text{-comod}) \cong \mathbb{Z}^{(I_C)}$ is the Grothendieck group of C-comod.

One of the main aims of this paper is to construct an inverse $\mathfrak{c}_C^{-1}$ (left and right) of the Cartan matrix $\mathfrak{c}_C$ and Coxeter transformations $\boldsymbol{\Phi}_C : \mathbb{Z}^{I_C}_{\blacktriangleright} \longrightarrow \mathbb{Z}^{I_C}_{\blacktriangleleft}$, $\boldsymbol{\Phi}_C^- : \mathbb{Z}^{I_C}_{\blacktriangleleft} \longrightarrow \mathbb{Z}^{I_C}_{\blacktriangleright}$, for any computable coalgebra C such that any simple left (and right) C-comodule admits a finite and socle-finite injective resolution. We prove that, under a suitable assumption on indecomposable C-comodules N and M, there exist almost split sequences $(*)$ in C-Comod and

$$\mathbf{dim}\, \tau_C^-(M) = \boldsymbol{\Phi}_C^-(\mathbf{dim}\, M) \quad \text{and} \quad \mathbf{dim}\, \tau_C(N) = \boldsymbol{\Phi}_C(\mathbf{dim}\, N),$$

where $\mathbf{dim}\, X \in \mathbb{Z}^{I_C}$ is the dimension vector (2.1) of the comodule X, compare with [1, Corollary IV.2.9] and [19, p. 67].

We recall from [12] that a coalgebra C is said to be left locally artinian if every indecomposable injective left C-comodule is artinian. Recall also that a coalgebra over an algebraically closed field is pointed if and only if it is basic, see [7].

1. Preliminaries on comodule categories

Let C be a K-coalgebra. We collect in this section basic facts concerning C-comodules, pseudocompact C^*-modules, the existence of almost split sequences in C-Comod, duality and injectives in the category of comodules.

We recall from [9,14,17] that any left C-comodule M is viewed as a rational (= discrete) right module over the pseudocompact algebra C^* and $M^* = D(M) = \operatorname{Hom}_K(M, K)$ is a pseudocompact left C^*-module. The functor $D(-)$ defines a duality $\widetilde{D} : C\text{-Comod} \longrightarrow C^*\text{-PC}$, where C^*-PC is the category of pseudocompact left C^*-modules. The quasi-inverse is the functor $(-)^\circ = \operatorname{hom}_K(-, K)$ that associates to any Y in C^*-PC the left C-comodule $Y^\circ = \operatorname{hom}_K(Y, K)$ consisting of all continuous K-linear maps $Y \longrightarrow K$. It follows from [10] that the algebra C^* is left (and right) topologically semiperfect, that is, every simple left C^*-module admits a projective cover in C^*-PC (see also [17]); equivalently, C^* admits a decomposition $C^* \cong \prod_{j\in I} C^* e_j$ in C^*-PC, where $\{e_j\}_{j\in I}$ is a topologically complete set of pairwise orthogonal primitive idempotents such that $e_j C^* e_j$ is a local algebra, for every $i \in I$. The decomposition is unique up to isomorphism and permutation.

The coalgebra C (or more generally, any C-C-bicomodule) can be viewed as a bimodule over the algebra C^* with respect to the right and the left hit actions of C^* on C, usually denoted by the symbols $\leftharpoonup$, $\rightharpoonup$ as in [9] and [14]. Here we omit these symbols and simply use juxtaposition, e.g., $eC = e \rightharpoonup C$ and $Ce = C \leftharpoonup e$, for any $e \in C^*$. Notice that Ce is an injective right C-comodule and eC is an injective left C-comodule, for any idempotent $e \in C^*$.

The following two simple lemmata are often used in the paper.

Lemma 1.1. *Assume that C is a coalgebra, $e = e^2$ is an idempotent in C^*, and $D(-) = \operatorname{Hom}_K(-, K)$.*

(a) *There is an isomorphism $\widetilde{D}(eC) \cong C^*e$ of left C^*-modules.*

(b) *If C is of finite dimension, then there is an isomorphism $D(C^*e) \cong eC$ of right C^*-modules.*
(c) $\mathrm{Hom}_C(C', Ce) = \mathrm{Hom}_C(C', C'e)$ *for every subcoalgebra C' of C.*

Proof. See [6] and [8]. □

Lemma 1.2. *Let C be a K-coalgebra. Given a left comodule M in C-*comod*, the K-dual space $D(M) = \mathrm{Hom}_K(M, K)$ admits a natural structure of right C-comodule and $D(-) = \mathrm{Hom}_K(-, K)$ defines the pair of dualities*

$$C\text{-comod} \underset{D}{\overset{D}{\rightleftarrows}} C^{op}\text{-comod}. \tag{1.3}$$

Proof. See [25]. □

An **injective copresentation** of a comodule M in C-Comod is an exact sequence

$$0 \longrightarrow M \longrightarrow E_0 \xrightarrow{g} E_1, \tag{1.4}$$

where E_0 and E_1 are injective comodules. We call a comodule M in C-Comod (socle) **finitely copresented** if M admits a socle-finite injective copresentation, that is, the injective comodules E_0 and E_1 have finite-dimensional socle. We denote by $C\text{-Comod}_{fc}$ the full subcategory of C-Comod whose objects are the finitely copresented comodules, and by C-inj the full subcategory of $C\text{-Comod}_{fc}$ whose objects are the socle-finite injective comodules. We set $C\text{-comod}_{fc} = C\text{-comod} \cap C\text{-Comod}_{fc}$. Finally, we denote by

$$C\text{-}\overline{\text{Comod}}_{fc} = \text{Comod}_{fc}/\mathcal{I}$$

the quotient category of $C\text{-Comod}_{fc}$ modulo the two-sided ideal $\mathcal{I} = [C\text{-inj}]$ consisting of all $f \in \mathrm{Hom}_C(N, N')$, with N and N' in $C\text{-Comod}_{fc}$, that have a factorisation through a socle-finite injective comodule, see [19] and [22].

It is observed in [12] that $C\text{-Comod}_{fc}$ is an abelian category if and only if C is left cocoherent. In this case $C\text{-Comod}_{fc}$ is closed under extensions in C-Comod and contains minimal injective resolutions of comodules M in $C\text{-Comod}_{fc}$, see [19, Section 3].

We recall that a comodule M is **quasi-finite** if $\dim_K \mathrm{Hom}_C(X, M)$ is finite, for any X in C-comod; equivalently, if the simple summands of $\mathrm{soc}\, M$ have finite (but perhaps unbounded) multiplicities [25]. It follows that every socle-finite comodule is quasi-finite. Hence all comodules in $C\text{-Comod}_{fc}$ are quasi-finite.

Given a left quasi-finite C-comodule M, the covariant **cohom functor**

$$h_C(M, -) : C\text{-Comod} \longrightarrow \mathrm{Mod}(K)$$

is defined by associating to any comodule N in C-Comod the vector space $h_C(M, N) = \varinjlim_\lambda D\,\mathrm{Hom}_C(N_\lambda, M)$, where $\{N_\lambda\}$ is the family of all finite-dimensional subcomodules of N [25].

Denote by $C^{op}\text{-Comod}_{fp}$ the full subcategory of C^{op}-Comod whose objects are the (injectively) finitely presented C^{op}-comodules, that is, the C^{op}-comodules L that admit a short exact sequence $E_1' \xrightarrow{g'} E_0' \longrightarrow L \longrightarrow 0$ in C^{op}-Comod, with socle-finite injective comodules E_1' and E_0', called a socle-finite injective presentation of L. Following [5, Section 3], we define a pair of contravariant left exact functors

$$C\text{-Comod}_{fc} \underset{\nabla_C'}{\overset{\nabla_C}{\rightleftarrows}} C^{op}\text{-Comod}_{fp} \tag{1.5}$$

to be the composite functors making the following diagrams commutative

$$\begin{tikzcd} C\text{-Comod}_{fc} \arrow[r, "\widetilde{D}", "\simeq"'] \arrow[d, "\nabla_C"'] & C^*\text{-PC}_{fp} \arrow[d, "(-)^+"'] \\ C^{op}\text{-Comod}_{fp} & C^{*op}\text{-PC}_{fc}, \arrow[l, "(-)^\circ"', "\simeq"] \end{tikzcd} \qquad \begin{tikzcd} C\text{-Comod}_{fc} & C^*\text{-PC}_{fp} \arrow[l, "(-)^\circ"', "\simeq"] \\ C^{op}\text{-Comod}_{fp} \arrow[u, "\nabla'_C"] \arrow[r, "\widetilde{D}", "\simeq"'] & C^{*op}\text{-PC}_{fc}, \arrow[u, "(-)^+"'] \end{tikzcd} \tag{1.6}$$

where C^*-PC_{fp} (resp. $C^{*\,op}$-PC_{fc}) is the category of pseudocompact (top-) finitely presented (resp. (top-) finitely copresented) modules (see [9,17]), $\widetilde{D} = \mathrm{Hom}_K(-, K)$,

$$(-)^+ = \hom_{C^*}\bigl(-, C^*\bigr) : C^*\text{-PC}_{fp} \longrightarrow C^{*\,op}\text{-PC}_{fc}$$

is a contravariant functor that associates to any X in C^*-PC_{fp}, with the top-finite pseudocompact projective presentation $P_1 \xrightarrow{f_1} P_0 \longrightarrow X \longrightarrow 0$, where P_1, P_0 are finite direct sums of indecomposable projective C^*-modules, the right C^*-module $X^+ = \hom_{C^*}(X, C^*)$ of all continuous C^*-homomorphisms $X \longrightarrow C^*$, with the top-finite pseudocompact projective copresentation

$$0 \longrightarrow X^+ \longrightarrow P_0^+ \xrightarrow{f_1^+} P_1^+.$$

Finally, $Y^\circ = \hom_K(Y, K)$ consists of all continuous K-linear maps $Y \longrightarrow K$ and $(-)^\circ$ associates to X^+ the right C-comodule $(X^+)^\circ$ in C^{op}-Comod_{fp}, with the socle-finite injective presentation

$$\bigl(P_1^+\bigr)^\circ \xrightarrow{(f_1^+)^\circ} \bigl(P_0^+\bigr)^\circ \longrightarrow \bigl(X^+\bigr)^\circ \longrightarrow 0.$$

The functors in the right-hand diagram of (1.6) are defined analogously. Sometimes, for simplicity of the notation, we write ∇_C instead of ∇'_C.

Following [5] and the classical construction of Auslander [2], we define the Auslander **transpose operator**

$$\mathrm{Tr} = \mathrm{Tr}_C : C\text{-Comod}_{fc} \longrightarrow C^{op}\text{-Comod}_{fc} \tag{1.7}$$

(on objects only!) that associates to any comodule M in C-Comod_{fc}, with a minimal socle-finite injective copresentation (1.4), the comodule

$$\mathrm{Tr}_C\, M = \mathrm{Ker}\bigl[\nabla_C E_1 \xrightarrow{\nabla_C(g)} \nabla_C E_0\bigr]$$

in C^{op}-Comod_{fc}. Basic properties of Tr_C are listed in [5, Proposition 3.2].

The existence of almost split sequences in C-Comod_{fc} essentially depends on the following theorem slightly extending some of the results in [5] and [6].

Theorem 1.8. *Let C be a K-coalgebra and ∇_C the functor* (1.5)*.*

(a) *There are functorial isomorphisms $\nabla_C M \cong \mathrm{Hom}_C(C, M)^\circ \cong h_C(M, C)$, for any comodule M in C-Comod_{fc}.*
(b) *The functors ∇_C, ∇'_C are left exact and restrict to the dualities*

$$C\text{-inj} \underset{\nabla_C}{\overset{\nabla_C}{\rightleftarrows}} C^{op}\text{-inj} \tag{1.9}$$

that are quasi-inverse to each other. Moreover, given an idempotent $e \in C^*$, the comodule Ce lies in C^{op}-inj, the comodule eC lies in C-inj, and there is an isomorphism $\nabla_C(Ce) \cong eC$ of left C-comodules.

(c) *For any comodule M in C-Comod$_{fc}$, with a minimal socle-finite injective copresentation* (1.4), *the comodules $\mathrm{Tr}_C\, M$, $\nabla_C E_1$, $\nabla_C E_0$ lie in C^{op}-Comod$_{fc}$, $\nabla_C M$ lies in C^{op}-Comod$_{fp}$, and the following sequence*

$$0 \longrightarrow \mathrm{Tr}_C\, M \longrightarrow \nabla_C E_1 \xrightarrow{\nabla_C(g)} \nabla_C E_0 \longrightarrow \nabla_C M \longrightarrow 0 \tag{1.10}$$

is exact in C^{op}-Comod.

(d) *The transpose operator Tr_C, together with the functor ∇_C, induces the equivalence of quotient categories $\mathrm{Tr}_C : C\text{-}\overline{\mathrm{Comod}}_{fc} \xrightarrow{\simeq} C^{op}\text{-}\overline{\mathrm{Comod}}_{fc}$.*

Proof. For our future purpose and the convenience of the reader, we outline the proof.

(a) Let $\{C_\lambda\}$ is the family of all finite-dimensional subcoalgebras of C and let M be a comodule in C-Comod$_{fc}$. Then M is quasi-finite, $C \cong \varinjlim_\lambda C_\lambda$, and we get isomorphisms

$$\begin{aligned}
\nabla_C M = \big((\widetilde{D}M)^+\big)^\circ &\cong \hom_{C^*}\big(\widetilde{D}M, C^*\big)^\circ \\
&\cong \mathrm{Hom}_C(C, M)^\circ \\
&\cong \Big[\varprojlim_\lambda \mathrm{Hom}_C(C_\lambda, M)\Big]^\circ \\
&\cong \varinjlim_\lambda \mathrm{Hom}_C(C_\lambda, M)^\circ \\
&\cong \varinjlim_\lambda D\,\mathrm{Hom}_C(C_\lambda, M) \\
&= h_C(M, C).
\end{aligned}$$

One can easily see that the composite isomorphism is functorial at M.

(b) Apply the definition of ∇_C.

To prove (c) and (d), we note that the exact functors $\widetilde{D} : C\text{-Comod}_{fp} \longrightarrow C^*\text{-PC}_{fp}$ and $(-)^\circ : C^{*\,op}\text{-PC}_{fc} \longrightarrow C^{op}\text{-Comod}_{fc}$ defining the functor ∇_C are equivalences of categories carrying injectives to projectives and projectives to injectives, respectively. Recall that C^* is a topologically semiperfect algebra. Now, given an indecomposable comodule M in C-Comod$_{fc}$, with a minimal socle-finite injective copresentation (1.4), we get a pseudocompact minimal top-finite projective presentation

$$\widetilde{D}E_1 \xrightarrow{\widetilde{D}g} \widetilde{D}E_0 \longrightarrow \widetilde{D}M \longrightarrow 0,$$

in C^*-PC, with $\widetilde{D}E_1 = E_1^*$, $\widetilde{D}E_0 = E_0^*$ finite direct sums of indecomposable projective C^*-modules, of the right pseudocompact C^*-module $\widetilde{D}M$. Hence, by applying the left exact functor $\hom_{C^*}(-, C^*)$, and the definition of the Auslander transpose $\mathrm{Tr}_{C^*}(\widetilde{D}M)$ of the pseudocompact left C^*-module $\widetilde{D}M$, we get the exact sequence

$$0 \longrightarrow (\widetilde{D}M)^+ \longrightarrow (\widetilde{D}E_0)^+ \xrightarrow{(\widetilde{D}g)^+} (\widetilde{D}E_1)^+ \longrightarrow \mathrm{Tr}_{C^*}(\widetilde{D}M) \longrightarrow 0 \tag{1.11}$$

in $C^{*\,op}$-PC and the projective copresentation $0 \longrightarrow (\widetilde{D}M)^+ \longrightarrow (DE_0)^+ \xrightarrow{(\widetilde{D}g)^+} (DE_1)^+$ of the right pseudocompact C^*-module $(\widetilde{D}M)^+$, where $(\widetilde{D}E_0)^+$ and $(\widetilde{D}E_1)^+$ are finitely generated projective top-finite right C^*-modules. The sequence (1.11) induces the sequence (1.10) and (c) follows. The statement (d) follows from the corresponding properties of the Auslander transpose operator $\mathrm{Tr}_{C^*} : C^*\text{-PC}_{fp} \longrightarrow C^{*\,op}\text{-PC}_{fp}$ on the pseudocompact finitely presented top-finite modules over C^*. Here we can follow the proof of [1, Proposition IV.2.2] or [3, Section IV.2]. □

We denote by $C\text{-Comod}^{\bullet}_{fc}$ and by $C\text{-Comod}^{\nu}_{fc}$ the full subcategory of $C\text{-Comod}_{fc}$ consisting of the comodules M such that $\dim_K \operatorname{Tr}_C(M)$ is finite and $\dim_K(\widetilde{D}M)^+$ is finite, respectively. Following the representation theory of finite-dimensional algebras, we define the (covariant) **Nakayama functor**

$$\nu_C : C\text{-Comod}^{\nu}_{fc} \longrightarrow C\text{-comod} \tag{1.12}$$

by the formula $\nu_C(-) = D\nabla_C(-)$.

A coalgebra C is said to be **left semiperfect** [13] if every simple left comodule has a projective cover, or equivalently, the injective envelope $E(X)$ of any finite-dimensional right C-comodule X is finite-dimensional.

It is easy to see that, for a left semiperfect coalgebra C, the functor ν_C restricts to the equivalence of categories

$$\nu_C : C\text{-inj} \xrightarrow{\simeq} C\text{-proj}, \tag{1.13}$$

where C-proj is the category of top-finite projective comodules in C-comod.

We denote by $C\text{-comod}_{f\mathcal{P}}$ the full subcategory of C-comod consisting of the left comodules N that, viewed as rational right C^*-modules, have a minimal top-finite projective presentation $P_1 \longrightarrow P_0 \longrightarrow N \longrightarrow 0$ in $C^{*op}\text{-PC} = \text{PC-}C^*$, that is, P_0 and P_1 are top-finite projective modules in PC-C^*. Here we make the identification

$$C\text{-comod} \equiv \text{rat-}C^* = \text{dis-}C^* \subseteq \text{PC-}C^*,$$

in the notation of [17, Section 4], where rat-C^* is the category of finite-dimensional rational right C^*-modules.

Finally, we denote by

$$C\text{-}\underline{\text{comod}}_{f\mathcal{P}} = C\text{-comod}_{f\mathcal{P}}/\mathcal{P}$$

the quotient category of $C\text{-comod}_{f\mathcal{P}}$ modulo the two-sided ideal $\mathcal{P}$ of $C\text{-comod}_{f\mathcal{P}}$ consisting of all $f \in \operatorname{Hom}_C(N, N')$, with N and N' in $C\text{-comod}_{f\mathcal{P}}$, that have a factorisation through a projective right C^*-module, when f is viewed as a C^*-homomorphism between the rational right C^*-modules N and N'.

If C is left semiperfect then, in view of the exact sequence (1.10) in C^{op}-Comod, we have $C\text{-comod}_{f\mathcal{P}} = C\text{-comod}$, $C\text{-Comod}^{\bullet}_{fc} = C\text{-Comod}_{fc}$, $C\text{-Comod}^{\nu}_{fc} = C\text{-Comod}_{fc}$ and, by applying ν_C to the sequence (1.10) we get the exact sequence

$$0 \longrightarrow \nu_C(M) \longrightarrow \nu_C(E_0) \xrightarrow{\nu_C(g)} \nu_C(E_1) \longrightarrow D\operatorname{Tr}_C(M) \longrightarrow 0 \tag{1.14}$$

in C-comod.

The following lemma is very useful.

Lemma 1.15. *Let C be a pointed K-coalgebra and let $_CQ$ be the left Gabriel quiver of C.*

(a) *The duality $D : C\text{-comod} \longrightarrow C^{op}\text{-comod}$ (1.3) restricts to the duality*

$$D : C\text{-comod}_{f\mathcal{P}} \longrightarrow C^{op}\text{-comod}_{fc} = C^{op}\text{-comod} \cap C^{op}\text{-Comod}_{fc}.$$

In particular, a left C-comodule N lies in $C\text{-comod}_{f\mathcal{P}}$ if and only if the right C-comodule $D(N)$ is finitely copresented.

(b) *The following four conditions are equivalent*:

(b1) $C\text{-comod}_{f\mathcal{P}} = C\text{-comod}$,

(b2) C^{op}-comod $\subseteq C^{op}$-Comod$_{fc}$,
(b3) *every simple comodule in* C^{op}-Comod *is finitely copresented,*
(b4) *the quiver* ${}_C Q$ *is right locally bounded, that is, for every vertex* a *of* ${}_C Q$ *there is only a finite number of arrows* $a \longrightarrow j$ *in* ${}_C Q$.

(c) *If* C *is right locally artinian, we have* C-comod$_{f\mathcal{P}} = C$-comod.

Proof. (a) Since we make the identification C-comod $\equiv$ rat-$C^* =$ dis-$C^* \subseteq$ PC-C^* (in the notation of [17, Section 4]), there is a commutative diagram

$$\begin{tikzcd} C\text{-comod} \arrow[r, "id"] \arrow[d, "D"', "\cong"] & \text{dis-}C^* \arrow[d, "(-)^\circ"', "\cong"] \arrow[r, phantom, "\subseteq"] & \text{PC-}C^* \arrow[d, "(-)^\circ"', "\cong"] \\ C^{op}\text{-comod} \arrow[r, "id"] & C^*\text{-dis} \arrow[r, phantom, "\subseteq"] & C^{op}\text{-Comod} \end{tikzcd}$$

Then a left C-comodule N lies in C-comod$_{f\mathcal{P}}$ if and only if there is an exact sequence $P_1 \longrightarrow P_0 \longrightarrow N \longrightarrow 0$ in PC-C^*, where P_0 and P_1 are top-finite projective modules in PC-C^*, or equivalently, N lies in C^{*op}-PC$_{fp}$ = PC$_{fp}$-C^*. By applying the duality $(-)^\circ : C^{*op}\text{-PC} \longrightarrow C^{op}\text{-Comod}$ (1.6), we get an exact sequence $0 \longrightarrow N^\circ \longrightarrow P_0^\circ \longrightarrow P_1^\circ$ in C^{op}-Comod. Since $\dim_K N$ is finite, we have $N^\circ = D(N)$. This shows that $D(N)$ lies in C^{op}-comod$_{fc}$, because P_0° and P_1° are socle-finite injective right C-comodules. It follows that the duality (1.3) restricts to the duality $D : C\text{-comod}_{f\mathcal{P}} \longrightarrow C^{op}\text{-comod}_{fc}$.

(b) By (a), the equality C-comod$_{f\mathcal{P}} = C$-comod holds if and only if the equality C^{op}-comod$_{fc} = C^{op}$-comod holds, that is, the conditions (b1) and (b2) are equivalent.

The implication (b2) $\Rightarrow$ (b3) is obvious. To prove the inverse implication (b3) $\Rightarrow$ (b2), we assume that the simple right C-comodules lie in C^{op}-Comod$_{fc}$ and let X be a comodule in C^{op}-comod. By standard arguments and the induction on the K-dimension of X, we show that X lies in C-Comod$_{fc}$ (apply the diagram in [5, p. 13]).

(b3) $\Rightarrow$ (b4) Let $\operatorname{soc} C_C = \bigoplus_{j \in I_C} \widehat{S}(j)$ be a direct sum decomposition of the right socle $\operatorname{soc} C_C$ of C, where I_C is an index set and $\{\widehat{S}(j)\}_{j \in I_C}$ is a set of pairwise non-isomorphic simple right C-coideals. Denote by $\widehat{E}(j) = E(\widehat{S}(j))$ the injective envelope of $\widehat{S}(j)$.

It follows from the dualities (1.9) and [18, Theorem 2.3(a)] that the quiver ${}_C Q$ is dual to the right Gabriel quiver Q_C of C. Hence, by the assumption (b2), for every vertex a of the quiver Q_C, there is only a finite number of arrows $j \longrightarrow a$ in Q_C. In other words, $\dim_K \operatorname{Exp}^1_C(\widehat{S}(j), \widehat{S}(a))$ is finite, for all $j \in I_C$, and $\dim_K \operatorname{Ext}^1_C(\widehat{S}(j), \widehat{S}(a)) = 0$, for all but a finite number of indices $j \in I_C$, see [17, Definition 8.6]. Fix $a \in I_C$ and let

$$0 \longrightarrow \widehat{S}(a) \longrightarrow \widehat{E}(a) \longrightarrow \widehat{E}_1 \longrightarrow \cdots$$

be a minimal injective resolution of $\widehat{S}(a)$ in C^{op}-Comod, with $\widehat{E}_1 \cong E(\operatorname{soc}(\widehat{E}(a)/\widehat{S}(a)))$. Given $j \in I_C$, we denote by $\mu_1(\widehat{S}(j), \widehat{S}(a))$ the number of times the comodule $\widehat{E}(j)$ appears as a direct summand in $\widehat{E}_1$. Since C is assumed to be pointed, $\dim_K \operatorname{End}_C \widehat{S}(j) = 1$ and

$$\mu_1\big(\widehat{S}(j), \widehat{S}(a)\big) = \dim_K \operatorname{Ext}^1_C\big(\widehat{S}(j), \widehat{S}(a)\big),$$

by [20, (4.23)]. Thus the injective C^{op}-comodule $\widehat{E}_1$ is socle finite, by the observation made earlier, and it follows that the simple right C-comodule $\widehat{S}(a)$ is finitely copresented. This shows that (b3) implies (b4). Since the inverse implication follows in a similar way, the proof of (b) is complete.

(c) Apply (b) and the easily seen fact that simple right comodules over any right locally artinian coalgebra are finitely copresented. □

Following [5], we get the following result.

Proposition 1.16. *Let C be a K-coalgebra and D the duality functors* (1.3)*.*

(a) *The transpose equivalence of Theorem* 1.8(d)*, defines the equivalence*

$$\mathrm{Tr}_C : C\text{-}\overline{\mathrm{Comod}}^{\bullet}_{fc} \xrightarrow{\simeq} C^{op}\text{-}\overline{\mathrm{comod}}_{fc},$$

and together with the duality $D : C^{op}\text{-}\mathrm{comod}_{fc} \xrightarrow{\simeq} C\text{-}\mathrm{comod}_{f\mathcal{P}}$ defined by (1.3)*, induces the translate operator*

$$\tau_C^- = D\,\mathrm{Tr}_C : C\text{-}\mathrm{Comod}^{\bullet}_{fc} \longrightarrow C\text{-}\mathrm{comod}_{f\mathcal{P}}, \tag{1.17}$$

and an equivalence of quotient categories $\overline{\tau}_C^- = D\,\mathrm{Tr}_C : C\text{-}\overline{\mathrm{Comod}}^{\bullet}_{fc} \xrightarrow{\simeq} C\text{-}\underline{\mathrm{comod}}_{f\mathcal{P}}$. Moreover, for any M in $\mathrm{Comod}^{\bullet}_{fc}$, with a presentation (1.4)*, the following sequence*

$$0 \longrightarrow (\widetilde{D}M)^+ \longrightarrow (\widetilde{D}E_1)^+ \xrightarrow{(\widetilde{D}g)^+} (\widetilde{D}E_0)^+ \longrightarrow \tau_C^- M \longrightarrow 0 \tag{1.18}$$

is exact in $C^{*\,op}$-PC and the comodule $\tau_C^- M$ lies in $C\text{-}\mathrm{comod}_{f\mathcal{P}} \equiv C^{*\,op}\text{-}\mathrm{rat}_{fp} \subseteq C^{*\,op}$-PC.

(b) *The duality* (1.3) *restricts to the duality $D : C\text{-}\mathrm{comod}_{f\mathcal{P}} \xrightarrow{\simeq} C^{op}\text{-}\mathrm{comod}_{fc}$ and together with the transpose operator $\mathrm{Tr}_{C^{op}} : C^{op}\text{-}\mathrm{comod}_{fc} \longrightarrow C\text{-}\mathrm{Comod}^{\bullet}_{fc}$ defines the translate operator*

$$\tau_C = \mathrm{Tr}_{C^{op}}\, D : C\text{-}\mathrm{comod}_{f\mathcal{P}} \longrightarrow C\text{-}\mathrm{Comod}^{\bullet}_{fc}, \tag{1.19}$$

and induces the equivalence of quotient categories $\overline{\tau}_C = \mathrm{Tr}_{C^{op}}\, D : C\text{-}\underline{\mathrm{comod}}_{f\mathcal{P}} \xrightarrow{\simeq} C\text{-}\overline{\mathrm{Comod}}^{\bullet}_{fc}$ that is quasi-inverse to the equivalence $\overline{\tau}_C^- = D\,\mathrm{Tr}_C : C\text{-}\overline{\mathrm{Comod}}^{\bullet}_{fc} \xrightarrow{\simeq} C\text{-}\underline{\mathrm{comod}}_{f\mathcal{P}}$ in (a)*.*

(c) *Let M be an indecomposable comodule in $C\text{-}\mathrm{Comod}^{\bullet}_{fc}$. Then $\tau_C^- M = 0$ if and only if M is injective. If $\tau_C^- M \neq 0$ then $\tau_C^- M$ is indecomposable, non-projective, of finite K-dimension, and there is an isomorphism $M \cong \tau_C \tau_C^- M$.*

(d) *Let N be an indecomposable comodule in $C-\mathrm{comod}_{f\mathcal{P}}$. Then $\tau_C N = 0$ if and only if N is projective. If $\tau_C N \neq 0$ then $\tau_C N$ is indecomposable, non-injective, finitely copresented, and there is an isomorphism $N \cong \tau_C^- \tau_C N$.*

Proof. By Lemma 1.15(a), the duality $D : C^{op}-\mathrm{comod} \xrightarrow{\simeq} C-\mathrm{comod}$ (1.3) restricts to the duality $D : C^{op}\text{-}\mathrm{comod}_{fc} \xrightarrow{\simeq} C\text{-}\mathrm{comod}_{f\mathcal{P}}$. One also shows, by applying foregoing definitions, that a homomorphism $f : X \longrightarrow X'$ in $C^{op}\text{-}\mathrm{comod}_{fc}$ has a factorisation through a socle-finite injective comodule if and only if the homomorphism $D(f) : D(X) \longrightarrow D(X')$ in $C\text{-}\mathrm{comod}_{f\mathcal{P}}$ belongs to $\mathcal{P}(D(X), D(X'))$. This shows that the duality $D : C^{op}\text{-}\mathrm{comod}_{fc} \xrightarrow{\simeq} C\text{-}\mathrm{comod}_{f\mathcal{P}}$ induces an equivalence of quotient categories $D : C^{op}\text{-}\overline{\mathrm{comod}}_{fc} \xrightarrow{\simeq} C\text{-}\underline{\mathrm{comod}}_{f\mathcal{P}}$. It follows from the definition of the category $C\text{-}\overline{\mathrm{Comod}}^{\bullet}_{fc}$ that the transpose equivalence of Theorem 1.8(d), defines the equivalence $\mathrm{Tr}_C : C\text{-}\overline{\mathrm{Comod}}^{\bullet}_{fc} \xrightarrow{\simeq} C^{op}\text{-}\overline{\mathrm{comod}}_{fc}$. This together with the earlier observation implies (a) and (b).

The statements (c) and (d) are obtained by a straightforward calculation and by using the definition of translates τ_C and τ_C^-, see [5] and consult [3]. The details are left to the reader. □

Following the terminology of representation theory of finite-dimensional algebras (see [1,3]), we call the operators $\tau_C = \mathrm{Tr}_{C^{op}}\, D$ (1.19) and $\tau_C^- = D\,\mathrm{Tr}_C$ (1.17), the **Auslander–Reiten translations** of C. It follows from Theorem 1.16 that the image of τ_C^- is the subcategory $C\text{-}\mathrm{comod}_{f\mathcal{P}}$ of the category C-comod.

By applying Theorem 4.2 and Corollary 4.3 in [5], we get the following result on the existence of almost split sequences in the category $C\text{-Comod}_{fc}$ of (socle) finitely copresented left C-comodules, under some assumption on the Gabriel quiver ${}_CQ$ of C.

Theorem 1.20. *Let C be a K-coalgebra such that its left Gabriel quiver ${}_CQ$ is left locally bounded, that is, for every vertex a of ${}_CQ$ there is only a finite number of arrows $j \longrightarrow a$ in ${}_CQ$.*

(a) *The following inclusion holds* $C\text{-comod} \subseteq C\text{-Comod}_{fc}$.
(b) *For any indecomposable non-injective comodule M in $C\text{-Comod}^{\bullet}_{fc}$, there exists a unique almost split sequence*

$$0 \longrightarrow M \longrightarrow M' \longrightarrow \tau_C^- M \longrightarrow 0 \tag{1.21}$$

in $C\text{-Comod}_{fc}$, with a finite-dimensional indecomposable comodule $\tau_C^- M$ lying in $C\text{-comod}_{f\mathcal{P}}$. The sequence (1.21) *is almost split in the whole comodule category* $C\text{-Comod}$.
(c) *For any indecomposable non-projective comodule N in $C\text{-comod}_{f\mathcal{P}} \subseteq C\text{-Comod}_{fc}$, there exists a unique almost split sequence*

$$0 \longrightarrow \tau_C N \longrightarrow N' \longrightarrow N \longrightarrow 0 \tag{1.22}$$

in $C\text{-Comod}_{fc}$, with an indecomposable comodule $\tau_C N$ lying in $C\text{-Comod}^{\bullet}_{fc}$. The sequence (1.22) *is almost split in the whole comodule category* $C\text{-Comod}$.
(d) *If, in addition, C is left semiperfect then $C\text{-Comod}^{\bullet}_{fc} = C\text{-Comod}_{fc}$, $C\text{-comod}_{f\mathcal{P}} = C\text{-comod}$, the Auslander–Reiten translate operators act as follows*

$$C\text{-Comod}_{fc} \underset{\tau_C}{\overset{\tau_C^-}{\rightleftarrows}} C\text{-comod}$$

and the almost split sequences (1.21) *and* (1.22) *do exist in the category $C\text{-Comod}_{fc}$, for any indecomposable non-injective comodule M in $C\text{-Comod}_{fc}$ and for any indecomposable non-projective comodule N in* $C\text{-comod}$. *Moreover, if the comodule M lies in* $C\text{-comod}$ *then the almost split sequence* (1.21) *lies in* $C\text{-comod}$.

Proof. (a) As in the proof of Lemma 1.15(b), we conclude from the assumption that ${}_CQ$ is left locally bounded that every simple left C-comodule admits a minimal socle-finite injective copresentation (1.4). Hence (a) follows as in Lemma 1.15(b).

The statements (b) and (c) follow from Theorem 4.2 and Corollary 4.3 in [5], because any comodule M lying in $C\text{-Comod}_{fc}$ is quasi-finite, Proposition 1.16(a) yields that $\tau_C^- M$ lies in $C\text{-comod}_{f\mathcal{P}}$, for any indecomposable comodule M in $C\text{-Comod}^{\bullet}_{fc}$, and the following inclusions hold $C\text{-comod}_{f\mathcal{P}} \subseteq C\text{-comod} \subseteq C\text{-Comod}_{fc}$, by (a).

(d) Assume that C is left semiperfect and let M be an indecomposable comodule in $C\text{-Comod}_{fc}$ with a minimal socle-finite injective copresentation (1.4). By Theorem 1.8, the induced sequence (1.10) is exact and the comodules $\nabla_C(E_0)$ and $\nabla_C(E_1)$ lie in $C^{op}\text{-inj}$. Since C is left semiperfect, the comodules $\nabla_C(E_0)$ and $\nabla_C(E_1)$ are finite-dimensional and, hence, $\dim_K \operatorname{Tr}_C(M)$ is finite, for any comodule M in $C\text{-Comod}_{fc}$. It follows that $C\text{-Comod}^{\bullet}_{fc} = C\text{-Comod}_{fc}$. Since C is left semiperfect, any comodule N in $C\text{-comod}$ has a projective presentation $P_1 \longrightarrow P_0 \longrightarrow N \longrightarrow 0$, with P_1, P_0 finite-dimensional projective C-comodules. It follows that N lies in $C\text{-comod}_{f\mathcal{P}}$ and, hence, the equality $C\text{-comod}_{f\mathcal{P}} = C\text{-comod}$ holds. This finishes the proof of the theorem. □

Corollary 1.23. *Let C be a pointed K-coalgebra such that the left Gabriel quiver ${}_CQ$ of C is both left and right locally bounded.*

(a) *The inclusions* $C\text{-comod}_{f\mathcal{P}} = C\text{-comod} \subseteq C\text{-Comod}_{fc}$ *hold and the Auslander–Reiten translate operators act as follows*

$$C\text{-Comod}^{\bullet}_{fc} \underset{\tau_C}{\overset{\tau_C^{-}}{\rightleftarrows}} C\text{-comod}.$$

(b) *For any indecomposable non-injective comodule* M *in* $C\text{-Comod}^{\bullet}_{fc}$, *there exists a unique almost split sequence*

$$0 \longrightarrow M \longrightarrow M' \longrightarrow \tau_C^{-} M \longrightarrow 0$$

in $C\text{-Comod}_{fc}$, *with an indecomposable comodule* $\tau_C^{-} M$ *lying in* $C\text{-comod}$.

(c) *For any indecomposable non-projective comodule* N *in* $C\text{-comod}$, *there exists a unique almost split sequence*

$$0 \longrightarrow \tau_C N \longrightarrow N' \longrightarrow N \longrightarrow 0$$

in $C\text{-Comod}_{fc}$, *with an indecomposable comodule* $\tau_C N$ *lying in* $C\text{-Comod}^{\bullet}_{fc}$.

(d) *The exact sequences in* (b) *and* (c) *are almost split in the whole comodule category* $C\text{-Comod}$.

Proof. Apply Lemma 1.15 and Theorem 1.20. □

Remark 1.24. (a) Under the assumption that the left Gabriel quiver ${}_CQ$ of C is both left and right locally bounded the almost split sequences (1.21) and (1.22) lie in $C\text{-Comod}_{fc}$. If we drop the assumption then the term $\tau_C M$ lies in $C\text{-comod}_{f\mathcal{P}} \subseteq C\text{-comod}$, but not necessarily lies in $C\text{-Comod}_{fc}$.

(b) We are mainly interested in the existence of almost split sequences in the category $C\text{-Comod}_{fc}$, because it is the part of C-Comod that plays a crucial role in the study of tameness of the coalgebra C, see [22].

Now we illustrate the existence of almost split sequences discussed in Corollary 1.23 by the following example.

Example 1.25. Let $Q = (Q_0, Q_1)$ be the infinite locally Dynkin quiver

$$Q: \quad \begin{tikzcd} & \overset{-1}{\bullet} & & & & & \\ \underset{0}{\bullet} & \underset{1}{\circ} \arrow[l] \arrow[u] \arrow[r] & \underset{2}{\bullet} \arrow[r] & \cdots \arrow[r] & \underset{s-1}{\bullet} \arrow[r] & \underset{s}{\bullet} \arrow[r] & \underset{s+1}{\bullet} \arrow[r] & \cdots \end{tikzcd}$$

of type $\mathbb{D}_\infty$ and let $C = K^{\square}Q$ be the hereditary path K-coalgebra of Q. Then $Q_0 = \{-1, 0, 1, 2, \ldots\}$ and C has the $Q_0 \times Q_0$ matrix form

$$C = \begin{bmatrix} K & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & K & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ K & K & K & K & K & K & K & K & K & \ldots \\ 0 & 0 & 0 & K & K & K & K & K & K & \ldots \\ 0 & 0 & 0 & 0 & K & K & K & K & K & \ldots \\ 0 & 0 & 0 & 0 & 0 & K & K & K & K & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & K & K & K & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & K & K & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & K & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

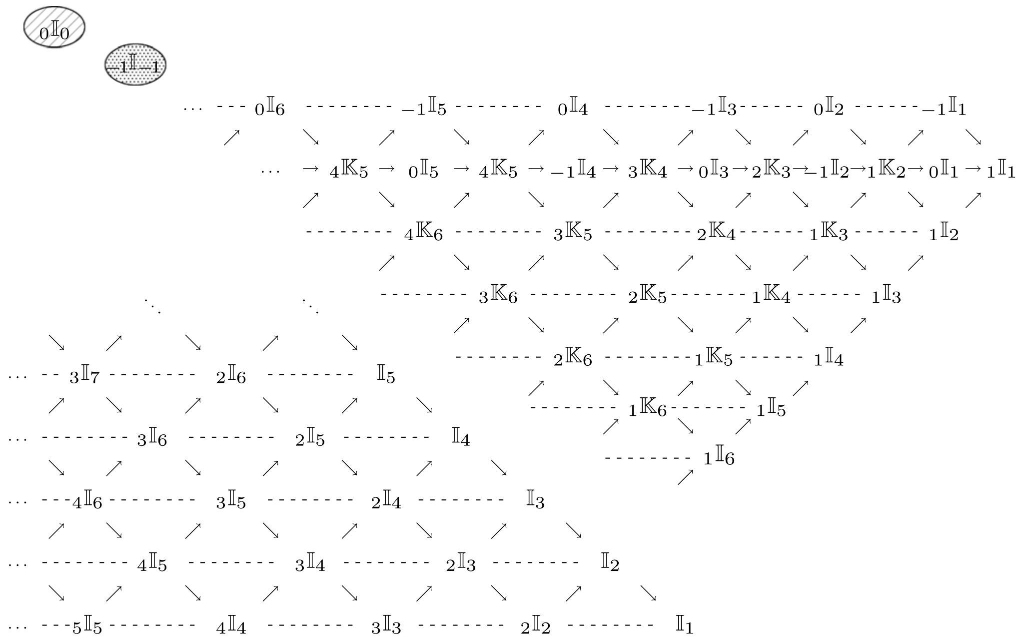


Fig. 0. The Auslander–Reiten quiver of the category C-comod $\cong \mathrm{rep}_K(Q)$.

and consists of the triangular $Q_0 \times Q_0$ square matrices with coefficients in K and at most finitely many non-zero entries. Then $\mathrm{soc}_C\, C = \bigoplus_{j \in Q_0} S(j)$, where $S(n) = Ke_n$ is the simple subcoalgebra of C spanned by the matrix $e_n \in C$ with 1 in the $n \times n$ entry, and zeros elsewere. Note that e_n is a group-like element of C.

Since the left Gabriel quiver ${}_CQ$ of C is the quiver Q, it follows from Lemma 1.15(b) that every simple right C-comodule is finitely copresented and the statements (a) and (c) of Corollary 1.23 hold for $C = K^{\square}Q$.

The coalgebra C is right locally artinian, right semiperfect, representation-directed in the sense of [19], and left pure semisimple, that is, every left C-comodule is a direct sum of finite-dimensional comodules (see [16] or [17]). It follows that $C\text{-Comod}^{\bullet}_{fc} = C\text{-comod}^{\bullet}$ and every indecomposable non-projective comodule N in C-comod admits an almost split sequence $0 \longrightarrow \tau_C N \longrightarrow N' \longrightarrow N \longrightarrow 0$ in C-comod.

Under the identification $C\text{-comod} = \mathrm{rep}_K(Q)$ of left C-comodules and K-linear representations of the quiver Q (see [5] or [17], [21, (3.1)]), the Auslander–Reiten translation quiver $\Gamma(C\text{-comod})$ of C-comod has four connected components (two of them are finite and two are infinite), and $\Gamma(C\text{-comod})$ has the following form (see [16]).

Here we use the terminology and notation introduced in [16, pp. 470–472]. Recall that the vertices of the Auslander–Reiten translation quiver $\Gamma(C\text{-comod})$ are representatives of the indecomposable left C-comodules in C-comod and the existence of an arrow $X \longrightarrow Y$ in $\Gamma(C\text{-comod})$ means that there exists an irreducible morphism $f : X \longrightarrow Y$ in C-comod, see also [23].

Each of the two finite components of $\Gamma(C\text{-comod})$ contains precisely one indecomposable simple projective C-comodule; namely the comodule ${}_0\mathbb{I}_0$ and ${}_{-1}\mathbb{I}_{-1}$, respectively. Each of the two infinite components contains no non-zero projective objects.

The indecomposable injective left C-comodules form the right-hand section

$$\begin{array}{c} \hphantom{\cdots \longrightarrow {}_1\mathbb{I}_6 \longrightarrow {}_1\mathbb{I}_5 \longrightarrow {}_1\mathbb{I}_4 \longrightarrow {}_1\mathbb{I}_3 \longrightarrow {}_1\mathbb{I}_2 \longrightarrow} {}_0\mathbb{I}_1 \hphantom{\longleftarrow {}_{-1}\mathbb{I}_1} \\ \hphantom{\cdots \longrightarrow {}_1\mathbb{I}_6 \longrightarrow {}_1\mathbb{I}_5 \longrightarrow {}_1\mathbb{I}_4 \longrightarrow {}_1\mathbb{I}_3 \longrightarrow {}_1\mathbb{I}_2 \longrightarrow} \downarrow \hphantom{\longleftarrow {}_{-1}\mathbb{I}_1} \\ \cdots \longrightarrow {}_1\mathbb{I}_6 \longrightarrow {}_1\mathbb{I}_5 \longrightarrow {}_1\mathbb{I}_4 \longrightarrow {}_1\mathbb{I}_3 \longrightarrow {}_1\mathbb{I}_2 \longrightarrow {}_1\mathbb{I}_1 \longleftarrow {}_{-1}\mathbb{I}_1 \end{array} \qquad (*)$$

of the infinite upper component of $\Gamma(C\text{-comod})$, and the indecomposable left C-comodules M in C-comod such that $\dim_K \mathrm{Tr}_C M$ is infinite are the two simple projective comodules ${}_0\mathbb{I}_0$, ${}_{-1}\mathbb{I}_{-1}$ and the comodules lying on the infinite right-hand section

$$\cdots \longrightarrow \mathbb{I}_6 \longrightarrow \mathbb{I}_5 \longrightarrow \mathbb{I}_4 \longrightarrow \mathbb{I}_3 \longrightarrow \mathbb{I}_2 \longrightarrow \mathbb{I}_1 \tag{$**$}$$

of the infinite lower component of $\Gamma(C\text{-comod})$. It follows that an indecomposable left C-comodule M lies in the category $C\text{-Comod}^{\bullet}_{fc} = C\text{-comod}^{\bullet}$ if and only if M lies in the infinite upper component of $\Gamma(C\text{-comod})$ or M lies in the infinite lower component of $\Gamma(C\text{-comod})$, but does not lie on the infinite section $(**)$. Every indecomposable comodule N lying in one of the infinite components of $\Gamma(C\text{-comod})$ has an almost split sequence $0 \longrightarrow \tau_C N \longrightarrow N' \longrightarrow N \longrightarrow 0$ in C-comod and it is given by the mesh in $\Gamma(C\text{-comod})$ terminating at N, compare with the examples given in Section 4.

2. Cartan matrix of an Euler coalgebra and its inverses

Throughout we assume that K is an arbitrary field and C is a **pointed** K-coalgebra. It follows that C is basic and there exists a direct sum decomposition $\mathrm{soc}_C C = \bigoplus_{j\in I_C} S(j)$ of the left socle $\mathrm{soc}\,{}_C C$ of C, where I_C is an index set and $\{S(j)\}_{j\in I_C}$ is a set of pairwise non-isomorphic simple left C-coideals, see [4] or [7]. Then $\{S(j)\}_{j\in I_C}$ is a set of representatives of the isomorphism classes of simple left C-comodules and $\dim_K S(j) = \dim_K \mathrm{End}_C S(j) = 1$, for any $j \in I_C$.

For every $j \in I_C$, let $E(j) = E(S(j))$ denote the injective envelope of $S(j)$. It follows that $E(j)$ is indecomposable, ${}_C C = \bigoplus_{j\in I_C} E(j)$, and there is a primitive idempotent $e_j \in C^*$ such that $E(j) \cong e_j C$. Working with right C-comodules, we have the simple right C-comodules $\widehat{S}(j) = DS(j)$, with injective envelopes $\widehat{E}(j) = \nabla_C(E(j)) \cong Ce_j$ in C^{op}-inj, see Theorem 1.8. Throughout we fix a set $\{e_j\}_{j\in I_C}$ of primitive idempotents of C^* such that $E(j) \cong e_j C$, for all $j \in I_C$.

Following the representation theory of finite-dimensional algebras, given a left C-comodule M (viewed as a rational right C^*-module), we define its **dimension vector**

$$\mathbf{dim}\, M = [\dim_K Me_j]_{j\in I_C}, \tag{2.1}$$

where $\dim_K Me_j$ has values in $\mathbb{Z} \cup \{\infty\}$. Since C is pointed, $\dim_K Me_j = \dim_K \mathrm{Hom}_C(M, e_j C) = \dim_K \mathrm{Hom}_C(M, E(j))$ and the dimension vector $\mathbf{dim}\, M$ coincides with the composition length vector $\mathbf{lgth}\, M = [\ell_j(M)]_{j\in I_C}$ of M (introduced in [19]), where

$$\ell_j(M) = \dim_K \mathrm{Hom}_C\big(M, E(j)\big) = \dim_K Me_j. \tag{2.2}$$

It follows from [19, Proposition 2.6] that $\ell_j(M) = \dim_K Me_j$ is the multiplicity the simple comodule $S(j)$ appears as a composition factor in the socle filtration $\mathrm{soc}^0 M \subseteq \mathrm{soc}^1 M \subseteq \cdots \subseteq \mathrm{soc}^m M \subseteq \cdots$ of M. Following [19], M is said to be **computable** if the composition length multiplicity $\ell_j(M) = \dim_K Me_j$ of $S(j)$ in M is finite, for every $j \in I_C$, or equivalently, $\mathbf{dim}\, M \in \mathbb{Z}^{I_C}$ (the product of I_C copies of the infinite cyclic group $\mathbb{Z}$). A pointed coalgebra C is defined to be **computable** if the injective comodule $E(i)$ is computable, or equivalently, if the dimension vector

$$\mathbf{e}(i) = \mathbf{dim}\, E(i) = [\dim_K e_i C e_j]_{j\in I_C} = \big[\dim_K \mathrm{Hom}_C\big(E(i), E(j)\big)\big]_{j\in I_C} \tag{2.3}$$

has finite coordinates, for every $i \in I_C$. Note that the class of computable coalgebras contains left semiperfect coalgebras, right semiperfect coalgebras and the incidence coalgebras $K^{\square}I$ of intervally finite posets I, see [19]. Moreover, if C is computable and left cocoherent then the K-category $C\text{-Comod}_{fc}$ is abelian, has enough injective objects, and is Ext-**finite**, that is, $\dim_K \mathrm{Ext}^m_C(M, N)$ is finite, for all $m \geqslant 0$ and all comodules M, N in $C\text{-Comod}_{fc}$

Given a pointed computable coalgebra C, with $\mathrm{soc}_C C = \bigoplus_{j\in I_C} S(j)$, we define the left **Cartan matrix** of C to be the integral $I_C \times I_C$ matrix

$$\mathfrak{c}_C = [\mathbf{c}_{ij}]_{i,j\in I_C} = \begin{bmatrix} \vdots \\ \mathbf{e}(i) \\ \vdots \end{bmatrix} \in \mathbb{M}_{I_C}(\mathbb{Z}), \tag{2.4}$$

whose $i \times j$ entry is the composition length multiplicity $\mathbf{c}_{ij} = \mathbf{e}(i)_j = \dim_K e_i C e_j$ of $S(j)$ in $E(i)$. In other words, the ith row of $\mathfrak{c}_C$ is the dimension vector $\mathbf{e}(i) = \mathbf{dim}\, E(i)$ of $E(i)$, see [19, Definition 4.1]. We say that a row (or a column) of a matrix is finite, if the number of its non-zero coordinates is finite. A matrix is called **row-finite** (or **column-finite**) if each of its rows (columns) is finite.

We start with the following simple observations.

Lemma 2.5. *Let C be a pointed computable K-coalgebra, with a fixed decomposition* $\mathrm{soc}_C C = \bigoplus_{j\in I_C} S(j)$, *and let $K_0(C) = K_0(C\text{-comod})$ be the Grothendieck group of C-comod.*

(a) *Given a C-comodule M in C-Comod, the dimension vector $\mathbf{dim}\, M$ has only a finite number of non-zero coordinates if and only if $\dim_K M$ is finite. If $\dim_K M < \infty$, then $\dim_K M = \sum_{j\in I_C} \dim_K Me_j$.*
(b) *The map $M \mapsto \mathbf{dim}\, M$ is an additive function on short exact sequences in C-Comod and induces the group isomorphism $\mathbf{dim} : K_0(C) \xrightarrow{\simeq} \mathbb{Z}^{(I_C)}$, $[M] \mapsto \mathbf{dim}\, M$, where $\mathbb{Z}^{(I_C)}$ is the direct sum of I_C copies of $\mathbb{Z}$. The group $K_0(C)$ is free abelian with the basis $\{[S(j)]\}_{j\in I_C}$ corresponding via $\mathbf{dim}$ to the standard basis vectors $e_j = \mathbf{dim}\, S(j)$ of $\mathbb{Z}^{(I_C)}$.*

Proof. (a) To prove the sufficiency, assume that $\dim_K M$ is finite. Then

$$M^* \cong \mathrm{Hom}_C(M, C) \cong \mathrm{Hom}_C\left(M, \bigoplus_{j\in I_C} E(j)\right) \cong \bigoplus_{j\in I_C} \mathrm{Hom}_C\big(M, E(j)\big) \cong \bigoplus_{j\in I_C} Me_j.$$

It follows that $\dim_K M = \dim_K M^* = \sum_{j\in I_C} \dim_K Me_j$. Hence, the sum is finite and $\mathbf{dim}\, M$ has only a finite number of non-zero coordinates. The converse implication follows in a similar way.

(b) The additivity of $\mathbf{dim}$ is obvious. Hence, $M \mapsto \mathbf{dim}\, M$ defines the group homomorphism $\mathbf{dim} : K_0(C) \xrightarrow{\simeq} \mathbb{Z}^{(I_C)}$. Since the set $\{[S(j)]\}_{j\in I_C}$ generates $K_0(C)$, the set $\{e_j\}_{j\in I_C}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^{(I_C)}$ and $\mathbf{dim}\, S(j) = e_j$ then the homomorphism $\mathbf{dim}$ is bijective. □

Lemma 2.6. *Let C be a pointed computable K-coalgebra and let $\mathfrak{c}_C \in \mathbb{M}_{I_C}(\mathbb{Z})$ be the left Cartan matrix* (2.4) *of C.*

(a) *$\mathfrak{c}_{C^{op}} = \mathfrak{c}_C^{tr}$, that is, $\mathfrak{c}_{C^{op}} \in \mathbb{M}_{I_C}(\mathbb{Z})$ is the transpose of the matrix $\mathfrak{c}_C$.*
(b) *The ith row of the matrix $\mathfrak{c}_C$ is finite if and only if the indecomposable injective left C-comodule $E(i)$ is finite-dimensional.*
(c) *The jth column of the matrix $\mathfrak{c}_C$ is finite if and only if the indecomposable injective right C-comodule $\widehat{E}(j) = \nabla_C(E(j))$ is finite-dimensional.*
(d) *The left Cartan matrix $\mathfrak{c}_C$ of C is row-finite if and only if C is right semiperfect.*
(e) *The left Cartan matrix $\mathfrak{c}_C$ of C is column-finite if and only if C is left semiperfect.*

Proof. (a) Let $\mathfrak{c}_{C^{op}} = [\bar{\mathbf{c}}_{ij}]_{i,j\in I_C}$, $\mathfrak{c}_C = [\mathbf{c}_{ij}]_{i,j\in I_C} \in \mathbb{M}_{I_C}(\mathbb{Z})$ be the Cartan matrices (2.4), of the coalgebra C^{op} and C, respectively. We recall from Theorem 1.8 that there is a duality $\nabla_C : C\text{-inj} \longrightarrow C^{op}\text{-inj}$ and $\widehat{E}(j) = \nabla_C(E(j))$. Hence, by applying (2.3), we get

$$\bar{\mathbf{c}}_{ij} = \dim_K \mathrm{Hom}_{C^{op}}\big(\widehat{E}(i), \widehat{E}(j)\big) = \dim_K \mathrm{Hom}_C\big(E(j), E(i)\big) = \mathbf{c}_{ji},$$

for every pair of elements $i, j \in I_C$. This yields the equality $\mathfrak{c}_{C^{op}} = \mathfrak{c}_C^{tr}$.

(b) We recall from (2.4) that the ith row of $\mathfrak{c}_C$ is the dimension vector $\mathbf{e}(i) = \mathbf{dim}\, E(i)$ of the injective left C-comodule $E(i)$. Then (a) follows by applying Lemma 2.5(a) to $M = E(i)$.

(c) By (a), the jth column of $\mathfrak{c}_C$ is the jth row of $\mathfrak{c}_{C^{op}}$. Hence, (c) follows from (b) applied to the coalgebra C^{op}.

(d) By (b), the matrix $\mathfrak{c}_C$ is row finite if and only if $\dim_K E(i)$ is finite, for any $i \in I_C$, or equivalently, if and only if the injective envelope of any simple left C-comodule is of finite K-dimension. But this property is equivalent to the right semiperfectness of C, see [13].

(e) According to (c), the matrix $\mathfrak{c}_C$ is column finite if and only if the injective envelope of any simple right C-comodule is of finite K-dimension. Since this property is equivalent to the left semiperfectness of C [13], the statement (e) follows and the proof is complete. □

In [19], a class of computable coalgebras C, called left Euler coalgebras, is defined in such a way that the left Cartan matrix $\mathfrak{c}_C \in \mathbb{M}_{I_C}(\mathbb{Z})$ of such a coalgebra C has a left inverse $\mathfrak{c}_C^-$ in the non-associative matrix algebra $\mathbb{M}_{I_C}(\mathbb{Z})$. Unfortunately, usually a left inverse $\mathfrak{c}_C^-$ is not row-finite or column-finite. Below, we introduce a class of Euler coalgebras C such that the left inverse $\mathfrak{c}_C^-$ of $\mathfrak{c}_C$ is a row-finite and a column-finite matrix.

We would like to remark here that the multiplication in the matrix algebra $\mathbb{M}_{I_C}(\mathbb{Z})$ is not associative. The matrices in $\mathbb{M}_{I_C}(\mathbb{Z})$ may have unequal left and right inverses and that one-sided inverse of a matrix may not be unique. Moreover, the left inverse may exist, without a right inverse existing. Also being invertible as a $\mathbb{Z}$-linear map is not equivalent to being invertible as a matrix, see [27].

We introduce a class of left Euler coalgebras as follows.

Definition 2.7. A K-coalgebra C is defined to be a **left** (resp. right) **sharp Euler coalgebra** if C has the following two properties.

(a) C is computable, that is, $\dim_K \mathrm{Hom}_C(E', E'')$ is finite, for every pair of indecomposable injective left C-comodules E' and E''.

(b) Every simple left (resp. right) C-comodule S admits a finite and socle-finite injective resolution

$$0 \longrightarrow S \longrightarrow E_0 \xrightarrow{h_1} E_1 \xrightarrow{h_2} \cdots \xrightarrow{h_n} E_n \longrightarrow 0, \tag{2.8}$$

that is, the injective comodules $E_0, \dots, E_n$ are socle-finite.

A K-coalgebra C is defined to be a **sharp Euler coalgebra** if it is both left and right sharp Euler coalgebra and the following condition is satisfied

(c) $\dim_K \mathrm{Ext}_C^m(S, S') = \dim_K \mathrm{Ext}_{C^{op}}^m(\widehat{S'}, \widehat{S})$, for all $m \geqslant 0$ and all simple left C-comodules S and S', where $\widehat{S} = DS$ and $\widehat{S'} = DS'$ are the dual simple right C-comodules.

Obviously, any sharp Euler coalgebra is an Euler coalgebra in the sense of [19]. Now we show that (one-sided) semiperfect coalgebras C with $\mathrm{gl.dim}\, C < \infty$ are sharp Euler coalgebras.

Lemma 2.9. *Assume that C is a pointed left or right semiperfect coalgebra, with a fixed decomposition* $\mathrm{soc}_C C = \bigoplus_{j \in I_C} S(j)$.

(a) $\mathrm{Ext}_C^m(S(a), S(b)) \cong \mathrm{Ext}_{C^{op}}^m(\widehat{S}(b), \widehat{S}(a))$, *for all $m \geqslant 0$ and any pair of simple left C-comodules $S(a)$ and $S(b)$, with $a, b \in I_C$, where $\widehat{S}(b) = DS(b)$ and $\widehat{S}(a) = DS(a)$ are the dual simple right C-comodules corresponding to $a, b \in I_C$.*

(b) *If the global dimension* $\mathrm{gl.dim}\, C$ *of C is finite, then C is a sharp Euler coalgebra.*

Proof. We prove the lemma in case C is a pointed left semiperfect coalgebra. The proof in case C is right semiperfect follows in a similar way.

(a) Let $S(a)$ and $S(b)$ be simple left C-comodules. Since C is left semiperfect, there is a minimal projective resolution $\mathbf{P}_*(a)$ of $S(a)$ in C-comod. By Lemma 1.2, there is a duality $D : C\text{-comod} \longrightarrow C^{op}\text{-comod}$ that carries $\mathbf{P}_*(a)$ to a minimal injective resolution $D\mathbf{P}_*(a)$ of $\widehat{S}(a)$ in the category C^{op}-comod and induces an isomorphism of chain complexes $\mathrm{Hom}_C(\mathbf{P}_*(a), S(b)) \cong \mathrm{Hom}_{C^{op}}(\widehat{S}(b), D\mathbf{P}_*(a))$. Hence, we get the induced isomorphism of the cohomology K-spaces

$$\mathrm{Ext}_C^m\big(S(a), S(b)\big) = H^m\big[\mathrm{Hom}_C\big(\mathbf{P}_*(a), S(b)\big)\big] \cong H^m\big[\mathrm{Hom}_{C^{op}}\big(\widehat{S}(b), D\mathbf{P}_*(a)\big)\big] = \mathrm{Ext}_{C^{op}}^m\big(\widehat{S}(b), \widehat{S}(a)\big),$$

and (a) follows.

(b) Assume that $\mathrm{gl.dim}\, C$ of C is finite. Since C is left semiperfect, the indecomposable injectives in C^{op}-Comod are finite-dimensional and therefore any simple right C-comodule has a finite injective resolution in C^{op}-comod. Hence, C is a right sharp Euler coalgebra. To prove that C is a left sharp Euler coalgebra, assume that S is a left simple C-comodule and let

$$0 \longrightarrow S \longrightarrow E_0 \xrightarrow{h_1} E_1 \xrightarrow{h_2} \cdots \xrightarrow{h_n} E_n \longrightarrow 0,$$

be a minimal injective resolution of S in C-Comod. We show that the injective comodules $E_0^{(j)}, \dots, E_n^{(j)}$ are socle-finite. Assume that $S = S(b)$. Since a minimal projective resolution $\mathbf{P}_*(b)$ of $S = S(b)$ lies in C-comod and is of finite length $\leqslant \mathrm{gl.dim}\, C$, it follows that, for $m \geqslant 0$, $\dim_K \mathrm{Ext}_C^m(S(a), S(b))$ is finite, for all $a \in I_C$, and $\dim_K \mathrm{Ext}_C^m(S(a), S(b)) = 0$, for all but a finite number of simple comodules $S(a)$. Since C is pointed, $\dim_K \mathrm{Ext}_C^m(S(a), S(b))$ is the Bass number $\mu_m(S(a), S(b))$ of the pair $(S(a), S(b))$, that is, $\mu_m(S(a), S(b))$ is the multiplicity the indecomposable injective comodule $E(a)$ appears in E_m, as a direct summand, see [20, (4.23)]. It follows that, for each $m \geqslant 0$, the number $\mu_m(S(a), S(b))$ is finite, and $\mu_m(S(a), S(b))$ is non-zero, for at most finitely many m and a finite number of indices $a \in I_C$. Consequently, the injective comodules $E_0^{(j)}, \dots, E_n^{(j)}$ are socle-finite, and the proof is complete. □

Next we give a description of sharp Euler path coalgebras $C = K^{\square}Q$, with Q a quiver.

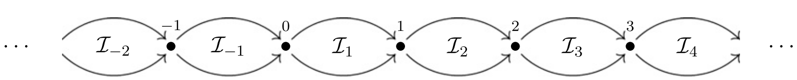
Lemma 2.10. *Assume that Q is a connected quiver and $K^{\square}Q$ is path K-coalgebra of a quiver Q. The following three conditions are equivalent.*

(a) *$K^{\square}Q$ is a sharp Euler coalgebra.*
(b) *$K^{\square}Q$ is left and right Euler coalgebra.*
(c) *The quiver Q is locally finite, that is, every vertex of Q has at most finitely many neighbours in Q.*

Proof. The equivalence of (b) and (c) follows from [19, Theorem 5.1(a)] and the implication (a) $\Rightarrow$ (b) is obvious. Since the coalgebras $C = K^{\square}Q$ and $C^{op} = (K^{\square}Q)^{op} \cong K^{\square}Q^{op}$ are hereditary then to prove the inverse implication (b) $\Rightarrow$ (a), it is enough to show that there is a K-linear isomorphism $\mathrm{Ext}_C^1(S(a), S(b)) \cong \mathrm{Ext}_{C^{op}}^1(\widehat{S}(b), \widehat{S}(a))$, for any pair of simple left C-comodules $S(a)$ and $S(b)$, with $a, b \in Q_0$, where $\widehat{S}(b) = DS(b)$ and $\widehat{S}(a) = DS(a)$ are the dual simple right C-comodules corresponding to $a, b \in Q_0$. Since the elements of $\mathrm{Ext}_C^1(S(a), S(b))$ can be interpreted as equivalence classes of one-fold extensions $0 \longrightarrow S(b) \longrightarrow N \longrightarrow S(a) \longrightarrow 0$ in C-comod and the duality $D : C\text{-comod} \longrightarrow C^{op}\text{-comod}$ carries $0 \longrightarrow S(b) \longrightarrow N \longrightarrow S(a) \longrightarrow 0$ to the exact sequence $0 \longrightarrow \widehat{S}(a) \longrightarrow DN \longrightarrow \widehat{S}(b) \longrightarrow 0$, it defines a K-linear isomorphism $\mathrm{Ext}_C^1(S(a), S(b)) \cong \mathrm{Ext}_{C^{op}}^1(\widehat{S}(b), \widehat{S}(a))$. This finishes the proof. □

Now we give examples of non-semiperfect sharp Euler coalgebras of infinite global dimension and of arbitrary large finite global dimension.

Example 2.11. Let I be the infinite poset of the form



directed from the left to the right, where $\mathcal{I}_m = \mathcal{G}_m$ is the garland of length $|m|+1$

$$\mathcal{G}_m\colon\quad \bullet \begin{array}{c} \nearrow \\ \\ \searrow \end{array} \begin{array}{c} \bullet \to \bullet - \cdots \to \bullet \to \bullet \\ \boxtimes \qquad\quad \boxtimes \;\; \boxtimes \\ \bullet \to \bullet - \cdots \to \bullet \to \bullet \end{array} \begin{array}{c} \searrow \\ \\ \nearrow \end{array} * \qquad (2|m|+2 \text{ vertices}, |m| \geqslant 1).$$

Obviously, I is an intervally finite poset. By the results given in [23], the incidence coalgebra $C = K^\square I$ of the poset I has the following properties (see also [19, Examples 4.25 and 4.26]):

1° C is a sharp Euler coalgebra and the global dimension $\mathrm{gl.dim}\, C$ of C is infinite.

2° If $S(a)$ is the simple left C-comodule corresponding to the maximal vertex $a = *$ then the injective dimension $\mathrm{inj.dim}\, S(a)$ of $S(a)$ equals $|m|+1$, for any $m \neq 0$.

3° $\dim_K \mathrm{Ext}^m_C(S(a), S(b)) = \dim_K \mathrm{Ext}^m_{C^{op}}(\widehat{S}(b), \widehat{S}(a))$, for all $m \geqslant 0$ and all $a, b \in I$, where $\widehat{S}(b) = DS(b)$ and $\widehat{S}(a) = DS(a)$ are the simple right C-comodules corresponding to the vertices a and b in I.

4° C is both left and right locally artinian, locally cocoherent, and the category $C\text{-Comod}_{fc}$ coincides with the full subcategory of C-Comod consisting of artinian objects.

5° The coalgebra C is neither left semiperfect nor right semiperfect.

6° The Cartan $\mathbb{Z} \times \mathbb{Z}$ square matrix $\mathfrak{c}_C \in \mathbb{M}_{\mathbb{Z}}(\mathbb{Z})$ of C is lower triangular and has no finite rows and no finite columns.

7° $\mathfrak{c}_C$ has a unique left inverse $\mathfrak{c}_C^- \in \mathbb{M}_{\mathbb{Z}}(\mathbb{Z})$, which is also a unique right inverse of $\mathfrak{c}_C$. The matrix $\mathfrak{c}_C^-$ is row-finite and column-finite.

8° Let $m_0 \geqslant 1$ be a fixed integer and let $\mathcal{G}_{m_0}$ be the garland of length $|m_0|+1$. If we take $\mathcal{I}_m = \mathcal{G}_{m_0}$, for each $m \in \mathbb{Z}$, in the construction of I then $C = K^\square I$ is a sharp Euler coalgebra, $\mathrm{gl.dim}\, C = m_0 + 1$ is finite, C is neither left semiperfect nor right semiperfect, and satisfies the conditions 4°, 6°, and 7°.

Now, given a pointed sharp Euler coalgebra C, we construct a left inverse and a right inverse of the Cartan matrix $\mathfrak{c}_C \in \mathbb{M}_{I_C}(\mathbb{Z})$. We follow the proof of Theorem 4.18 in [19] and [24, Theorem 3.4], and we use the notation introduced there. Given a left (resp. right) sharp Euler coalgebra C, we fix a finite minimal injective resolution

$$0 \longrightarrow S(j) \xrightarrow{h_0^{(j)}} E_0^{(j)} \xrightarrow{h_1^{(j)}} E_1^{(j)} \xrightarrow{h_2^{(j)}} \cdots \xrightarrow{h_n^{(j)}} E_n^{(j)} \longrightarrow 0, \tag{2.12}$$

of the simple left C-comodule $S(j)$ in $C\text{-Comod}_{fc}$, with $E_0^{(j)} = E(j)$, and a finite minimal injective resolution

$$0 \longrightarrow \widehat{S}(j) \xrightarrow{\hat{h}_0^{(j)}} \widehat{E}_0^{(j)} \xrightarrow{\hat{h}_1^{(j)}} \widehat{E}_1^{(j)} \xrightarrow{\hat{h}_2^{(j)}} \cdots \xrightarrow{\hat{h}_{\hat{n}}^{(j)}} \widehat{E}_{\hat{n}}^{(j)} \longrightarrow 0, \tag{2.13}$$

of the simple right C-comodule $\widehat{S}(j) = DS(j)$ in $C^{op}\text{-Comod}_{fc}$, with $\widehat{E}_0^{(j)} = \widehat{E}(j)$, respectively. We fix finite direct sum decompositions

$$E_m^{(j)} = \bigoplus_{p \in I_C} E(p)^{d_{mp}^{(j)}} = \bigoplus_{p \in I_m^{(j)}} E(p)^{d_{mp}^{(j)}}, \qquad \widehat{E}_m^{(j)} = \bigoplus_{p \in I_C} \widehat{E}(p)^{\hat{d}_{mp}^{(j)}} = \bigoplus_{p \in \widehat{I}_m^{(j)}} \widehat{E}(p)^{\hat{d}_{mp}^{(j)}} \tag{2.14}$$

of $E_m^{(j)}$ and $\widehat{E}_m^{(j)}$, for $m \geqslant 0$, where $I_m^{(j)}$ and $\widehat{I}_m^{(j)}$ are a finite subsets of I_C, $d_{mp}^{(j)}$ and $\hat{d}_{mp}^{(j)}$ are a positive integers, for each $p \in I_m^{(j)}$ and each $p \in \widehat{I}_m^{(j)}$, respectively, and we set $d_{mp}^{(j)} = 0$, for any $p \in I_C \setminus I_m^{(j)}$, and $\hat{d}_{mp}^{(j)} = 0$, for any $p \in I_C \setminus \widehat{I}_m^{(j)}$.

Theorem 2.15. *Let C be a pointed computable K-coalgebra, with a fixed decomposition* $\mathrm{soc}\,_C C = \bigoplus_{j \in I_C} S(j)$, *and let $\mathfrak{c}_C = [\mathbf{c}_{ij}]_{i,j \in I_C} \in \mathbb{M}_{I_C}(\mathbb{Z})$ be the left Cartan matrix* (2.4) *of C.*

(a) *If C is a left sharp Euler coalgebra then the matrix $\mathfrak{c}_C^{\leftarrow} = [\mathbf{c}_{ij}^{-}]_{i,j\in I_C} \in \mathbb{M}_{I_C}(\mathbb{Z})$, with $\mathbf{c}_{jp}^{-} = \sum_{m=0}^{\infty}(-1)^m d_{mp}^{(j)} \in \mathbb{Z}$, is row-finite and is a left inverse of $\mathfrak{c}_C$ in $\mathbb{M}_{I_C}(\mathbb{Z})$, where $d_{mp}^{(j)}$ is the integer defined by the decomposition* (2.14) *of the mth term $E_m^{(j)}$ of the minimal injective resolution* (2.12) *of the simple left C-comodule $S(j)$. Moreover, for each $j \in I_C$, we have $\mathbf{dim}\,\widehat{E}(j)\cdot(\mathfrak{c}_C^{\leftarrow})^{tr} = \mathbf{dim}\,S(j) = e_j$, where $\mathbf{dim}\,\widehat{E}(j)$ is the jth column of $\mathfrak{c}_C$.*

(b) *If C is a right sharp Euler coalgebra then the matrix $\mathfrak{c}_C^{\rightarrow} = [\hat{\mathbf{c}}_{ij}^{-}]_{i,j\in I_C} \in \mathbb{M}_{I_C}(\mathbb{Z})$, with $\hat{\mathbf{c}}_{jp}^{-} = \sum_{m=0}^{\infty}(-1)^m \hat{d}_{mj}^{(p)} \in \mathbb{Z}$, is column-finite and is a right inverse of $\mathfrak{c}_C$ in $\mathbb{M}_{I_C}(\mathbb{Z})$, where $\hat{d}_{mj}^{(p)}$ is the integer defined by the decomposition* (2.14) *of the mth term $\widehat{E}_j^{(p)}$ of the minimal injective resolution* (2.13) *of the simple right C-comodule $\widehat{S}(p) = DS(p)$. Moreover, for each $j \in I_C$, we have $\mathbf{dim}\,E(j)\cdot\mathfrak{c}_C^{\rightarrow} = \mathbf{dim}\,S(j) = e_j$.*

(c) *If C is a sharp Euler coalgebra then the matrix*

$$\mathfrak{c}_C^{-1} := \mathfrak{c}_C^{\leftarrow} = \mathfrak{c}_C^{\rightarrow} = \big[\mathbf{c}_{ij}^{-}\big]_{i,j\in I_C} \in \mathbb{M}_{I_C}(\mathbb{Z}), \tag{2.16}$$

with $\mathbf{c}_{ij}^{-} = \hat{\mathbf{c}}_{ij}^{-} = \sum_{m=0}^{\infty}(-1)^m d_{mi}^{(j)} = \sum_{m=0}^{\infty}(-1)^m \hat{d}_{mj}^{(i)}$, is both row-finite and column-finite, and $\mathfrak{c}_C^{-}$ is a left inverse of $\mathfrak{c}_C$ and a right inverse of $\mathfrak{c}_C$.

Proof. (a) Assume that C is a left sharp Euler coalgebra. Then the minimal injective resolution (2.13) of $S(j)$ is finite and the injective comodules $E_0^{(j)},\dots,E_n^{(j)}$ are socle-finite. Hence the sum $\mathbf{c}_{jp}^{-} = \sum_{m=0}^{\infty}(-1)^m d_{mp}^{(j)}$ is an integer, the matrix $\mathfrak{c}_C^{\leftarrow} = [\mathbf{c}_{ij}^{-}]_{i,j\in I_C}$ is well defined, and each of its row is finite, because the set $I_0^{(j)} \cup I_1^{(j)} \cup \ldots \cup I_n^{(j)} \subseteq I_C$ is finite and

$$\mathbf{c}_{jp}^{-} = \sum_{m=0}^{\infty}(-1)^m d_{mp}^{(j)} = \sum_{m=0}^{n}(-1)^m d_{mp}^{(j)} = 0, \quad \text{for all } p \notin I_0^{(j)} \cup I_1^{(j)} \cup \cdots \cup I_n^{(j)}.$$

To prove the equality $\mathfrak{c}_C^{\leftarrow}\cdot\mathfrak{c}_C = \mathbf{E}$ (the identity matrix), we note that, by the additivity of the function **dim**, the exact sequence (2.12) together with the decomposition (2.14) yields

$$e_j = \mathbf{dim}\,S(j) = \sum_{m=0}^{\infty}(-1)^m\,\mathbf{dim}\,E_m^{(j)} = \sum_{m=0}^{\infty}(-1)^m \sum_{p\in I_C} d_{mp}^{(j)}\cdot\mathbf{dim}\,E(p) = \sum_{p\in I_C}\mathbf{c}_{jp}^{-}\cdot\mathbf{dim}\,E(p).$$

Hence the equality $\mathfrak{c}_C^{\leftarrow}\cdot\mathfrak{c}_C = \mathbf{E}$ follows, because the pth row of the matrix $\mathfrak{c}_C$ is the dimension vector $\mathbf{e}(p) = \mathbf{dim}\,E(p)$ of $E(p)$, see (2.4).

By applying the matrix transpose $(-)^{tr} : \mathbb{M}_{I_C}(\mathbb{Z}) \longrightarrow \mathbb{M}_{I_C}(\mathbb{Z})$ and the equality $\mathfrak{c}_{C^{op}} = \mathfrak{c}_C^{tr}$, we get $\mathbf{E} = \mathbf{E}^{tr} = \mathfrak{c}_C^{tr}\cdot(\mathfrak{c}_C^{\leftarrow})^{tr} = \mathfrak{c}_{C^{op}}\cdot(\mathfrak{c}_C^{\leftarrow})^{tr}$ and, in view of Lemma 2.6, the equality $\mathbf{dim}\,\widehat{E}(j)\cdot(\mathfrak{c}_C^{\leftarrow})^{tr} = \mathbf{dim}\,S(j) = e_j$ follows.

(b) Assume that C is a right sharp Euler coalgebra. Then C^{op} is a left sharp Euler coalgebra and, by (a) with C and C^{op} interchanged, the matrix $\mathfrak{c}_{C^{op}}^{\leftarrow} = [\hat{\mathbf{c}}_{ij}^{\leftarrow}]_{i,j\in I_C} \in \mathbb{M}_{I_C}(\mathbb{Z})$, with $\mathbf{c}_{jp}^{\leftarrow} = \sum_{m=0}^{\infty}(-1)^m \hat{d}_{mp}^{(j)} \in \mathbb{Z}$, is row-finite and is a left inverse of $\mathfrak{c}_{C^{op}} = \mathfrak{c}_C^{tr}$ in $\mathbb{M}_{I_C}(\mathbb{Z})$, where $\hat{d}_{mp}^{(j)}$ is the integer defined by the decomposition (2.14) of the mth term $\widehat{E}_m^{(j)}$ of the minimal injective resolution of the simple right C-comodule $\widehat{S}(j) = DS(j)$. It follows that $\mathbf{c}_{jp}^{\leftarrow} = \hat{\mathbf{c}}_{pj}^{-}$, for all $j,p \in I_C$, and consequently, we get $(\mathfrak{c}_{C^{op}}^{\leftarrow})^{tr} = \mathfrak{c}_C^{\rightarrow}$.

By Lemma 2.6, we get $\mathfrak{c}_{C^{op}} = \mathfrak{c}_C^{tr}$ and the pth row $\mathbf{dim}\,\widehat{E}(j)$ of $\mathfrak{c}_{C^{op}}$ is the pth column of $\mathfrak{c}_C$. Since the equality $\mathfrak{c}_{C^{op}}^{\leftarrow}\cdot\mathfrak{c}_{C^{op}} = \mathbf{E}$ holds, the matrix transpose yields $\mathbf{E} = \mathbf{E}^{op} = \mathfrak{c}_{C^{op}}^{tr}\cdot(\mathfrak{c}_{C^{op}}^{\leftarrow})^{tr} = \mathfrak{c}_C\cdot\mathfrak{c}_C^{\rightarrow}$, that is, $\mathfrak{c}_C^{\rightarrow}$ is a right inverse of $\mathfrak{c}_C$. Hence (b) follows.

(c) Assume that C is a sharp Euler coalgebra, that is, C is left and right sharp and the equality $\dim_K \mathrm{Ext}^m_C(S(a), S(b)) = \dim_K \mathrm{Ext}^m_{C^{op}}(\widehat{S}(b), \widehat{S}(a))$ holds, for all $a, b \in I_C$. We show that $\mathfrak{c}^{\leftarrow}_C = \mathfrak{c}^{\rightarrow}_C$. Since C is pointed, we have

$$d^{(j)}_{mi} = \dim_K \mathrm{Ext}^m_C\big(S(i), S(j)\big) \quad \text{and} \quad \hat{d}^{(i)}_{mj} = \dim_K \mathrm{Ext}^m_{C^{op}}\big(\widehat{S}(i), \widehat{S}(j)\big), \tag{2.17}$$

see [20, (4.23)], and therefore $d^{(j)}_{mi} = \hat{d}^{(i)}_{mj}$. It follows that, given $i, j \in I_C$, we have $\mathbf{c}^{-}_{ji} = \sum_{m=0}^{\infty}(-1)^m \times d^{(j)}_{mi} = \sum_{m=0}^{\infty}(-1)^m \hat{d}^{(i)}_{mj} = \hat{\mathbf{c}}^{-}_{ji}$. This shows that $\mathfrak{c}^{\leftarrow}_C = \mathfrak{c}^{\rightarrow}_C$ and, according to (a) and (b), the matrix $\mathfrak{c}^{-1}_C := \mathfrak{c}^{\leftarrow}_C = \mathfrak{c}^{\rightarrow}_C$ (2.16) is row-finite and column-finite, and is both left and right inverse of $\mathfrak{c}_C$. This finishes the proof of the theorem. □

Corollary 2.18. *Assume that C is a pointed sharp Euler coalgebra as in Theorem* 2.15*, with the Cartan matrix $\mathfrak{c}_C$ and its inverse $\mathfrak{c}^{-1}_C$* (2.16)*.*

(a) *The matrix $\mathfrak{c}^{-1}_C$ is row-finite and column-finite, and, given $a \in I_C$, we have*:

$$\mathbf{dim}\, E(a) \cdot \mathfrak{c}^{-1}_C = \mathbf{dim}\, S(a) \quad \text{and} \quad \mathbf{dim}\, S(a) \cdot \mathfrak{c}_C = \mathbf{dim}\, E(a),$$
$$\mathbf{dim}\, \widehat{E}(a) \cdot (\mathfrak{c}^{-1}_C)^{tr} = \mathbf{dim}\, S(a) \quad \text{and} \quad \mathfrak{c}_C \cdot (\mathbf{dim}\, S(a))^{tr} = (\mathbf{dim}\, \widehat{E}(a))^{tr}.$$

(b) *The subsets $\{\mathbf{dim}\, E(a)\}_{a \in I_C}$, $\{\mathbf{dim}\, \widehat{E}(a)\}_{a \in I_C}$ of the group $\mathbb{Z}^{I_C}$ are $\mathbb{Z}$-linearly independent.*
(c) *For each $j \in I_C$, the vector $e_j = \mathbf{dim}\, S(j)$ belongs to the subgroup generated by the set $\{\mathbf{dim}\, E(a)\}_{a \in I_C}$, and to the subgroup generated by the set $\{\mathbf{dim}\, \widehat{E}(a)\}_{a \in I_C}$.*

Proof. The equalities in (a) follow from Theorem 2.15, and (b) is a consequence of (a), because the vectors $\mathbf{dim}\, S(a) = e_a \in \mathbb{Z}^{I_C}$, with $a \in I_C$, are $\mathbb{Z}$-linearly independent.

(c) We recall that the ath row of $\mathfrak{c}_C$ is the vector $\mathbf{dim}\, E(a)$. Since the matrix $\mathfrak{c}^{-1}_C$ is row-finite, the equality $\mathfrak{c}^{-1}_C \cdot \mathfrak{c}_C = \mathbf{E}$ yields $e_j = \mathbf{dim}\, S(j) = \sum_{a \in I_C} \mathbf{c}^{-}_{ja} \cdot \mathbf{dim}\, E(a)$ and the first part of (c) follows. The second one follows in a similar way from the equality $\mathfrak{c}_C \cdot \mathfrak{c}^{-1}_C = \mathbf{E}$. □

Corollary 2.19. *If C is pointed and left semiperfect* (*resp. right semiperfect*) *of finite global dimension then the Cartan matrix $\mathfrak{c}_C$ of C is column-finite* (*resp. row-finite*) *and the matrix $\mathfrak{c}^{-1}_C$* (2.16) *is a two-sided inverse of $\mathfrak{c}_C$. Moreover, $\mathfrak{c}^{-1}_C$ is column-finite and row-finite, and the equalities of Corollary* 2.18(a) *hold.*

Proof. By Lemma 2.6 (d) and (e), the Cartan matrix $\mathfrak{c}_C$ of C is column-finite (resp. row-finite), if C is left semiperfect (resp. right semiperfect). Since, according to Lemma 2.9, C is a sharp Euler coalgebra, the corollary follows from Corollary 2.18. □

3. Coxeter transformation for a sharp Euler coalgebra

We study in this section the properties of the Coxeter transformations defined in [19, Definition 4.27] for pointed Euler coalgebras. Here, we also follow [1, Definition III.3.14]. We modify [19, Definition 4.27] as follows.

Definition 3.1. Assume that C is a pointed sharp Euler K-coalgebra with fixed decomposition $\mathrm{soc}_C C = \bigoplus_{j \in I_C} S(j)$. Let $\mathfrak{c}_C \in \mathbb{M}_{I_C}(\mathbb{Z})$ be the Cartan matrix of C and let $\mathfrak{c}^{-1}_C$ be the two-sided inverse (2.16) of $\mathfrak{c}_C$.

(a) The **Coxeter matrix** of C is the $I_C \times I_C$ square matrix $\boldsymbol{\Phi}_C = -\mathfrak{c}^{-tr}_C \cdot \mathfrak{c}_C$, where we set $\mathfrak{c}^{-tr}_C = (\mathfrak{c}^{-1}_C)^{tr} = (\mathfrak{c}^{tr}_C)^{-1}$.

(b) The **Coxeter transformations** of C are the group homomorphisms

$$\mathbb{Z}_{\blacktriangleright}^{I_C} \underset{\boldsymbol{\Phi}_C^-}{\overset{\boldsymbol{\Phi}_C}{\rightleftarrows}} \mathbb{Z}_{\blacktriangleleft}^{I_C} \tag{3.2}$$

defined by the formulas $\boldsymbol{\Phi}_C(x) = -(x \cdot \mathfrak{c}_C^{-tr}) \cdot \mathfrak{c}_C$, for $x \in \mathbb{Z}_{\blacktriangleright}^{I_C}$, and $\Phi_C^-(y) = -(y \cdot \mathfrak{c}_C^{-1}) \cdot \mathfrak{c}_C^{tr}$, for $y \in \mathbb{Z}_{\blacktriangleleft}^{I_C}$, where $\mathbb{Z}_{\blacktriangleright}^{I_C} \subseteq \mathbb{Z}^{I_C}$ is the subgroup of $\mathbb{Z}^{I_C}$ generated by the subset $\{\widehat{\mathbf{e}}(a) = \mathbf{dim}\,\widehat{E}(a)\}_{a \in I_C}$ and $\mathbb{Z}_{\blacktriangleleft}^{I_C} \subseteq \mathbb{Z}^{I_C}$ is the subgroup of $\mathbb{Z}^{I_C}$ generated by the subset $\{\mathbf{e}(a) = \mathbf{dim}\,E(a)\}_{a \in I_C}$.

By Corollary 2.18, the sets $\{\mathbf{dim}\,E(a)\}_{a \in I_C}$ and $\{\mathbf{dim}\,\widehat{E}(a)\}_{a \in I_C}$ are $\mathbb{Z}$-linearly independent in $\mathbb{Z}^{I_C}$ and therefore they form $\mathbb{Z}$-bases of $\mathbb{Z}_{\blacktriangleleft}^{(I_C)}$ and $\mathbb{Z}_{\blacktriangleright}^{(I_C)}$, respectively. Note also that $M \mapsto \mathbf{dim}\,M$ defines the group isomorphism of the Grothendieck group $K_0^{\blacktriangleright}(C) = K_0(C^{op}\text{-inj})$ and $\mathbb{Z}_{\blacktriangleright}^{(I_C)}$, and the group isomorphism of the Grothendieck group $K_0^{\blacktriangleleft}(C) = K_0(C\text{-inj})$ and $\mathbb{Z}_{\blacktriangleleft}^{I_C}$. Note that, by Corollary 2.18,

$$\boldsymbol{\Phi}_C\big(\mathbf{dim}\,\nabla_C E(a)\big) = \boldsymbol{\Phi}_C\big(\hat{\mathbf{e}}(a)\big) = -\big(\hat{\mathbf{e}}(a) \cdot \mathfrak{c}_C^{-tr}\big) \cdot \mathfrak{c}_C = -e_a \cdot \mathfrak{c}_C = -\mathbf{e}(a) = -\,\mathbf{dim}\,E(a),$$
$$\boldsymbol{\Phi}_C^-\big(\mathbf{dim}\,E(a)\big) = \boldsymbol{\Phi}_C^-\big(\mathbf{e}(a)\big) = -e_a \cdot \mathfrak{c}_C^{-1} = -\hat{\mathbf{e}}(a) - \mathbf{dim}\,\nabla_C E(a).$$

It follows that the transformations (3.2) are well-defined and mutually inverse.

The following theorem is the main result of this section (compare with [1, Corollary IV.2.9]).

Theorem 3.3. *Assume that C is a pointed sharp Euler K-coalgebra with fixed decomposition* $\operatorname{soc}_C C = \bigoplus_{j \in I_C} S(j)$. *Let $\boldsymbol{\Phi}_C$ and $\boldsymbol{\Phi}_C^-$ be the Coxeter transformations* (3.2) *of C.*

(a) *Let M be an indecomposable left C-comodule in* $C\text{-Comod}_{fc}^{\bullet}$ *such that* $\operatorname{inj.dim} M = 1$ *and* $\operatorname{Hom}_C(C, M) = 0$. *If*

$$0 \longrightarrow M \longrightarrow M' \longrightarrow \tau_C^- M \longrightarrow 0$$

is the unique almost split sequence (1.21) *in* $C\text{-Comod}_{fc}$, *with an indecomposable comodule $\tau_C^- M$ lying in* $C\text{-comod}_{f\mathcal{P}}$ *then*

$$\mathbf{dim}\,\tau_C^- M = \boldsymbol{\Phi}_C^-(\mathbf{dim}\,M).$$

(b) *Assume that N is an indecomposable non-projective left C-comodule in* $C\text{-comod}_{f\mathcal{P}} \subseteq C\text{-Comod}_{fc}$ *such that* $\operatorname{inj.dim} DN = 1$ *and* $\operatorname{Hom}_C(C, DN) = 0$. *If*

$$0 \longrightarrow \tau_C N \longrightarrow N' \longrightarrow N \longrightarrow 0$$

is the unique almost split sequence (1.22) *in* $C\text{-Comod}_{fc}$, *with an indecomposable comodule $\tau_C N$ lying in* $C\text{-Comod}_{fc}^{\bullet}$, *then*

$$\mathbf{dim}\,\tau_C N = \boldsymbol{\Phi}_C(\mathbf{dim}\,N).$$

Proof. (a) Assume that M is an indecomposable left C-comodule in $C\text{-Comod}_{fc}^{\bullet}$ such that $\operatorname{inj.dim} M = 1$. Then M admits a minimal injective copresentation

$$0 \longrightarrow M \longrightarrow E_0 \xrightarrow{g} E_1 \longrightarrow 0$$

in $C\text{-Comod}_{fc}$ where E_0 and E_1 are socle-finite injective comodules. It follows that $\mathbf{dim}\,M = \mathbf{dim}\,E_1 - \mathbf{dim}\,E_0$. Since $\boldsymbol{\Phi}_C^-(\mathbf{dim}\,E(a)) = -\,\mathbf{dim}\,E(a) \cdot \mathfrak{c}_C^{-1} = -\,\mathbf{dim}\,\widehat{E}(a)$, for every $a \in I_C$ and the comodules E_0

and E_1 are finite direct sums of the comodules $E(a)$, with $a \in I_C$, we get $\boldsymbol{\Phi}_C^-(\mathbf{dim}\, E_0) = -\,\mathbf{dim}\, \nabla_C(E_0)$, $\boldsymbol{\Phi}_C^-(\mathbf{dim}\, E_1) = -\,\mathbf{dim}\, \nabla_C(E_1)$ and, by applying $\boldsymbol{\Phi}_C^-$, the equality $\mathbf{dim}\, M = \mathbf{dim}\, E_1 - \mathbf{dim}\, E_0$ yields

$$\boldsymbol{\Phi}_C^-(\mathbf{dim}\, M) = \boldsymbol{\Phi}_C^-(\mathbf{dim}\, E_1) - \boldsymbol{\Phi}_C^-(\mathbf{dim}\, E_0) = \mathbf{dim}\, \nabla_C(E_0) - \mathbf{dim}\, \nabla_C(E_1).$$

On the other hand, the exact sequence (1.10) in C^{op}-Comod, induced by the injective copresentation of M, has the form

$$0 \longrightarrow \mathrm{Tr}_C(M) \longrightarrow \nabla_C(E_1) \xrightarrow{\nabla_C(g)} \nabla_C(E_0) \longrightarrow 0,$$

because the assumption $\mathrm{Hom}_C(C, M) = 0$ yields $\nabla_C(M) = \mathrm{Hom}_C(C, M)^\circ = 0$, see Theorem 1.8(a). Since $\dim_K \mathrm{Tr}_C(M)$ is finite, we have $\mathbf{dim}\, D\mathrm{Tr}_C(M) = \mathbf{dim}\, \mathrm{Tr}_C(M)$ and the exact sequence yields

$$\mathbf{dim}\, \tau_C^- M = \mathbf{dim}\, D\,\mathrm{Tr}_C(M) = \mathbf{dim}\, \mathrm{Tr}_C(M) = \mathbf{dim}\, \nabla_C(E_0) - \mathbf{dim}\, \nabla_C(E_1) = \boldsymbol{\Phi}_C^-(\mathbf{dim}\, M)$$

and (a) follows.

(b) By Proposition 1.16(b), there is a duality $D : C\text{-comod}_{f\mathcal{P}} \xrightarrow{\simeq} C^{op}\text{-comod}_{fc}$ that carries the indecomposable left C-comodule in C-comod$_{f\mathcal{P}}$ to the indecomposable right C-comodule in C^{op}-comod$_{fc}$. Since we assume inj.dim $DN = 1$, the comodule DN is not injective and there is a minimal socle-finite injective copresentation

$$0 \longrightarrow DN \longrightarrow E_0' \xrightarrow{g'} E_1' \longrightarrow 0$$

of DN in C^{op}-Comod$_{fc}$. By an obvious $\nabla_{C^{op}}$ version of Theorem 1.8, there is a short exact sequence

$$0 \longrightarrow \mathrm{Tr}_{C^{op}}(DN) \longrightarrow \nabla_{C^{op}}\big(E_1'\big) \xrightarrow{\nabla_{C^{op}}(g')} \nabla_{C^{op}}\big(E_0'\big) \longrightarrow \nabla_{C^{op}}(DN) \longrightarrow 0$$

in C-Comod$_{fc}$. The assumption $\mathrm{Hom}_C(C, DN) = 0$ yields $\nabla_{C^{op}}(DN) = \mathrm{Hom}_C(C, DN)^\circ = 0$. Then, by applying the arguments used in the proof of (a), we get

$$\begin{aligned} \mathbf{dim}\, \mathrm{Tr}_{C^{op}}(DN) &= \mathbf{dim}\, \nabla_{C^{op}}\big(E_1'\big) - \mathbf{dim}\, \nabla_{C^{op}}\big(E_0'\big) \\ &= \boldsymbol{\Phi}_C\big(\mathbf{dim}\, E_0'\big) - \boldsymbol{\Phi}_C\big(\mathbf{dim}\, E_1'\big) \\ &= \boldsymbol{\Phi}_C(\mathbf{dim}\, DN) = \boldsymbol{\Phi}_C(\mathbf{dim}\, N), \end{aligned}$$

because $\dim_K N$ is finite. This finishes the proof. □

Remark 3.4. If C is a sharp Euler coalgebra such that gl.dim $C = 1$ and M (resp. N) is an indecomposable non-injective comodule (resp. non-projective comodule), we have inj.dim $M = 1$ and $\mathrm{Hom}_C(C, M) = 0$ (resp. inj.dim $DN = 1$ and $\mathrm{Hom}_C(C, DN) = 0$), and Theorem 3.3 applies to M (resp. to N).

4. Examples

In this section we illustrate previous results by concrete examples.

Example 4.1. Let Q be the infinite locally Dynkin quiver

$$Q: \quad \underset{0}{\bullet} \longrightarrow \underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \underset{3}{\bullet} \longrightarrow \underset{4}{\bullet} \longrightarrow \underset{5}{\bullet} \longrightarrow \cdots$$

of type $\mathbb{A}_\infty$ and let $C = K^\square Q$ be the path K-coalgebra of Q, see [4,17,26]. Then C has the upper triangular matrix form

$$C = \begin{bmatrix} K & K & K & K & K & K & K & K & \dots \\ 0 & K & K & K & K & K & K & K & \dots \\ 0 & 0 & K & K & K & K & K & K & \dots \\ 0 & 0 & 0 & K & K & K & K & K & \dots \\ 0 & 0 & 0 & 0 & K & K & K & K & \dots \\ 0 & 0 & 0 & 0 & 0 & K & K & K & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & K & K & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & K & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and consists of the upper triangular $\mathbb{N} \times \mathbb{N}$ square matrices with coefficients in K with at most finitely many non-zero entries. Then $\mathrm{soc}_C C = \bigoplus_{j \in I_C} S(j)$, where $I_C = \mathbb{N} = \{0, 1, 2, \dots\}$ and $S(n) = Ke_n$ is the simple subcoalgebra spanned by the matrix $e_n \in C$ with 1 in the $n \times n$ entry, and zeros elsewere. Note that e_n is a group-like element of C.

The Cartan matrix $\mathfrak{c}_C \in \mathbb{M}_\mathbb{N}(\mathbb{Z})$ of C and its inverse $\mathfrak{c}_C^{-1}$ have the lower triangular forms

$$\mathfrak{c}_C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & \dots \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \dots \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad \mathfrak{c}_C^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Hence the Coxeter matrices $\boldsymbol{\Phi}_C = -\mathfrak{c}_C^{-tr} \cdot \mathfrak{c}_C$ and $\boldsymbol{\Phi}_C^{-1} = -\mathfrak{c}_C^{-1} \cdot \mathfrak{c}_C^{tr}$ are of the forms

$$\boldsymbol{\Phi}_C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad \boldsymbol{\Phi}_C^{-1} = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The coalgebra C is pointed, representation-directed in the sense of [19], right semiperfect and hereditary, that is, $\mathrm{gl.dim}\, C = 1$ [15]. Hence C is a sharp Euler K-coalgebra. Every left C-comodule is a direct sum of finite-dimensional ones [17] and therefore every indecomposable left C-comodule is finite-dimensional. The left C-comodules in C-comod can be identified with the finite-dimensional K-linear representations of the quiver Q. Under the identification $C\text{-comod} = \mathrm{rep}_K(Q)$, the Auslander–Reiten quiver of C-comod has the form

$$\begin{array}{cccccccccccccccccccc}
 & \cdots & & {}_4\mathbb{I}_4 & & \text{- - - -} & & {}_3\mathbb{I}_3 & & \text{- - - -} & & {}_2\mathbb{I}_2 & & \text{- - - -} & & {}_1\mathbb{I}_1 & & \text{- - - -} & & {}_0\mathbb{I}_0 \\
 & & & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
 & & & \text{- - - -} & & {}_3\mathbb{I}_4 & & \text{- - - -} & & {}_2\mathbb{I}_3 & & \text{- - - -} & & {}_1\mathbb{I}_2 & & \text{- - - -} & & {}_0\mathbb{I}_1 & & \\
 & & & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & & \\
 & \cdots & & {}_3\mathbb{I}_5 & & \text{- - - -} & & {}_2\mathbb{I}_4 & & \text{- - - -} & & {}_1\mathbb{I}_3 & & \text{- - - -} & & {}_0\mathbb{I}_2 & & & & \\
 & & & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & & & & \\
 & & & \text{- - - -} & & {}_2\mathbb{I}_5 & & \text{- - - -} & & {}_1\mathbb{I}_4 & & \text{- - - -} & & {}_0\mathbb{I}_3 & & & & & & \\
 & & & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & & & & & & \\
 & & & \vdots & & & & \vdots & & & & \vdots & & & & & & & &
\end{array}$$

Fig. 1. The Auslander–Reiten quiver of the category C-comod $\cong \mathrm{rep}_K(Q)$.

see [16], where

$${}_n\mathbb{I}_m : 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow K_n \xrightarrow{id} K_{n+1} \xrightarrow{id} \cdots \xrightarrow{id} K_m \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots\cdots$$

$K_n = K_{n+1} = \ldots = K_m = K$ and $n \leqslant m$. Note that ${}_m\mathbb{I}_m = S(m)$ is simple and ${}_0\mathbb{I}_m = E(m)$ is the injective envelope of $S(m)$, for each $m \geqslant 0$. Hence, the indecomposable injectives in the category C-comod form the right-hand section

$$\cdots \longrightarrow {}_0\mathbb{I}_6 \longrightarrow {}_0\mathbb{I}_5 \longrightarrow {}_0\mathbb{I}_4 \longrightarrow {}_0\mathbb{I}_3 \longrightarrow {}_0\mathbb{I}_2 \longrightarrow {}_0\mathbb{I}_1 \longrightarrow {}_0\mathbb{I}_0$$

of Fig. 1. Note also that C-comod contains no non-zero projective objects. Thus

$$0 \longrightarrow \mathrm{Tr}_C({}_n\mathbb{I}_m) \longrightarrow \nabla_C E(n-1) \longrightarrow \nabla_C E(m-1)$$

is an injective copresentation yielding $\tau_C^-({}_n\mathbb{I}_m) = D\,\mathrm{Tr}_C({}_n\mathbb{I}_m) \cong {}_{n-1}\mathbb{I}_{m-1}$, for $n \geqslant 1$. The almost split sequences are

$$0 \longrightarrow {}_n\mathbb{I}_m \longrightarrow {}_{n-1}\mathbb{I}_m \oplus {}_n\mathbb{I}_{m-1} \longrightarrow {}_{n-1}\mathbb{I}_{m-1} \longrightarrow 0$$

with irreducible morphisms ${}_n\mathbb{I}_m \longrightarrow {}_n\mathbb{I}_{m+1}$ and ${}_{n-1}\mathbb{I}_m \longrightarrow {}_n\mathbb{I}_m$ being the obvious monomorphism into the first summand and epimorphism onto the second summand. The map on the right is given by natural epimorphism and monomorphism with alternate signs. This means that $\tau_C({}_{n-1}\mathbb{I}_{m-1}) \cong {}_n\mathbb{I}_m$ and $\tau_C^-({}_n\mathbb{I}_m) \cong {}_{n-1}\mathbb{I}_{m-1}$, if $n \geqslant 1$. Note also that $\mathbf{dim}\,\tau_C({}_{n-1}\mathbb{I}_{m-1}) = \boldsymbol{\Phi}_C(\mathbf{dim}\,{}_n\mathbb{I}_m)$ and $\mathbf{dim}\,\tau_C^-({}_n\mathbb{I}_m) = \boldsymbol{\Phi}_C^{-1}(\mathbf{dim}_{n-1}\mathbb{I}_{m-1})$, if $n \geqslant 1$ (compare with Theorem 3.3).

Example 4.2. Let Q be the infinite locally Dynkin quiver

$$Q: \quad \cdots \longrightarrow \underset{-2}{\bullet} \longrightarrow \underset{-1}{\bullet} \longrightarrow \underset{0}{\bullet} \longrightarrow \underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \underset{3}{\bullet} \longrightarrow \underset{4}{\bullet} \longrightarrow \underset{4}{\bullet} \longrightarrow \cdots$$

of type ${}_\infty\mathbb{A}_\infty$ and let $C = K^\square Q$ be the path K-coalgebra of Q. Then C has the upper triangular matrix form

$$C = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \dots & K & K & K & K & K & K & K & K & \dots \\ \dots & 0 & K & K & K & K & K & K & K & \dots \\ \dots & 0 & 0 & K & K & K & K & K & K & \dots \\ \dots & 0 & 0 & 0 & K & K & K & K & K & \dots \\ \dots & 0 & 0 & 0 & 0 & K & K & K & K & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & K & K & K & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & 0 & K & K & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & K & \dots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and consists of the upper triangular $\mathbb{Z} \times \mathbb{Z}$ square matrices with coefficients in K with at most finitely many non-zero entries. Then $\operatorname{soc}_C C = \bigoplus_{j \in I_C} S(j)$, where $I_C = \mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ and $S(n) = Ke_n$ is the simple subcoalgebra spanned by the matrix $e_n \in C$ with 1 in the $n \times n$ entry, and zeros elsewere. Note that e_n is a group-like element of C. The coalgebra C is pointed, hereditary, left and right locally artinian and, by Corollary 2.10, C is a sharp Euler K-coalgebra. Obviously, C is neither right semiperfect nor left semiperfect.

The Cartan matrix $\mathfrak{c}_C \in \mathbb{M}_\mathbb{N}(\mathbb{Z})$ of C and its inverse $\mathfrak{c}_C^{-1}$ have the lower triangular forms

$$\mathfrak{c}_C = \begin{bmatrix} \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \dots & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ \dots & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & \dots \\ \dots & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & \dots \\ \dots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \dots \\ \dots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad \mathfrak{c}_C^{-1} = \begin{bmatrix} \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \ddots & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \ddots & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & \\ \dots & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & \ddots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Hence the Coxeter matrices $\boldsymbol{\Phi}_C = -\mathfrak{c}_C^{-tr} \cdot \mathfrak{c}_C$ and $\boldsymbol{\Phi}_C^{-1} = -\mathfrak{c}_C^{-1} \cdot \mathfrak{c}_C^{tr}$ are of the forms

$$\boldsymbol{\Phi}_C = \begin{bmatrix} \ddots & \ddots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \ddots & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \\ \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots \end{bmatrix} \qquad \boldsymbol{\Phi}_C^{-1} = \begin{bmatrix} \ddots & \ddots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \ddots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \ddots & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \\ \dots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \ddots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots \end{bmatrix}.$$

It is known that there is an equivalence of categories $K^\square Q\text{-comod} \cong \operatorname{rep}_K(Q)$ and we view it as an identification, see [5] or [17], [21, Proposition 3.3]. We recall from [16] that any finite-dimensional

K-linear representation $N \in \mathrm{rep}_K(Q)$ of the infinite quiver Q restricts to a representation of a finite convex linear quiver $Q^N = \mathbf{supp}(N)$ (the support of N) of the Dynkin type $\mathbb{A}_n$ and is isomorphic to a finite interval representation of the form

$$ {}_n\mathbb{I}_m : \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow K_n \xrightarrow{id} K_{n+1} \xrightarrow{id} \cdots \xrightarrow{id} K_m \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots $$

where $-\infty < m \leqslant t < \infty$ and $K_j = K$, for all $m \leqslant j \leqslant t$. It is easy to see that the indecomposable injective $K^{\square}Q$-comodules are infinite-dimensional. Hence the category C-comod contains no non-zero injective objects and no non-zero projective objects.

By Corollary 1.23, every indecomposable object N of C-comod has an almost split sequence in C-comod starting from N and has an almost split sequence in C-comod terminating in N. Moreover the Auslander–Reiten translation quiver $\Gamma(C\text{-comod})$ of the category C-comod has the form

$$
\begin{array}{ccccccccccccc}
\cdots {}_4\mathbb{I}_4 & \text{------} & {}_3\mathbb{I}_3 & \text{------} & {}_2\mathbb{I}_2 & \text{------} & {}_1\mathbb{I}_1 & \text{------} & {}_0\mathbb{I}_0 & \text{------} & & & \\
& \searrow & & \nearrow \; \searrow & & \nearrow \; \searrow & & \nearrow \; \searrow & & \nearrow \; \searrow & & \nearrow & \\
\cdots\text{---} & {}_3\mathbb{I}_4 & \text{------} & {}_2\mathbb{I}_3 & \text{------} & {}_1\mathbb{I}_2 & \text{------} & {}_0\mathbb{I}_1 & \text{------} & {}_{-1}\mathbb{I}_0 & \text{--} & & \\
& \nearrow & & \searrow \; \nearrow & & \searrow \; \nearrow & & \searrow \; \nearrow & & \searrow \; \nearrow & & \searrow & \\
\cdots {}_3\mathbb{I}_5 & \text{------} & {}_2\mathbb{I}_4 & \text{------} & {}_1\mathbb{I}_3 & \text{------} & {}_0\mathbb{I}_2 & \text{------} & {}_{-1}\mathbb{I}_1 & \text{------} & & & \\
& \searrow & & \nearrow \; \searrow & & \nearrow \; \searrow & & \nearrow \; \searrow & & \nearrow \; \searrow & & \nearrow & \\
\cdots\text{---} & {}_2\mathbb{I}_5 & \text{------} & {}_1\mathbb{I}_4 & \text{------} & {}_0\mathbb{I}_3 & \text{------} & {}_{-1}\mathbb{I}_2 & \text{------} & {}_{-2}\mathbb{I}_1 & \text{--} & & \\
& \nearrow & & \searrow \; \nearrow & & \searrow \; \nearrow & & \searrow \; \nearrow & & \searrow \; \nearrow & & \searrow & \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & & &
\end{array}
$$

Fig. 2. The Auslander–Reiten quiver of the category C-comod $\cong \mathrm{rep}_K(Q)$.

Note that the Coxeter transformation $\boldsymbol{\Phi}_C : \mathbb{Z}^{I_C}_{\blacktriangleright} \longrightarrow \mathbb{Z}^{I_C}_{\blacktriangleleft}$ (3.2) extends to the isomorphism $\boldsymbol{\Phi}_C : \mathbb{Z}^{\mathbb{Z}} \longrightarrow \mathbb{Z}^{\mathbb{Z}}$ defined by the formula $\boldsymbol{\Phi}_C(x) = x \cdot \boldsymbol{\Phi}_C$. It carries any vector $x = (x_n)_{n\in\mathbb{Z}} \in \mathbb{Z}^{\mathbb{Z}}$ to the vector $\Phi_C(x) = \hat{x} = (\hat{x}_n)_{n\in\mathbb{Z}} \in \mathbb{Z}^{(\mathbb{Z})}$, with $\hat{x}_n = x_{n-1}$, for all $n \in \mathbb{Z} = I_C$. This means that $\boldsymbol{\Phi}_C$ shifts any vector $x \in \mathbb{Z}^{\mathbb{Z}}$ by one step to the right. It follows that the inverse $\boldsymbol{\Phi}_C^{-1} : \mathbb{Z}^{\mathbb{Z}} \longrightarrow \mathbb{Z}^{\mathbb{Z}}$ of $\boldsymbol{\Phi}_C$ shifts any vector $x \in \mathbb{Z}^{\mathbb{Z}}$ by one step to the left.

Hence, by applying the Auslander–Reiten quiver shown in Fig. 2, we conclude that, given an indecomposable N in C-comod, there exist almost split sequences

$$ 0 \longrightarrow \tau_C N \longrightarrow Y \longrightarrow N \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow N \longrightarrow Z \longrightarrow \tau_C^{-1} N \longrightarrow 0 $$

in C-comod and the following equalities hold (compare with Theorem 3.3)

$$ \mathbf{dim}(\tau_C N) = \boldsymbol{\Phi}_C(\mathbf{dim}\, N) \quad \text{and} \quad \mathbf{dim}\big(\tau_C^{-1} N\big) = \boldsymbol{\Phi}_C^{-1}(\mathbf{dim}\, N). $$

Let us also look at the (abelian) category C-Comod_{fc} of finitely copresented C-comodules. It consists of artinian C-comodules and, by applying [19, Proposition 2.13(a)], one can show that every indecomposable comodule M of C-Comod_{fc} is either injective, with $\dim_K M = \infty$, or M is finite-dimensional isomorphic to one of the comodules listed in Figure 2 and $\dim_K \mathrm{Tr}_C M$ is finite. It follows that C-$\mathrm{Comod}^{\bullet}_{fc} = C$-$\mathrm{Comod}_{fc}$ and C-$\mathrm{comod}^{\bullet}_{fc} = C$-comod. One can also show that the Grothendieck group $K_0(C\text{-}\mathrm{Comod}_{fc})$ of C-Comod_{fc} is isomorphic to the Grothendieck group $K_0^{\blacktriangleleft}(C) = K_0(C\text{-inj}) \cong \mathbb{Z}^{I_C}_{\blacktriangleleft}$ of the category C-inj. Moreover, the Auslander–Reiten quiver $\Gamma(C\text{-}\mathrm{Comod}_{fc})$ of the category C-Comod_{fc} has two connected components:

(a) the component shown in Fig. 2 consisting of all indecomposable C-comodules of finite dimension, and

(b) the following component consisting of all indecomposable injective C-comodules:

$$\cdots \longrightarrow E(-2) \longrightarrow E(-1) \longrightarrow E(0) \longrightarrow E(1) \longrightarrow E(2) \longrightarrow E(3) \longrightarrow \cdots,$$

with $\dim_K E(j) = \infty$, for all $j \in \mathbb{Z}$.

Example 4.3. Let $Q = (Q_0, Q_1)$ be the infinite locally Dynkin quiver of type $\mathbb{D}_\infty$ presented in Example 1.25, with $Q_0 = \{-1, 0, 1, 2, 3, \ldots\}$, and let $C = K^{\square}Q$ be the path K-coalgebra of C. Then C has a $Q_0 \times Q_0$ square matrix form shown in Example 1.25. The Cartan matrix $\mathfrak{c}_C \in \mathbb{M}_{Q_0}(\mathbb{Z})$ of C and its inverse $\mathfrak{c}_C^{-1}$ have the following forms

$$\mathfrak{c}_C = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \ldots \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad \mathfrak{c}_C^{-1} = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

Hence the Coxeter matrices $\boldsymbol{\Phi}_C = -\mathfrak{c}_C^{-tr} \cdot \mathfrak{c}_C$ and $\boldsymbol{\Phi}_C^{-1} = -\mathfrak{c}_C^{-1} \cdot \mathfrak{c}_C^{tr}$ are of the forms

$$\boldsymbol{\Phi}_C = \begin{bmatrix} -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix} \qquad \boldsymbol{\Phi}_C^{-1} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & \ldots \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

The coalgebra C is pointed, right semiperfect and hereditary, that is, $\mathrm{gl.dim}\, C = 1$. Hence C is a sharp Euler K-coalgebra. Every left C-comodule is a direct sum of finite-dimensional ones [17] and therefore every indecomposable left C-comodule is finite-dimensional. If N is a terminus of a mesh in the Auslander–Reiten quiver $\Gamma(C\text{-comod})$ shown in Fig. 0 then N is the right-hand term of an almost split sequence $0 \longrightarrow \tau_C N \longrightarrow N' \longrightarrow N \longrightarrow 0$ in C-comod and $\mathbf{dim}(\tau_C N) = \boldsymbol{\Phi}_C(\mathbf{dim}\, N)$, by Theorem 3.3. It follows that the dimension vectors of the modules lying in each of two infinite components of $\Gamma(C\text{-comod})$ shown in Figure 0 can be computed from the dimension vectors of the modules lying on the sections $(*)$ and $(**)$ (presented in Example 1.25) by applying the iterations $\boldsymbol{\Phi}_C^m$, with $m \geqslant 1$, of the Coxeter transformation $\boldsymbol{\Phi}_C$. Obviously, C is representation-directed in the sense of [19] and every indecomposable C-comodule N is uniquely determined by its dimension vector $\mathbf{dim}\, N$, see [17–19].

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