MATHEMATICS

SEPARATION AXIOMS BETWEEN T_0 AND T_1

BY

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1. Introduction. The first systematic treatment of separation axioms is due to URYSOHN [7]. A more detailed discussion was given by FREUDENTHAL and VAN EST [2] in 1951. Both of these investigations are concerned with separation axioms stronger than T_1. The only separation axiom between T_0 and T_1 known heretofore was introduced by J. W. T. YOUNGS [9] who encountered it in the study of locally connected spaces. Another axiom was suggested to us by an observation of C. T. YANG (see [4, p. 56]) that the derived set of every set is closed iff the derived set of every point is closed.

After introducing some tools which will play an important role throughout the article, we use these to give a new proof of a result of M. H. STONE [6] which states that by identifying "undistinguishable" points in a topological space every space can be made into a T_0-space. Next, we introduce a number of new separation axioms, giving equivalent forms for some, analyse their inclusion relations, and observe that they all can be described in terms of the behavior of derived sets of points. In the remaining sections some applications are considered. These include consideration of homogeneity, normality, behavior under a strengthening of the topology, and relation to discrete spaces of Alexandroff.

We employ the terminology and notation used by KELLEY [4]. By a degenerate set we shall mean a set which contains at most one point.

2. Preliminary considerations. To characterize separation axioms between T_0 and T_1 it is convenient to take as a point of departure the concept of weak separation [8, p. 78]. A set A is said to be *weakly separated* from B (notation: $A \vdash B$) iff there exists an open set $G \supset A$ such that $G \cap B = 0$. Instead of $[x] \vdash [y]$ we shall simply write $x \vdash y$ and say that x can be weakly separated from y. It is then immediately clear that:

The closure of a point x (or more precisely of the set $[x]$), which will be denoted by $[\bar{x}]$, consists of those and only those points y of the space for which $y \not\vdash x$. The derived set of a point, denoted by $[x]'$, consists of all $y \neq x$ for which $y \not\vdash x$.

It is of interest also to characterize those sets for which, for a fixed x, $x \not\vdash y$. We are thus led to the following definitions.

Definition 2.1. The set of all y for which $x \vdash\!/\!- y$ is called the *kernel* of x and is denoted by $[\hat{x}]$. The set $[\hat{x}] \sim [x]$ is called the *shell* of x and is denoted by $[\widehat{x}]$.

It is well known that

$$(2.1) \qquad [\bar{x}] = \cap [C : C \text{ closed}, \ x \in C].$$

An analogous result holds for $[\hat{x}]$.

$$(2.2) \qquad [\hat{x}] = \cap [G : G \text{ open}, \ x \in G].$$

Before proceeding to give some of the basic properties of point closures, derived sets, kernels, and shells we make one more set of definitions.

Definition 2.2. By N_D we shall mean the set of all x in a topological space $(x, \mathscr{T})$ for which $[x]' = 0$. Similarly N_S is the set of all x for which $[\widehat{x}] = 0$. Finally, we define

$$\langle x \rangle = [\bar{x}] \cap [\hat{x}].$$

The following basic relations hold in arbitrary topological spaces.

Theorem 2.1. *Let x and y be elements of a topological space $(X, \mathscr{T})$ then*

(a) $y \in [\bar{x}]$ *implies* $[\bar{y}] \subset [\bar{x}]$,
(b) $y \in [\hat{x}]$ *implies* $[\hat{y}] \subset [\hat{x}]$,
(c) $y \in [\hat{x}]$ *iff* $x \in [\bar{y}]$,
(d) $y \in [\widehat{x}]$ *iff* $x \in [y]'$,
(e) *for every* $x \in X$ $[\widehat{x}]$ *is degenerate iff for all* $x, y \in X$, $x \neq y$ $[x]' \cap [y]' = 0$,
(f) $y \in \langle x \rangle$ *implies* $\langle y \rangle = \langle x \rangle$,
(g) *for all* $x, y \in X$ *either* $\langle x \rangle \cap \langle y \rangle = 0$ *or* $\langle x \rangle = \langle y \rangle$.

Proof: Assertions (a) and (b) follow from formulas (2.1) and (2.2), respectively. Statement (c) is an immediate consequence of the definition of $[\bar{x}]$ and $[\hat{x}]$. (d) is derived from (c) if one observes that $y \in [\widehat{x}]$ implies $y \neq x$. (e) is established by observing that $z \in [x]' \cap [y]'$ is true *iff* x, $y \in [\widehat{z}]$. To prove (f) we note that $y \in [\bar{x}]$ implies $[\bar{y}] \subset [\bar{x}]$ and $x \in [\hat{y}]$. The last relation yields $[\hat{x}] \subset [\hat{y}]$. Similarly, $y \in [\hat{x}]$ leads to $[\hat{y}] \subset [\bar{x}]$ and $[\bar{x}] \subset [\bar{y}]$. Putting these results together we obtain $[\bar{x}] = [\bar{y}]$ and $[\hat{x}] = [\hat{y}]$ from which (f) follows. Statement (g) is an immediate consequence of (f).

In terms of the sets $\langle x \rangle$ we can now prove the result of Stone mentioned earlier. We shall call two topological spaces $(X, \mathscr{T})$ and $(Y, \mathscr{U})$ *lattice equivalent* iff a one-to-one order preserving map can be established between the elements of $\mathscr{T}$ and those of $\mathscr{U}$.

Theorem 2.2. *Let $(X, \mathscr{T})$ be an arbitrary topological space. Let R be the equivalence relation on $X \times X$ defined by $(x, y) \in R$, iff $y \in \langle x \rangle$. Then X/R with the quotient topology $\mathscr{T}_R$ is a T_0-space, and the two spaces are lattice equivalent.*

Proof: Recall that the quotient topology $\mathscr{T}_R$ of X/R consists of those and only those collections G_R of sets $\langle x\rangle$ which satisfy

$$\bigcup [\langle x\rangle : \langle x\rangle \in G_R] = G \in \mathscr{T}.$$

To show that the two topologies $\mathscr{T}$ and $\mathscr{T}_R$ are lattice equivalent we define

$$\varphi(G) = [\langle x\rangle : x \in G],$$

where the $\langle x\rangle$ are considered as elements of X/R, and $\varphi(G)$ is thus a subset of X/R. $\varphi(G) \in \mathscr{T}_R$, for $x \in G$ implies $[\hat{x}] \subset G$, which in turn implies $\langle x\rangle \subset G$. So

$$\bigcup [\langle x\rangle : x \in G] = G,$$

here the $\langle x\rangle$ are considered to be subsets of X. $\varphi(G)$ is one-to-one, because in view of the above relation $\varphi(G_1) = \varphi(G_2)$ implies $G_1 = G_2$. $\varphi(G)$ is onto $\mathscr{T}_R$. Let $G_R \in \mathscr{T}_R$ then $\bigcup [\langle x\rangle : \langle x\rangle \in G_R] = G \in \mathscr{T}$ and hence $G_R = \varphi(G)$. Clearly φ is order preserving.

Let $\langle x\rangle \neq \langle y\rangle$ be two elements of X/R. If $\langle x\rangle \not\vdash \langle y\rangle$ then every open set G_R which contains $\langle x\rangle$ also contains $\langle y\rangle$. In view of the discussion given above this means that every open set G of $\mathscr{T}$ which contains x also contains y. Hence $y \in [\hat{x}]$. Similarly the assumption $\langle y\rangle \not\vdash \langle x\rangle$ leads to the conclusion $x \in [\hat{y}]$, or equivalently $y \in [\bar{x}]$. It follows that $y \in \langle x\rangle$ which is a contradiction. Hence either $\langle x\rangle \vdash \langle y\rangle$ or $\langle y\rangle \vdash \langle x\rangle$ must hold and $(X/R, \mathscr{T}_R)$ is a T_0-space.

For our later work it will be helpful to have various equivalent forms for the T_0 and T_1 separation axioms.

Theorem 2.3. *A space $(X, \mathscr{T})$ is a T_0-space iff one of the following conditions holds*:

(a) *for arbitrary $x, y \in X$, $x \neq y$ either $x \vdash y$ or $y \vdash x$,*
(b) *$y \in [\bar{x}]$ implies $x \notin [\bar{y}]$,*
(b)′ *$y \in [x]'$ implies $[\bar{y}] \subset [x]'$,*
(c) *$y \in [\hat{x}]$ implies $x \notin [\hat{y}]$,*
(c)′ *$y \in [\hat{x}]$ implies $[\hat{y}] \subset [\hat{x}]$,*
(d) *$([\bar{x}] \cap [y]) \cup ([\bar{y}] \cap [x])$ is degenerate* (Youngs [9]),
(e) *for every $x \in X$ $[x]'$ is the union of closed sets,*
(f) *for every $x \in X$ $[x]' \cap [\hat{x}] = 0$,*
(g) *for every $x \in X$ $\langle x\rangle = [x]$.*

Proof: It is easily seen that each one of these conditions, except (e), is a simple restatement of condition (a) in a different terminology. To prove statement (e) we observe that for every $z \in [x]'$ in a T_0-space there must exist an open set G containing x and not containing z. But then $z \in \sim G$ which is closed and does not contain x. Also $z \in (\sim G) \cap [\bar{x}]$ which is a closed set contained in $[x]'$. If condition (e) is satisfied then every $z \in [x]'$ is contained in some closed set C not containing x, but then $\sim C$ is an open set containing x and not z. If $z \notin [x]'$ and $z \neq x$ then $z \in \sim[\bar{x}]$ which is an open set not containing x. It follows that the space is T_0.

Theorem 2.4. *A space $(X, \mathscr{T})$ is a T_1-space iff one of the following conditions holds:*

(a) *for arbitrary* $x, y \in X, x \neq y, \quad x \vdash y,$

(b) *for every* $x \in X \quad [\check{x}] = [x],$

(b′) *for every* $x \in X \quad [x]' = 0,$

(c) *for every* $x \in X \quad [\hat{x}] = [x],$

(c)′ *for every* $x \in X \quad [\widehat{x}] = 0,$

(d) *for arbitrary* $x, y \in X, x \neq y \quad [\check{x}] \cap [\check{y}] = 0,$

(e) *for arbitrary* $x, y \in X, x \neq y \quad [\hat{x}] \cap [\hat{y}] = 0,$

(f) $N_D = X,$

(g) $N_S = X.$

Proof: Most of these statements are clearly equivalent to statement (a). To clarify those conditions stated in terms of kernels or shells we might point out that by Theorem 2.1 (c) and (d) $[\check{x}]$ can contain points other than x iff there exists a $[\hat{y}]$ which contains $x \neq y$, and that $[x]'$ is non-empty iff there exists a non-empty $[\widehat{y}]$. That conditions (d) and (e) are satisfied in a T_1-space is clear. That they ensure that the space is T_1 follows from the fact that the conditions imply, respectively, $x \notin [\check{y}]$ and $y \notin [\hat{x}]$, each of which means $x \vdash y$.

3. New separation axioms. The first group of new axioms which we are about to introduce is based on the observation that in a T_0-space $[x]'$ is the union of closed sets for every x (Theorem 2.3 (e)), while in a T_1-space $[x]' = 0$, and hence closed, for all x. Also in a T_1-space $[x]' \cap [y]' = 0$ for $x \neq y$. This suggests introducing three new axioms as follows.

Definition 3.1. A space $(X, \mathscr{T})$ will be called a T_{UD}-space iff for every $x \in X$

$[x]'$ is the union of disjoint closed sets.

The space will be called a T_D-space [1] iff for every $x \in X$

$[x]'$ is a closed set.

The space will be called a T_{DD}-space iff it is a T_D-space and iff in addition for all $x, y \in X, x \neq y.$

$$[x]' \cap [y]' = 0.$$

It is clear that the following inclusion relations hold among these spaces

$$T_1 \subset T_{DD} \subset T_D \subset T_{UD} \subset T_0.$$

Here, and later, the notation $T_\alpha \subset T_\beta$ is understood to mean that every T_α-space is a T_β-space.

For the separation axiom T_D there is an interesting equivalent formulation.

Theorem 3.1. *A space $(X, \mathscr{T})$ is a T_D-space iff for every $x \in X$ there exists an open set G and a closed set C in the space such that*

$$[x] = G \cap C.$$

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1) The combination of axioms b), 1^0, 2^0, 5^0 mentioned by Frechet in an article in Bull. des Sc. Math. 42, p. 156 (1918) can be proved to be equivalent to those of a T_D-space.

Proof: If the space is T_D it suffices to set $G = \sim[x]'$ and $C = [\bar{x}]$. If the conditions of the theorem are satisfied then one can replace C in $[x] = G \cap C$ by $[\bar{x}]$. Hence

$$[x]' = [\bar{x}] \sim [x] = [\bar{x}] \sim G \cap [\bar{x}] = [\bar{x}] \cap (X \sim G).$$

Thus $[x]'$ is the intersection of two closed sets, and hence is closed.

The two axioms in the second group arise from a consideration of weak separation of finite sets. It is easily seen that, given two finite sets F_1 and F_2, with $F_1 \cap F_2 = 0$, in a T_1-space then $F_1 \vdash F_2$ and $F_2 \vdash F_1$.

Definition 3.2. A space $(X, \mathscr{T})$ will be called a T_F-space iff, given any point x and any finite set F in X, such that $x \notin F$, then either $x \vdash F$ or $F \vdash x$. A space will be called a T_{FF}-space iff, given two arbitrary finite sets F_1 and F_2 in X, with $F_1 \cap F_2 = 0$, then either $F_1 \vdash F_2$ or $F_2 \vdash F_1$.

Equivalent forms for the two axioms are given in the following theorems.

Theorem 3.2. *A space $(X, \mathscr{T})$ is a T_F-space iff either*

(a) *for every x and every set F consisting of at most two points only $x \vdash F$ or $F \vdash x$ holds*

or

(b) *for every $x \in X$, $y \in [x]'$ implies $[y]' = 0$.*

Proof: We first show that (a) implies (b). For a given x let $y \in [x]$, and $z \in [x]'$ and denote $[x, z]$ by F. Then $y \vdash F$ is impossible. Hence $F \vdash y$ must hold. It follows that for every $y \in [x]'$ there exists a closed set $C_{y,z}$ which contains y and does not contain x and z. Hence

$$[y] = \cap [C_{y,z},\ z \in [x]',\ z \neq y] \cap [\bar{x}] \supset [\bar{y}],$$

and $y \in [x]'$ implies $[x]' = 0$. If $[x]'$ contains only one element y then $[x] \vdash [y]$ holds and hence $[\bar{y}] \subset [x]'$, that is $[y]' = 0$.

Now assume that (b) holds and consider x and a finite set

$$F = [y_1, \ldots, y_k;\ z_1, \ldots, z_k]$$

where y_1 to y_k are assumed to be in $[x]'$ and z_1 to z_m in $\sim[\bar{x}]$. By (b) x can be in the closure of another point only if $[x]' = 0$. Hence, unless the set of y is empty, x cannot be in the closure of a z_k, so that

$$(\bigcup_{n=1}^{k} \sim [y_n]) \cup (\bigcup_{s=1}^{m} \sim [\bar{z}_s])$$

is an open set (the sets $[y_n) = [\overline{y_n}]$ by (b)) containing x and not F. If there are no y then $\sim[\bar{x}]$ is an open set containing F and not x. To complete the proof of the theorem it suffices to point out that the original definition of a T_F-space clearly implies condition (a).

By analogous arguments one establishes the result below.

Theorem 3.3. *A space $(X, \mathscr{T})$ is a T_{FF} space iff one of the following requirements is satisfied.*
(a) *Given any two sets F_1 and F_2, in X which both consist of at most two points and which are such that $F_1 \cap F_2 = 0$ then either $F_1 \vdash F_2$ or $F_2 \vdash F_1$.*
(b) *either of the following cases holds:*
(i) $[x]' = 0$ *for all but at most one* $x \in X$,
(ii) $[x] = 0$ *for all but at most one* $x \in X$.

Condition (b) of Theorem 3.2 can be rephrased as follows: if x has a non empty derived set it is not in the derived set of another point, which is another way of saying that its shell is empty. This in turn is equivalent to the statement

$$N_S \cup N_D = X.$$

Assume that X is a T_F space and that $y \in [\widehat{x}]$. Then $x \in [y]'$. If $[\widehat{y}]$ is not empty, then y is in the derived set of some point, but then $[y]'$ is empty, which is a contradiction. Hence $y \in [\widehat{x}]$ implies $[\widehat{y}] = 0$. In the above argument one can replace shell by derived set and derived set by shell and thus conclude that if $y \in [\widehat{x}]$ implies $[\widehat{y}] = 0$ we have a T_F-space.

The condition $[x]' \cap [\widehat{y}] = 0$ for all $x \neq y$ also characterizes a T_F-space. For, if $z \in [x]' \cap [\widehat{y}]$ in a T_F-space then $[z]' = 0$ so that z cannot be in the shell of another point. This contradicts the assumption $z \in [\widehat{y}]$. Hence in a T_F $[x]' \cap [y] = 0$. If this condition holds then any point which is in the derived set of one point cannot be in the shell of another, that is the first point must have an empty derived set and the space is a T_F-space. We have now established the result below.

Theorem 3.4. *A space $+X, \mathscr{T})$ is a T_F-space iff one of the conditions:*
(a) $N_S \cup N_D = X$,
(a)′ *for all* $x, y \in X$ $\quad [\hat{x}] \cap [\hat{y}]$ *is degenerate or* $[\bar{x}] \cap [\bar{y}]$ *is degenerate*
(b) *for every* $x \in X$ $\quad y \in [\widehat{x}]$ *implies* $[\widehat{y}] = 0$,
(c) *for all* $x, y \in X, x \neq y$ $\quad [x]' \cap [\widehat{y}] = 0$,
holds.

Condition (b) of Theorem 3.3 is easily seen to be equivalent to the requirement given below.

Theorem 3.5. *A space is a T_{FF}-space iff either*

$$N_S = X \sim [a] \text{ or } N_D = X \sim [b].$$

Finally, the inclusion relation

$$T_1 \subset T_{FF} \subset T_F \subset T_0$$

is self evident.

A third group of separation axioms is formed by Youngs' axiom and a slightly stronger version.

Definition 3.3. A space $(X, \mathscr{T})$ is called a T_Y-space iff for all $x, y \in X, x \neq y$ $[\bar{x}] \cap [\bar{y}]$ is degenerate. A space is called a T_{YS}-space iff for all $x, y \in X, x \neq y$, $[\bar{x}] \cap [\bar{y}]$ is either 0 or $[x]$ or $[y]$.

Equivalent formulations of these two axioms are stated in the next two theorems.

Theorem 3.6. *A space* $(X, \mathscr{T})$ *is a* T_Y*-space iff one of the following conditions is satisfied.*

(a) *The space is* T_F *and for all* $x, y \in X, x \neq y$ $[x]' \cap [y]'$ *is degenerate,*

(b) *the space is* T_F *and for all* $x, y \in X, x \neq y$ $[\hat{x}] \cap [\hat{y}]$ *is degenerate,*

(c) *for all* $x, y \in X, x \neq y$ $[\hat{x}] \cap [\hat{y}]$ *is degenerate.*

Proof: Let $y \in [\bar{x}]$ in a T_Y-space then $[\bar{y}] \cap [\bar{x}] = [y]$ so that $[y]' = 0$. Hence every T_Y-space (and also every T_{YS}) is a T_F-space. That $[x]' \cap [y]'$ is degenerate in a T_Y-space is obvious. If condition (a) is satisfied then either $[x]' \cap [y]' = 0$ in which case the space is T_Y since x and y cannot both be in $[\bar{x}] \cap [\bar{y}]$; or $[x]' \cap [y]' = [z]$ then $[\bar{x}] \cap [\bar{y}] = [z]$. Assume it also contained x then $x \in [y]'$ and hence (since the space is T_F) $[\bar{x}] = [x]$ so that $[\bar{x}] \cap [\bar{y}] = [x]$ which is a contradiction since $z \neq x$. This proves the equivalence of condition (a).

We now turn to condition (c). If the space is T_Y then the assumption $z, w \in [\hat{x}] \cap [\hat{y}]$ implies $x, y \in [\bar{z}] \cap [\bar{w}]$ which is a contradiction. Hence in a T_Y-space condition (c) is satisfied. If (c) is satisfied one can turn around the argument above to conclude that the space is T_Y. Finally one shows the equivalence of (b) and (c) by an argument analogous to the one used to establish (a).

Theorem 3.7. *A space* $(X, \mathscr{T})$ *is a* T_{YS}*-space iff one of the following conditions holds:*

(a) *the space is* T_F *and for all* $x, y \in X, x \neq y$ $[x]' \cap [y]' = 0$,

(b) *the derived sets of any two distinct points are separated,*

(c) *the closure of the derived sets of any two distinct points are disjoint.*

Proof: That condition (a) is equivalent to the defining condition for a T_{YS}-space is established by an argument very similar to the one given for condition (a) of Theorem 3.6. Now in a T_{YS}-space either one of the points has an empty derived set or $[\bar{x}] \cap [\bar{y}] = 0$ hence (b) and (c) are satisfied. Clearly (c) implies (b) so it suffices to prove that (b) implies T_{YS}. If $[x]'$ is closed $([\bar{x}]') \cap [y]' = 0$ implies $[x]' \cap [\bar{y}] = 0$ or $[y]$. If it is $[y]$ then $[y]$ is closed and $[\bar{x}] \cap [\bar{y}] = [y]$. If $[x]' \cap [\bar{y}] = 0$ then $[\bar{x}] \cap [\bar{y}] = 0$ or $[x]$. If $[x]'$ is not closed then its closure is $[\bar{x}]$ so that we have $[\bar{x}] \cap [y]' = 0$ but then $[\bar{x}] \cap [\bar{y}]$ is at most $[y]$ so that in each case the space is T_{YS}.

It is of interest to note that the axioms T_F, T_{FF} and T_Y are self dual under the exchange "shell" for "derived set of point" and "kernel" for "point closure". That T_{YS} does not have this property follows from the example below.

Example 3.1. Let X be the set of real numbers. In addition to X and the null set let all $[x]$, $x \neq 0$, and their unions be open.

An analysis of the axioms introduced above shows that most of them can be built up of certain basic requirements expressed in terms of the behaviour of derived sets and shells of points. These basic axioms are the following:

O : for every $x \in X$ $[x]'$ is the union of closed sets,
Δ : for every $x \in X$ $[x]'$ is the union of disjoint closed sets,
Γ : for every $x \in X$ $[x]'$ is closed,
α : for every $x \in X$ $[x]'$ consists of points y such $[y]'=0$,
β : for every $x \in X$ $[x]'$ is degenerate,
γ : for every $x \in X$ $[x]'$ is the union of disjoint point closures,
δ : for every $x \in X$ $[x]'$ is a point closure,
ε : for all $x, y \in X, x \neq y$ $[x]' \cap [y]'$ is degenerate.

For the last five conditions dual statements in terms of kernels and shells can also be made, they are listed below. For the first three statements no meaningful dual statements can be given since in general the kernel of a point is neither open nor closed.

α' : for every $x \in X$ $[\widehat{x}]$ consists of points y such that $[\widehat{y}]=0$,
β' : for every $x \in X$ $[\widehat{x}]$ is degenerate,
γ' : for every $x \in X$ $[\widehat{x}]$ is the union of disjoint kernels,
δ' : for every $x \in X$ $[\widehat{x}]$ is a kernel,
ε' : for all $x, y \in X, x \neq y$ $[\widehat{x}] \cap [\widehat{y}]$ is degenerate.

Recall (Theorem 2.1 (e)) that "$[\widehat{x}]$ is degenerate for every $x \in X$" is equivalent to "for all $x, y \in X, x \neq y$ $[x]' \cap [y]'=0$". Similarly "$[x]'$ is degenerate" is equivalent to "$[\widehat{x}] \cap [\widehat{y}]=0$". If by $T(\ )$ we denote a space that satisfies the requirements listed in the parentheses, we arrive at the following relations.

$$\begin{aligned}
T_D &= T(\Gamma),\\
T_F &= T(\alpha)=T(\alpha)',\\
T_Y &= T(\alpha, \varepsilon)=T(\alpha', \varepsilon'),\\
T_{UD} &= T(\Delta),\\
T_{DD} &= T(\Gamma, \beta'),\\
T_{YS} &= T(\alpha, \beta').
\end{aligned}$$

T_{PF} cannot be stated solely in terms of these conditions. Other combinations of these basic conditions lead to further new separation axioms. Those involving only β, β', ε, ε' may not be T_0. All the remaining combinations are clearly between T_0 and T_1. Note that different combinations do not always lead to different axioms.

The conditions γ and δ and their duals, which we had not encountered before, were included, because in the spaces satisfying them the partial ordering (see [8, p. 83]) that can be defined by $x \subset y$ iff $x \in [y]'$ has a very interesting discrete structure. The axioms γ and γ' can also be thought of as weaker forms of T_F.

4. Ordering of the axioms. On the basis of the results of the preceding section the following inclusion relations between the separation axioms we have discussed are evident.

$$T_0 \supset T_{UD} \supset T_D \supset T_{DD} \supset T_1,$$
$$T_{UD} \supset T_F \supset T_Y \supset T_{FF} \supset T_1,$$
$$T_Y \supset T_{YS} \supset T_{DD} \supset T_1.$$

We might also mention the following additional inclusion relations.

$$T_{UD} \supset T(\gamma) \supset T_F,$$
$$T(\gamma) \supset T(\delta),$$
$$T_D \supset T(\delta) \supset T(\alpha, \beta).$$

Next we construct a number of examples to show that, at least our main axioms, are distinct. In all examples X is the set of real numbers and it is understood that the null set and the set X are closed.

Example 4.1. Let the closed sets be all $[x]$ for $x \neq 0$ and all finite unions of these sets.

This space is T_{UD}, T_F, T_Y, and T_{YS} but not T_D (and not T_{DD}). Hence T_{UD} and T_D are distinct as are T_{YS} and T_{DD}.

Example 4.2. Let every set containing $x=0$ as an element be closed.

This is a T_{FF}, T_D, and T_Y-space but not a T_{YS} space. We thus see that T_Y is distinct from T_{YS} and that T_{FF} is distinct from T_{YS}.

Example 4.3. Let the sets $[x, -x]$ and $[x]$, for $x>0$, as well as their finite unions be closed.

This space is T_{DD}, T_Y, T_{YS} but not T_{FF}. It follows that T_{FF} is distinct from T_{DD} and T_Y.

Example 4.4. Let the sets C_a and D_a, defined respectively by $x \geqslant a$ and $x>a$, be closed.

This is an example of a space which is T_{UD} and T_D but not T_F and T_{DD}. Thus T_F is distinct from T_{UD} and T_D and T_D is distinct from T_{DD}.

Example 4.5. Let only the sets C_a of the previous example be closed.

This is a T_0-space which is not a T_{UD}-space.

Example 4.6. Let A be the set of all reals which are not integers. Let the closed sets of the space be all finite subsets of A and all sets of the form $A \cup N$, where N is a finite set of integers.

This space is T_F and T_D but not T_Y.

Since none of the examples given are T_1-spaces, T_1 is distinct from all of them. The following chart shows the ordering relation between our main separation axioms.

$$\begin{tikzcd}
T_1 \arrow[rr] \arrow[dd] & & T_{DD} \arrow[rr] \arrow[d] & & T_D \arrow[dd] & \\
& & T_{YS} \arrow[d] & & & \\
T_{FF} \arrow[rr] & & T_Y \arrow[r] & T_F \arrow[r] & T_{UD} \arrow[r] & T_0
\end{tikzcd}$$

We saw in Example 4.1 that there exist T_Y-spaces which are not T_D-spaces. Example 4.4 shows that there are also T_D-spaces which are not T_F and hence surely not T_Y so that no inclusion relation can exist between T_D and T_Y. In Example 4.2 we encountered a T_{FF}-space which is not a T_{DD}-space. In Example 4.3 we have a T_{DD}-space which is not a T_{FF}. Hence no inclusion relation can exist between this pair of axioms. All other conceivable inclusion relations, other than the ones already established, are also excluded. It may also be of interest to note that $T_Y \cap T_D \neq T_{DD}$ and that $T_{FF} \cap T_{DD} \neq T_1$. This is shown in the examples given below. However, it is easily seen that $T_{YS} \cap T_D = T_{DD}$.

Example 4.7. With A as defined in Example 4.6, let $[x]$ be ciosed for all $x \in A$, in addition let the sets $n = {}^1/_2 \leqslant x \leqslant n + {}^1/_2 = N_n$ and $N_n - [n]$ be closed for all integers n. Finally let all finite unions of sets of these types be closed.

This space is T_D and T_Y but not T_{DD}.

Example 4.8. Let all sets not containing $x=0$ be closed.

This space is T_{FF} and T_{DD} but not T_1.

It is clear from these examples that another source of new separation axioms between T_0 and T_1 is provided by certain unions or intersections of axioms already defined.

5. Further properties of T_D-spaces. The theorem of YANG mentioned in the introduction can now be restated as follows.

Theorem 5.1. *A space $(X, \mathscr{I})$ is a T_D-space iff for all sets $A \subset X$ A' is closed.*

Proof: If every A' is closed then $[\bar{x}]'$ is closed for all $x \in X$ and the space is T_D. Now assume that the space is a T_D-space. Let x be a limit point of A', then every N_x and in particular $(\sim[x]') \cap N_x$ contains points of A' other than x. Let $y \neq x$ be in A' and in $(\sim[x]') \cap N_x$ then $(\sim[\bar{x}]) \cap N_x$ is a neighborhood of y which is contained in N_x. Since $y \in A'$ this neighborhood contains points of A other than y, say z. Clearly $z \neq x$. Hence every N_x contains points of A (z in particular) other than x it follows that $x \in A'$ and A' is closed.

A space is called a *discrete space of Alexandroff* iff it is a T_0-space and for every $A \subset X$

$$\bar{A} = \cup [[\bar{x}] : x \in A].$$

Theorem 5.2. *Every discrete space of Alexandroff is a T_D-space.*

Proof: In a T_0-space $y \in [x]'$ implies $[\bar{y}] \subset [x]'$ (Theorem 2.3(b)). Hence in a discrete space of Alexandroff

$$[x]' = \cup\ [[y] : y \in [x]'] = ([\bar{x}]').$$

Corollary 5.1. *Every finite T_0-space is a T_D-space.*

That not all T_D-spaces are discrete spaces of Alexandroff is shown by Example 4.3. However this space could have been made into such a space by extending the family of closed sets to include arbitrary unions of point closures.

It is also of interest to note that there exist homogeneous T_D-spaces which are not T_1-spaces. A space is called *homogeneous* [5, p. 53] iff given two points x and y of the space there exists a homeomorphism of the space onto itself carrying x into y. Example 4.4 describes a homogeneous T_D-space which is not a T_1-space.

We conclude this section by stating a result, the proof of which will appear elsewhere.

Theorem 5.3. *If the spaces $(X, \mathscr{T})$ and $(Y, \mathscr{U})$ are lattice equivalent and are both T_D-spaces then they are homeomorphic.*

6. Further properties of T_F-spaces. Contrary to T_D-spaces T_F-spaces differ but little from T_1-spaces. This is so in at least two ways.

Theorem 6.1. *Every homogeneous T_F-space is a T_1-space.*

Proof: Clearly a point with a non-empty derived set cannot correspond under a homeomorphism to a point with an empty derived set. Every T_F-space contains points of the second kind, so in order to be homogeneous it cannot contain any points of the first kind, that is it must be a T_1-space.

Theorem 6.2. *If $(X, \mathscr{T})$ is a T_F-space then $(X, \mathscr{T}')$, where $\mathscr{T}'$ is the topology generated by $\mathscr{T} \cup [N_D]$ as a subbase, is a T_1-space.*

Proof: Assume $[x]' \neq 0$ in $(X, \mathscr{T})$. Let $y \in [x]'$ then $y \in N_D$. Hence in $(X, \mathscr{T}')$ $y \in [x]'$. It follows that $[x]' = 0$ in $(X, \mathscr{T}')$.

7. Miscellaneous observations. It is known that the property of being a T_0 or a T_1-space is preserved under a strengthening of the topology. The same is true for our major new separation axioms. In most cases this follows from the original definitions of the axioms. For the case of the T_D axiom it becomes clear if one considers the equivalent form given in Theorem 3.1. To what extent this invariance under strengthening of the topology should be considered as essential for a separation axiom is open to question. T_2 satisfies the requirement but T_3 and T_4 (see [8, pp. 80, 87]) do not. The requirement does not seem to be essential even for axioms between T_0 and T_1. The axiom $T(\delta)$ does not satisfy this condition as the following example shows.

Example 7.1. Let X be the set of natural numbers. Let the closed sets of $(X, \mathscr{T})$ be all sets of the form $x \geqslant n$. Then $(X, \mathscr{T})$ is a $T(\delta)$-space. Now add as new closed sets the sets of even numbers $\geqslant 2n$. In the new space the derived set of $2n-1$ is the set $x \geqslant 2n$ which is not a point closure.

A T_3-space is usually defined as a regular T_1-space. It is known (see

[3, p. 109]) that it can also be characterized as a regular T_0-space. This is immediately clear if one considers condition (e) of Theorem 2.3 as the definition of a T_0-space. The question has been raised whether there exists a separation axiom T_α weaker than T_1 such that a normal T_α-space is T_4. None of the separation axioms introduced here can serve as such a T_α.

The best result we can prove along this line is the following.

Theorem 7.1. *Every normal T_{UD}-space is a T_D-space, and every normal T_F-space is a T_Y and a $T_{(\alpha,\beta)}$-space. A completely normal T_F-space is a T_{YS} and T_{DD}-space.*

Proof: The second assertion easily follows from the first. To prove the first assertion we observe that in a normal space $[x]'$ cannot contain two disjoint closed sets. A completely normal T_F-space is a T_Y-space. One shows that $[\bar{x}] \cap [\bar{y}] = z \neq x, y$ is impossible if the space is to be completely normal. It follows that the space is a T_{YS}-space and since it is also a T_D-space it is a T_{DD}-space.

Note that there are normal T_D-spaces which are not T_1 and hence not regular. It suffices that the space have no disjoint closed sets. Such a space is described in Example 4.4.

We conclude with a final remark about the T_D-axiom which shows that, in a certain sense, this axiom is an asymmetric form of the T_1-axiom.

A topological space is a T_D-space (T_1-space) iff given any two points x and y of the space there exists a closed set $A \subset X$ such that $x \in A$ and $(A \sim [x]) \cup [y]$ is closed or (and) there exists a closed set B, with $y \in B$ such that $(B \sim [y]) \cup [x]$ is closed.

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