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The Ramsey numbers for a cycle of length six or seven versus a clique of order seven

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Abstract

For two given graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ is the smallest integer n such that for any graph G of order n, either G contains G_1 or the complement of G contains G_2. Let C_m denote a cycle of length m and K_n a complete graph of order n. It was conjectured that $R(C_m, K_n) = (m-1)(n-1)+1$ for $m \geqslant n \geqslant 3$ and $(m, n) \neq (3, 3)$. We show that $R(C_6, K_7) = 31$ and $R(C_7, K_7) = 37$, and the latter result confirms the conjecture in the case when $m = n = 7$.

Keywords: Ramsey number; Cycle; Complete graph

1. Introduction

All graphs considered in this paper are finite simple graphs without loops. For two given graphs G_1 and G_2, the *Ramsey number* $R(G_1, G_2)$ is the smallest integer n such that for any graph G of order n, either G contains G_1 or $\overline{G}$ contains G_2, where $\overline{G}$ is the complement of G. The *neighborhood* $N(v)$ of a vertex v is the set of vertices adjacent to v in G and $N[v] = N(v) \cup \{v\}$. The *minimum degree* of G is denoted by $\delta(G)$. Let $V_1, V_2 \subseteq V(G)$. We use $E(V_1, V_2)$ to denote the set of the edges between V_1 and V_2. The *independence number* of a graph G is denoted by $\alpha(G)$. For $U \subseteq V(G)$, we write $\alpha(U)$ for $\alpha(G[U])$, where $G[U]$ is the subgraph induced by U in G. A cycle and a path of order n are denoted by C_n and P_n, respectively. A *clique* or *complete graph* of order n is denoted by K_n. We use mK_n to denote the union of m vertex disjoint K_n's. For two vertex disjoint graphs G and H, $G + H$ denote the graph with its vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv | u \in V(G) \text{ and } v \in V(H)\}$. A *wheel* of order $n+1$ is $W_n = K_1 + C_n$ and W_n^- is a graph obtained from W_n by deleting a spoke from W_n. A *fan* $F_n = K_1 + nK_2$ is a graph of order $2n+1$ and a *book* $B_n = K_2 + \overline{K_n}$ is a graph of order $n+2$. For notations not defined here, we follow [2].

For the Ramsey number $R(C_m, K_n)$, it has been determined for the cases $n \leqslant 6$; $m = 3$ and $7 \leqslant n \leqslant 9$; $m = 4$ and $n = 7, 8$; $m = 5$ and $n = 7$; and some other cases such as $n \geqslant 4m+2$, and so on. For details, see the dynamic survey [8]. In 1978, Erdös et al. [4] posed the following.

Conjecture 1 (*Erdös et al. [4]*). $R(C_m, K_n) = (m-1)(n-1)+1$ for $m \geqslant n \geqslant 3$ and $(m, n) \neq (3, 3)$.

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The conjecture was confirmed for $n = 3$ in early works on Ramsey theory [5,9]. Yang et al. [11] proved the conjecture for $n = 4$.

Theorem 1 (*Yang et al. [11]*). $R(C_m, K_4) = 3m - 2$ *for* $m \geqslant 4$.

Bollobás et al. [1] showed that the conjecture is true for $n = 5$.

Theorem 2 (*Bollobás et al. [1]*). $R(C_m, K_5) = 4m - 3$ *for* $m \geqslant 5$.

Recently, Schiermeyer [10] confirmed the conjecture for $n = 6$.

Theorem 3 (*Schiermeyer [10]*). $R(C_m, K_6) = 5m - 4$ *for* $m \geqslant 6$.

Until now, the conjecture is still open. Researchers are interested in determining all the values of the Ramsey number $R(C_m, K_7)$. In this paper our main purpose is to determine the values of $R(C_m, K_7)$ when $m = 6, 7$, which is our first step towards calculating the values of $R(C_m, K_7)$ for all m. The main results of this paper are as follows.

Theorem 4. $R(C_6, K_7) = 31$.

Theorem 5. $R(C_7, K_7) = 37$.

Obviously, Theorem 5 confirms Conjecture 1 for the case when $m = n = 7$.

2. Some lemmas

In order to prove Theorems 4 and 5, we need the following lemmas.

Lemma 1 (*Graver and Yackel [6] and Kalbfleisch [7]*). $R(K_3, K_7) = 23$.

Lemma 2 (*Dirac [3]*). *Let G be a graph of order n. If $\delta(G) \geqslant n/2$, then G is hamiltonian.*

The following lemma can be deduced from the known Ramsey numbers, see [8].

Lemma 3. $R(B_2, K_7) \leqslant 34$.

Lemma 4. *Let G be a graph of order $6n - 5$ $(n \geqslant 6)$ with $\alpha(G) \leqslant 6$. If G contains no C_n, then $\delta(G) \geqslant n - 1$.*

Proof. If there is some vertex v such that $d(v) \leqslant n - 2$, then $G' = G - N[v]$ is a graph of order at least $5n - 4$. By Theorem 2, $\alpha(G') \geqslant 6$. Thus, an independent set of order at least 6 in G' together with v form an independent set of order at least 7 in G, which contradicts $\alpha(G) \leqslant 6$. □

Lemma 5. *Let G be a graph of order $6n - 5$ $(n \geqslant 6)$ with $\alpha(G) \leqslant 6$. If G contains no C_n, then G contains no W_{n-2}.*

Proof. Suppose to the contrary that G contains a $W_{n-2} = \{w_0\} + C$, where $C = w_1 \cdots w_{n-2}$ is a cycle of length $n - 2$. Set $U = V(G) - V(W_{n-2})$. By Lemma 4, $\delta(G) \geqslant n - 1$. Thus, we have $N_U(w_i) \neq \emptyset$ for $0 \leqslant i \leqslant n - 2$. Let $v_i \in N_U(w_i)$ and $V_i = N_U[v_i]$, where $0 \leqslant i \leqslant n - 2$. Since G contains no C_n, we have

$$N(V_i) \cap V(W_{n-2}) = \{w_i\} \quad \text{for } 0 \leqslant i \leqslant n - 2, \tag{1}$$

$$V_i \cap V_j = \emptyset \quad \text{for } 0 \leqslant i < j \leqslant n - 2, \tag{2}$$

and

$$E(V_0, V_i) = \emptyset \quad \text{for } 1 \leqslant i \leqslant n - 2. \tag{3}$$

By (1), we have $d_{W_{n-2}}(v_i) = 1$, which implies $|V_i| \geqslant n-1$ for $0 \leqslant i \leqslant n-2$ since $\delta(G) \geqslant n-1$. By (2), we have $n(n-1) \leqslant |V(W_{n-2}) \cup \left(\bigcup_{i=0}^{n-2}\right)| \leqslant 6n-5$, which implies $n \leqslant 6$, and hence $n=6$. In this case, $|G|=31$. Thus, by (2), we have $5 \leqslant |V_i| \leqslant 6$ for $0 \leqslant i \leqslant 4$. If there is some V_i such that $|V_i|=6$, then $V(G)=V(W_4) \cup \left(\bigcup_{i=0}^{4} V_i\right)$. By (1) and (3), we have $N(V_0) \subseteq V_0 \cup \{w_0\}$. If $|V_0|=6$, then since $\delta(G) \geqslant 5$, we have $\delta(G[V_0]) \geqslant 4$. By Lemma 2, $G[V_0]$ contains a C_6, a contradiction. If $|V_0|=5$, then $G[V_0 \cup \{w_0\}] = K_6$ since $\delta(G) \geqslant 5$, a contradiction again. If $|V_i|=5$ for $0 \leqslant i \leqslant 4$, then $V(G)-(V(W_4) \cup \left(\bigcup_{i=0}^{4} V_i\right))$ contains exactly one vertex, say y. By (1) and (3), we have $N(V_0) \subseteq V_0 \cup \{w_0, y\}$. Noting that $\delta(G) \geqslant 5$, we have $d_{V_0}(w_0) \geqslant 3$ or $d_{V_0}(y) \geqslant 3$, which implies that either $G' = G[V_0 \cup \{w_0\}]$ or $G'' = G[V_0 \cup \{y\}]$ is a graph of order 6 with a minimum degree of at least 3. By Lemma 2, either G' or G'' contains a C_6, again a contradiction. □

3. Proof of theorems

Proof of Theorem 4. Let G be a graph of order 31. Suppose to the contrary that neither G contains a C_6 nor $\overline{G}$ contains a K_7. By Lemma 4, we have $\delta(G) \geqslant 5$.

Before starting to prove Theorem 4, we first show the following claims.

Claim 1.1. *G contains no K_4.*

Proof. Suppose to the contrary that G contains a K_4 with vertex set $\{v_1, v_2, v_3, v_4\}$ and $U = V(G) - \{v_1, v_2, v_3, v_4\}$. Set $N_U(v_i) = U_i$ for $1 \leqslant i \leqslant 4$. Since $\delta(G) \geqslant 5$, we have $|U_i| \geqslant 2$ for $1 \leqslant i \leqslant 4$. Let $u_i \in U_i$ and $V_i = N_U(u_i)$ for $1 \leqslant i \leqslant 4$.

If $U_i \cap U_j = \emptyset$ for $1 \leqslant i < j \leqslant 4$, then since $\delta(G) \geqslant 5$, we have $|V_i| \geqslant 4$ for $1 \leqslant i \leqslant 4$. By Lemma 5, $G[V_i]$ contains no C_4, which implies $\alpha(V_i) \geqslant 2$. On the other hand, since G contains no C_6, we have $V_i \cap V_j = \emptyset$ and $E(V_i, V_j) = \emptyset$ for $1 \leqslant i < j \leqslant 4$. Thus, we have $\alpha\left(\bigcup_{i=1}^{4} V_i\right) \geqslant 8$, a contradiction.

If there are some U_i and U_j with $i \neq j$ such that $U_i \cap U_j \neq \emptyset$, we assume without of loss of generality that $U_3 \cap U_4 \neq \emptyset$. Let $U_0 = U_3 \cap U_4$ and $U_i' = U_i - U_0$ for $i = 3, 4$. By Lemma 5, $U_0 \cap (U_1 \cup U_2) = \emptyset$. Thus, noting that G contains no C_6, we have $U_i \cap U_j = \emptyset$ for $i = 1, 2$ and all $j \neq i$. This implies that $|V_i| \geqslant 4$ for $i = 1, 2$. By Lemma 5, we have $\alpha(V_i) \geqslant 2$ for $i = 1, 2$. If $|U_0| \geqslant 2$, we assume without loss of generality that $u_3, u_4 \in U_0$. In this case, we have $E\left(\{v_3\}, \bigcup_{i=1}^{4} V_i\right) = \emptyset$, $V_i \cap V_j = \emptyset$ and $E(V_i, V_j) = \emptyset$ for $1 \leqslant i < j \leqslant 4$ for otherwise G contains a C_6. Thus, we have $\alpha(\{v_3\} \cup \left(\bigcup_{i=1}^{4} V_i)\right) \geqslant 7$, a contradiction. If $|U_0| = 1$, we assume $U_0 = \{u_0\}$. Since $|U_i| \geqslant 2$ for $1 \leqslant i \leqslant 4$, we may assume $u_i \in U_i'$ for $i = 3, 4$. Let $V_0 = \{u_0, u_3, u_4\}$. Since G contains no C_6, we see that V_0 is an independent set, $V_i \cap V_j = \emptyset$ and $E(V_i, V_j) = \emptyset$ for $0 \leqslant i < j \leqslant 2$, which implies that $\alpha\left(\bigcup_{i=0}^{2} V_i\right) \geqslant 7$, again a contradiction. □

Claim 1.2. *G contains no $K_1 + P_4$.*

Proof. Suppose G contains $K_1 + P_4$,say,$P = v_1v_2v_3v_4$ is a path and $V(P) \subseteq N(v_0)$. Set $U = V(G) - \{v_i | 0 \leqslant i \leqslant 4\}$ and $U_i = N_U(v_i)$ for $1 \leqslant i \leqslant 4$. By Lemma 5,$v_1v_4 \neq E(G)$. By Claim 1.1,$v_1v_3, v_2v_4 \neq E(G)$. Thus,noting that $\delta(G) \geqslant 5$,we have $|U_i| \geqslant 3$ for $i = 1, 4$ and $|U_i| \geqslant 2$ for $i = 2, 3$. Since G contains no C_6,we have $U_i \cap U_j = \emptyset$ and $E(U_i, U_j) = \emptyset$ for $1 \leqslant i < j \leqslant 4$. By Claim 1.1,$\alpha(U_i) \geqslant 2$ for $i = 1, 4$. If $\alpha(U_2) \geqslant 2$ or $\alpha(U_3) \geqslant 2$,then we have $\alpha\left(\bigcup_{i=1}^{4} U_i\right) \geqslant 7$,a contradiction. If$\alpha(U_2) = \alpha(U_3) = 1$, then by Claim 1.1, we have $G[U_2] = G[U_3] = K_2$. In this case, we have $E\left(\{v_0\}, \bigcup_{i=1}^{4} U_i\right) = \emptyset$ for otherwise G contains a C_6. This implies that $\alpha(\{v_0\} \cup \left(\bigcup_{i=1}^{4} U_i)\right) \geqslant 7$, again a contradiction. □

Claim 1.3. *G contains no B_3.*

Proof. Assume that G contains a B_3, say, $v_1v_2 \in E(G)$ and $v_3, v_4, v_5 \in N(v_1) \cap N(v_2)$. Set $U = V(G) - \{v_i | 1 \leqslant i \leqslant 5\}$ and $U_i = N_U(v_i)$ for $3 \leqslant i \leqslant 5$. By Claim 1.1, $v_iv_j \neq E(G)$ for $3 \leqslant i < j \leqslant 5$. Thus, noting that $\delta(G) \geqslant 5$, we have $|U_i| \geqslant 3$

for $3 \leqslant i \leqslant 5$. Since G contains no C_6, we have $U_i \cap U_j = \emptyset$ and $E(U_i, U_j) = \emptyset$ for $3 \leqslant i < j \leqslant 5$. By Claim 1.1, we have $\alpha(U_i) \geqslant 2$ for $3 \leqslant i \leqslant 5$. By Claim 1.2, we have $E\left(\{v_1\}, \bigcup_{i=3}^5 U_i\right) = \emptyset$. Thus we obtain that $\alpha(\{v_1\} \cup \left(\bigcup_{i=3}^5 U_i\right)) \geqslant 7$, a contradiction.

Claim 1.4. *G contains no* W_4^-.

Proof. Suppose that G contains a W_4^-, say, $W_4^- = \{v_5\} + C - \{v_1v_5\}$, where $C = v_1v_2v_3v_4$ is a cycle. Set $U = V(G) - \{v_i | 1 \leqslant i \leqslant 5\}$ and $U_i = N_U(v_i)$ for $1 \leqslant i \leqslant 5$. Since G contains no C_6, we have $U_1 \cap \left(\bigcup_{i=2}^5 U_i\right) = \emptyset$. By Claims 1.2 and 1.3, we see that U_3, U_4, U_5 are pairwise disjoint and U_2, U_3, U_5 are pairwise disjoint. Thus, we have $U_4 \cap (U_1 \cup U_3 \cup U_5) = \emptyset$ and $U_i \cap \left(\bigcup_{1 \leqslant j \leqslant 5 \text{ and } j \neq i} U_j\right) = \emptyset$ for $i = 3, 5$. Let $u_i \in U_i$ for $i = 3, 4, 5$. Set $V_3 = N_U(u_3) - \{u_5\}$, $V_4 = N_U(u_4)$ and $V_5 = N_U(u_5) - \{u_3\}$. Since $\delta(G) \geqslant 5$, by the arguments above, we have $|V_i| \geqslant 3$ for $i = 3, 4, 5$. By Claim 1.1, $\alpha(V_i) \geqslant 2$ for $3 \leqslant i \leqslant 5$. Note that G contains no C_6, we see that $E\left(\{v_1\}, \bigcup_{i=3}^5 V_i\right) = \emptyset$, $V_i \cap V_j = \emptyset$ and $E(V_i, V_j) = \emptyset$ for $3 \leqslant i < j \leqslant 5$. This implies that $\alpha(\{v_1\} \cup \left(\bigcup_{i=3}^5 V_i\right)) \geqslant 7$, a contradiction. □

Claim 1.5. *G contains no* B_2.

Proof. Suppose G contains a B_2, say $v_1v_2v_3v_4$ is a cycle with diagonal v_2v_4. Set $U = V(G) - \{v_1, v_2, v_3, v_4\}$ and $N_U(v_i) = U_i$ for $1 \leqslant i \leqslant 4$. By Claim 1.2, $U_1 \cap U_2 = U_2 \cap U_3 = U_3 \cap U_4 = U_4 \cap U_1 = \emptyset$. By Claim 1.3, $U_2 \cap U_4 = \emptyset$. By Claim 1.4, $U_1 \cap U_3 = \emptyset$. Thus, we have $U_i \cap U_j = \emptyset$ for $1 \leqslant i < j \leqslant 4$. Let $u_i \in U_i$ for $i = 2, 4$, $V_2 = N_U(u_2) - \{u_4\}$ and $V_4 = N_U(u_4) - \{u_2\}$. Noting that $\delta(G) \geqslant 5$, we have $|U_i| \geqslant 3$ for $i = 1, 3$ and $|V_i| \geqslant 3$ for $i = 2, 4$. Since G contains no C_6, it is easy to check that U_1, V_2, U_3, V_4 are pairwise disjoint and there is no edge between any two of them. By Claim 1.1, we have $\alpha(U_i) \geqslant 2$ for $i = 1, 3$ and $\alpha(V_i) \geqslant 2$ for $i = 2, 4$. Thus, we obtain that $\alpha(U_1 \cup V_2 \cup U_3 \cup V_4) \geqslant 8$, a contradiction. □

Claim 1.6. *G contains no* F_2.

Proof. Suppose that G contains an F_2, say, $v_0v_1v_2$ and $v_0v_3v_4$ are two triangles with v_0 in common. Let $U = V(G) - \{v_i | 0 \leqslant i \leqslant 4\}$ and $U_i = N_U(v_i)$ for $0 \leqslant i \leqslant 4$. By Claim 1.2, we have $E(\{v_1, v_2\}, \{v_3, v_4\}) = \emptyset$, which implies that $|U_i| \geqslant 3$ for $1 \leqslant i \leqslant 4$ since $\delta(G) \geqslant 5$. By Claim 1.1, $\alpha(U_i) \geqslant 2$ for $1 \leqslant i \leqslant 4$. By Claim 1.5, $U_1 \cap U_2 = U_3 \cap U_4 = \emptyset$ and $U_0 \cap U_i = \emptyset$ for $1 \leqslant i \leqslant 4$. Since G contains no C_6, we see that $(U_1 \cup U_2) \cap (U_3 \cup U_4) = \emptyset$ and $E(U_1 \cup U_2, U_3 \cup U_4) = \emptyset$. If $E(U_1, U_2)$ or $E(U_3, U_4)$ contains a $2K_2$, then G contains a C_6, a contradiction. Thus, noting that $\alpha(U_i) \geqslant 2$ for $1 \leqslant i \leqslant 4$, we have $\alpha(U_1 \cup U_2) \geqslant 3$ and $\alpha(U_3 \cup U_4) \geqslant 3$, and hence $\alpha\left(\bigcup_{i=1}^4 U_i\right) \geqslant 6$. By Claim 1.5, we get that $\alpha(\{v_0\} \cup \left(\bigcup_{i=1}^4 U_i\right)) \geqslant 7$, again a contradiction. □

We now begin to prove Theorem 4.

By Lemma 1, G contains a triangle $v_1v_2v_3$. Let $U = V(G) - \{v_1, v_2, v_3\}$ and $U_i = N_U(v_i)$ for $1 \leqslant i \leqslant 3$. Since $\delta(G) \geqslant 5$, we have $|U_i| \geqslant 3$ for $1 \leqslant i \leqslant 3$. By Claim 1.5, $U_i \cap U_j = \emptyset$ for $1 \leqslant i < j \leqslant 3$. By Claim 1.6, U_i is an independent set for $1 \leqslant i \leqslant 3$. If $E(U_i, U_j) = \emptyset$ for $1 \leqslant i < j \leqslant 3$, then $\alpha(\bigcup_{i=1}^3 U_i) \geqslant 9$, a contradiction. Hence, we may assume without loss of generality that $v_4 \in U_2$, $v_5 \in U_3$ and $v_4v_5 \in E(G)$. Let $X = \{v_i | 1 \leqslant i \leqslant 5\}$, $Y = V(G) - X$ and $Y_i = N_Y(v_i)$ for $1 \leqslant i \leqslant 5$. By Claim 1.5, we have $v_1v_4, v_1v_5, v_2v_5, v_3v_4 \neq E(G)$, which implies that $|Y_i| \geqslant 3$ for $i = 1, 4, 5$. By Claim 1.1, $\alpha(Y_i) \geqslant 2$ for $i = 4, 5$. By Claim 1.6, $\alpha(Y_1) \geqslant 3$. Since G contains no C_6, it is easy to obtain that $Y_i \cap Y_j = \emptyset$ and $E(Y_i, Y_j) = \emptyset$ for $i, j \in \{1, 4, 5\}$ and $i \neq j$. Thus, we have $\alpha(Y_1 \cup Y_4 \cup Y_5) \geqslant 7$, again a contradiction.

Up to now, we have shown that $R(C_6, K_7) \leqslant 31$. On the other hand, since $6K_5$ contains no C_6 and its complement contains no K_7, we have $R(C_6, K_7) \geqslant 31$, and hence $R(C_6, K_7) = 31$. □

Proof of Theorem 5. Let G be a graph of order 37. Suppose to the contrary that neither G contains a C_7 nor $\overline{G}$ contains a K_7. By Lemma 4, we have $\delta(G) \geqslant 6$.

In order to prove Theorem 5, we need the following claims.

Claim 2.1. *G contains no* $K_1 + P_5$.

Proof. Suppose that G contains $K_1 + P_5$, say, $P = v_1 \cdots v_5$ and $V(P) \subseteq N(v_0)$. Let $U = V(G) - \{v_i | 0 \leqslant i \leqslant 5\}$ and $N_U(v_i) = U_i$ for $0 \leqslant i \leqslant 5$. Because of $\delta(G) \geqslant 6$, we have $U_i \neq \emptyset$ for $0 \leqslant i \leqslant 5$.

If $U_2 \cap U_4 \neq \emptyset$, then we let $v_6 \in U_2 \cap U_4$, $X = \{v_i | 0 \leqslant i \leqslant 6\}$ and $Y = V(G) - X$. Set $Y_i = N_Y(v_i)$, $z_i \in Y_i$ and $Z_i = N_Y(z_i)$ for $0 \leqslant i \leqslant 6$. Since G contains no C_7, it is easy to check that $Y_i \cap Y_j = \emptyset$ for $i = 1, 5, 6$ and $j \neq i$, and $E(Y_i, Y_j) = \emptyset$ for $i, j \in \{1, 5, 6\}$ and $i \neq j$, which implies that $|Z_i| \geqslant 5$ for $i = 1, 5, 6$. For the same reason, we have $E(\{v_0\}, Z_1 \cup Z_5 \cup Z_6) = \emptyset$, $Z_i \cap Z_j = \emptyset$ and $E(Z_i, Z_j) = \emptyset$ for $i, j \in \{1, 5, 6\}$ and $i \neq j$. By Lemma 5, $\alpha(Z_i) \geqslant 2$ for $i = 1, 5, 6$. Thus, we have $\alpha(\{v_0\} \cup Z_1 \cup Z_5 \cup Z_6) \geqslant 7$, a contradiction. Hence, we have $U_2 \cap U_4 = \emptyset$.

Noting that $U_2 \cap U_4 = \emptyset$ and G contains no C_7, it is easy to check that $U_i \cap U_j = \emptyset$ and $E(U_i, U_j) = \emptyset$ for $1 \leqslant i < j \leqslant 5$. Let $u_i \in U_i$ and $V_i = N_U(u_i)$ for $i = 1, 5$, then we have $|V_i| \geqslant 5$. By Lemma 5, $\alpha(V_i) \geqslant 2$ for $i = 1, 5$. Since G contains no C_7, we have $V_1 \cap V_5 = \emptyset$, $E(V_1, V_5) = \emptyset$, $V_i \cap \left(\bigcup_{i=2}^{4} U_i\right) = \emptyset$ and $E\left(V_i, \bigcup_{i=2}^{4} U_i\right) = \emptyset$ for $i = 1, 5$. This implies that $\alpha(V_1 \cup V_5 \cup \left(\bigcup_{i=2}^{4} U_i\right)) \geqslant 7$, a contradiction. □

Claim 2.2. *G contains no* W_5^-.

Proof. Suppose that G contains a W_5^-, say, $C = v_1 \cdots v_5$ and $W_5^- = \{v_0\} + C - \{v_0 v_1\}$. Let $U = V(G) - \{v_i | 0 \leqslant i \leqslant 5\}$ and $U_i = N_U(v_i)$ for $0 \leqslant i \leqslant 5$. Since $\delta(G) \geqslant 6$, we have $U_i \neq \emptyset$. Noting that G contains no C_7, we have $U_i \cap U_j = \emptyset$ and $E(U_i, U_j) = \emptyset$ for $2 \leqslant i < j \leqslant 4$, and $U_i \cap U_j = \emptyset$ and $E(U_i, U_j) = \emptyset$ for $i = 0, 1$ and all $j \neq i$. Take $u_i \in U_i$ and set $V_i = N_U(u_i)$ for $i = 0, 1$, then since $\delta(G) \geqslant 6$, we have $|V_i| \geqslant 5$ for $i = 0, 1$. By Lemma 5, $\alpha(V_i) \geqslant 2$. Since G contains no C_7, we have $V_0 \cap V_1 = \emptyset$ and $E(V_0, V_1) = \emptyset$. For the same reason, we have $V_i \cap U_j = \emptyset$ and $E(V_i, U_j) = \emptyset$ for $i = 0, 1$ and $j = 2, 3, 4$. Thus, by the arguments above, we have $\alpha(V_0 \cup V_1 \cup \left(\bigcup_{i=2}^{4} U_i\right)) \geqslant 7$, a contradiction. □

Claim 2.3. *G contains no* W_4.

Proof. Suppose that G contains a W_4, say $C = v_1 \cdots v_4$ is a cycle and $V(C) \subseteq N(v_0)$. Let $U = V(G) - \{v_i | 0 \leqslant i \leqslant 4\}$ and set $U_i = N_U(v_i)$ for $0 \leqslant i \leqslant 4$. Obviously, $U_i \neq \emptyset$. By Claim 2.1, $U_0 \cap U_i = \emptyset$ for $1 \leqslant i \leqslant 4$. By Claim 2.2, $U_1 \cap U_2 = U_2 \cap U_3 = U_3 \cap U_4 = U_4 \cap U_1 = \emptyset$. Since G contains no C_7, we have $E(U_i, U_j) = \emptyset$ for $0 \leqslant i < j \leqslant 4$. If $U_1 \cap U_3 \neq \emptyset$, then $U_2 \cap U_4 = \emptyset$ for otherwise there is a C_7 in G. By symmetry, we may assume $U_1 \cap U_3 = \emptyset$. Let $u_i \in U_i$ and $V_i = N_U(u_i)$ for $i = 0, 1, 3$. By the arguments above, we have $|V_i| \geqslant 5$ for $i = 0, 1, 3$. Since G contains no C_7, we see that $E(\{v_2\}, V_0 \cup V_1 \cup V_3) = \emptyset$, V_0, V_1 and V_3 are pairwise disjoint and there is no edge between any two of them. By Lemma 5, we have $\alpha(V_i) \geqslant 2$ for $i = 0, 1, 3$, which implies that $\alpha(\{v_2\} \cup V_0 \cup V_1 \cup V_3) \geqslant 7$, a contradiction. □

Claim 2.4. *G contains no* K_4.

Proof. Suppose that G contains a K_4, say $S = \{v_1, v_2, v_3, v_4\}$ is a clique. Set $U = V(G) - S$ and $U_i = N_U(v_i)$ for $1 \leqslant i \leqslant 4$. Since $\delta(G) \geqslant 6$, we have $|U_i| \geqslant 3$.

If there are U_i and U_j with $i \neq j$ such that $U_i \cap U_j \neq \emptyset$, we assume without loss of generality that $v_5 \in U_3 \cap U_4$. Let $X = S \cup \{v_5\}$, $Y = V(G) - X$ and $Y_i = N_Y(v_i)$ for $1 \leqslant i \leqslant 5$. By Claim 2.1, we have $(Y_3 \cup Y_4) \cap (Y_1 \cup Y_2 \cup Y_5) = \emptyset$. By Claim 2.2, $Y_5 \cap (Y_1 \cup Y_2) = \emptyset$. Since G contains no C_7, we have $E(Y_i, Y_j) = \emptyset$ for $i, j \in \{2, 3, 5\}$ and $i \neq j$. Let $z_i \in Y_i$ and $Z_i = N_Y(z_i)$ for $i = 2, 3, 5$, then by the arguments above, we have $|Z_i| \geqslant 4$ for $i = 2, 3, 5$. By Claim 2.3, $\alpha(Z_i) \geqslant 2$. Noting that G contains no C_7, we see that $E(\{v_1\}, Z_2 \cup Z_3 \cup Z_5) = \emptyset$, $Z_i \cap Z_j = \emptyset$ and $E(Z_i, Z_j) = \emptyset$ for $i, j \in \{2, 3, 5\}$ and $i \neq j$, which implies that $\alpha(\{v_1\} \cup Z_2 \cup Z_3 \cup Z_5) \geqslant 7$, a contradiction. Hence, we have $U_i \cap U_j = \emptyset$ for $1 \leqslant i < j \leqslant 4$.

Take $u_i \in U_i$ for $1 \leqslant i \leqslant 4$. Set $T = \{u_1, u_2, u_3, u_4\}$, $U' = U - T$ and $N_{U'}(u_i) = V_i$ for $1 \leqslant i \leqslant 4$. If $\Delta(G[T]) \geqslant 2$, then G contains a C_7, and hence we may assume $\Delta(G[T]) \leqslant 1$. Thus, noting that $U_i \cap U_j = \emptyset$ for $1 \leqslant i < j \leqslant 4$, we have $|V_i| \geqslant 4$ for $1 \leqslant i \leqslant 4$. By Claim 2.3, $\alpha(V_i) \geqslant 2$. Since G contains no C_7, it is easy to see that $V_i \cap V_j = \emptyset$ and $E(V_i, V_j) = \emptyset$ for $1 \leqslant i < j \leqslant 4$, which implies that $\alpha\left(\bigcup_{i=1}^{4} V_i\right) \geqslant 8$, a contradiction. □

Claim 2.5. *G contains no* $K_1 + P_4$.

Proof. Suppose that G contains $K_1 + P_4$, say $P = v_1v_2v_3v_4$ is a path and $V(P) \subseteq N(v_0)$. Set $S = \{v_i|0 \leqslant i \leqslant 4\}$, $U = V(G) - S$ and $U_i = N_U(v_i)$ for $0 \leqslant i \leqslant 4$.

We first show that $U_1 \cap U_2 = U_3 \cap U_4 = \emptyset$. By symmetry, we need only to show $U_3 \cap U_4 = \emptyset$. If not, we let $v_5 \in U_3 \cap U_4$. Set $X = S \cup \{v_5\}$, $Y = V(G) - X$ and $Y_i = N_Y(v_i)$ for $0 \leqslant i \leqslant 5$. Since G contains no C_7, we have $Y_1 \cap Y_i = \emptyset$ for $i \neq 1$, $Y_2 \cap Y_i = \emptyset$ for $i \neq 0, 2$ and $Y_4 \cap Y_i = \emptyset$ for $i \neq 3, 4$. For the same reason, we have $E(Y_i, Y_j) = \emptyset$ for $i, j \in \{1, 2, 4\}$ and $i \neq j$. Let $z_i \in Y_i$ and $Z_i = N_Y(z_i)$ for $i = 1, 2, 4$. By the arguments above, we have $|Z_1| \geqslant 5$ and $|Z_i| \geqslant 4$ for $i = 2, 4$. Noting that G contains no C_7, we see that Z_1, Z_2 and Z_4 are pairwise disjoint and there is no edge between any of them. By Claims 2.1, 2.3 and 2.4, we have $\alpha(Z_1) \geqslant 3$ and $\alpha(Z_i) \geqslant 2$ for $i = 2, 4$, which implies that $\alpha(Z_1 \cup Z_2 \cup Z_4) \geqslant 7$, a contradiction. Hence, we have $U_1 \cap U_2 = U_3 \cap U_4 = \emptyset$.

Next, we show that $U_1 \cap U_3 = U_2 \cap U_4 = \emptyset$. By symmetry, we need only to show $U_2 \cap U_4 = \emptyset$. If not, we let $v_5 \in U_2 \cap U_4$. Set $X = S \cup \{v_5\}$, $Y = V(G) - X$ and $Y_i = N_Y(v_i)$ for $0 \leqslant i \leqslant 5$. Since G contains no C_7, we have $Y_i \cap Y_j = \emptyset$ for $i = 1, 5$ and all $j \neq i$. Let $z_i \in Y_i$ and $Z_i = N_Y(z_i)$ for $i = 1, 5$, then by the arguments above, we have $|Z_i| \geqslant 5$ for $i = 1, 5$. By Claims 2.1, 2.3 and 2.4, we have $\alpha(Z_i) \geqslant 3$ for $i = 1, 5$. If $Z_1 \cap Z_5 \neq \emptyset$ or $E(Z_1, Z_5) \neq \emptyset$ or $E(\{v_0\}, Z_1 \cup Z_5) \neq \emptyset$, then G contains a C_7, a contradiction. Thus, we have $\alpha(\{v_0\} \cup Z_1 \cup Z_5) \geqslant 7$, a contradiction. Hence, we have$U_1 \cap U_3 = U_2 \cap U_4 = \emptyset$.

By the arguments above, we have $(U_1 \cup U_4) \cap (U_2 \cup U_3) = \emptyset$. By Claim 2.1, $U_0 \cap (U_1 \cup U_4) = \emptyset$. By Claim 2.2, $U_1 \cap U_4 = \emptyset$. Thus, we have $U_i \cap U_j = \emptyset$ for $i = 1, 4$ and all $j \neq i$. Let $u_i \in U_i$ and $V_i = N_U(u_i)$ for $i = 1, 4$, then we have $|V_i| \geqslant 5$ for $i = 1, 4$. By Claims 2.1, 2.3 and 2.4, we have $\alpha(V_i) \geqslant 3$ for $i = 1, 4$. Since G contains no C_7, it is easy to see that $E(\{v_0\}, V_1 \cup V_4) = \emptyset$, $V_1 \cap V_4 = \emptyset$ and $E(V_1, V_4) = \emptyset$. Thus, we have $\alpha(\{v_0\} \cup V_1 \cup V_4) \geqslant 7$, again a contradiction. □

Claim 2.6. *G contains no B_3.*

Proof. Assume that G contains a B_3, say, $v_1v_2 \in E(G)$ and $v_3, v_4, v_5 \in N(v_1) \cap N(v_2)$. Set $U = V(G) - \{v_i|1 \leqslant i \leqslant 5\}$ and $U_i = N_U(v_i)$ for $i = 3, 4, 5$.

We first show that $U_i \cap U_j = \emptyset$ for $3 \leqslant i < j \leqslant 5$. If not, we assume $v_6 \in U_3 \cap U_4$. Set $X = \{v_i|1 \leqslant i \leqslant 6\}$, $Y = V(G) - X$ and $Y_i = N_Y(v_i)$ for $1 \leqslant i \leqslant 6$. Since G contains no C_7, we see that $Y_5 \cap Y_i = \emptyset$ for $i \neq 5$ and $Y_i \cap Y_j = \emptyset$ for $i = 3, 4$ and all $j \neq 3, 4$. By Claim 2.4, $v_iv_j \notin E(G)$ for $3 \leqslant i < j \leqslant 5$, which implies $|Y_i| \geqslant 3$ since $\delta(G) \geqslant 6$. Thus, we can take $z_i \in Y_i$ for $3 \leqslant i \leqslant 5$ such that $z_3 \neq z_4$. Note that G contains no C_7, $z_iz_j \notin E(G)$ for $3 \leqslant i < j \leqslant 5$. Set $Z_i = N_Y(z_i)$ for $3 \leqslant i \leqslant 5$. By the arguments above, we have $|Z_5| \geqslant 5$ and $|Z_i| \geqslant 4$ for $i = 3, 4$. By Claims 2.1, 2.3 and 2.4, we have $\alpha(Z_5) \geqslant 3$ and $\alpha(Z_i) \geqslant 2$ for $i = 3, 4$. Because G contains no C_7, we have $Z_i \cap Z_j = \emptyset$ and $E(Z_i, Z_j) = \emptyset$ for $3 \leqslant i < j \leqslant 5$. Thus we get $\alpha(\bigcup_{i=3}^{5} Z_i) \geqslant 7$, a contradiction. Hence, we have $U_i \cap U_j = \emptyset$ for $3 \leqslant i < j \leqslant 5$.

By Claim 2.4, $v_iv_j \notin E(G)$ for $3 \leqslant i < j \leqslant 5$. Since G contains no C_7, we have $E(U_i, U_j) = \emptyset$ for $3 \leqslant i < j \leqslant 5$. Thus, noting that $\delta(G) \geqslant 6$, we have $|U_i| \geqslant 4$ for $3 \leqslant i \leqslant 5$. By Claim 2.3, $\alpha(U_i) \geqslant 2$ for $3 \leqslant i \leqslant 5$. By Claim 2.5, $E(\{v_1\}, \bigcup_{i=3}^{5} U_i) = \emptyset$. Thus, noting that $U_i \cap U_j = \emptyset$ for $3 \leqslant i < j \leqslant 5$, we have $\alpha(\{v_1\} \cup (\bigcup_{i=3}^{5} U_i)) \geqslant 7$, again a contradiction. □

Claim 2.7. *G contains no W_4^-.*

Proof. Suppose G contains a W_4^-, say, $W_4^- = \{v_5\} + C - \{v_1v_5\}$, where $C = v_1v_2v_3v_4$ is a cycle. Set $S = \{v_i|1 \leqslant i \leqslant 5\}$, $U = V(G) - S$ and $U_i = N_U(v_i)$ for $1 \leqslant i \leqslant 5$.

We first show that $U_1 \cap (U_3 \cup U_5) = \emptyset$. By symmetry, we need only to show that $U_1 \cap U_5 = \emptyset$. If not, we let $v_6 \in U_1 \cap U_5$. Set $X = S \cup \{v_6\}$, $Y = V(G) - X$ and $Y_i = N_Y(v_i)$ for $1 \leqslant i \leqslant 6$. Since G contains no C_7, we have $E(Y_4, Y_6) = \emptyset$ and $Y_i \cap Y_j = \emptyset$ for $i = 4, 6$ and all $j \neq i$. Let $z_i \in Y_i$ and $Z_i = N_Y(z_i)$ for $i = 4, 6$. By the arguments above, we have $|Z_i| \geqslant 5$. By Claims 2.1, 2.3 and 2.4, we have $\alpha(Z_i) \geqslant 3$ for $i = 4, 6$. Because G contains no C_7, we have $Z_4 \cap Z_6 = \emptyset$, $E(Z_4, Z_6) = \emptyset$ and $E(\{v_1\}, Z_4 \cup Z_6) = \emptyset$, which implies that $\alpha(\{v_1\} \cup Z_4 \cup Z_6) \geqslant 7$, a contradiction. Hence, we have$U_1 \cap (U_3 \cup U_5) = \emptyset$.

Next, we show that $U_1 \cap (U_2 \cup U_4) = \emptyset$. By symmetry, we need only to show that $U_1 \cap U_4 = \emptyset$. If not, we let $v_6 \in U_1 \cap U_4$. Set $X = S \cup \{v_6\}$, $Y = V(G) - X$ and $Y_i = N_Y(v_i)$ for $1 \leqslant i \leqslant 6$. Since G contains no C_7, we have $E(Y_3, Y_6) = \emptyset$ and $Y_6 \cap Y_i = \emptyset$ for $i \neq 6$. By Claim 2.5, $Y_3 \cap (Y_2 \cup Y_4) = \emptyset$. By Claim 2.6, $Y_3 \cap Y_5 = \emptyset$. If $Y_3 \cap Y_1 \neq \emptyset$, then G contains a C_7, a contradiction. Thus, we have $Y_3 \cap Y_i = \emptyset$ for $i \neq 3$. Let $z_i \in Y_i$ and $Z_i = N_Y(z_i)$ for $i = 3, 6$, then $|Z_i| \geqslant 5$. By Claims 2.1, 2.3 and 2.4, we have $\alpha(Z_i) \geqslant 3$ for $i = 3, 6$. Note that since G contains no C_7, we have

$Z_3 \cap Z_6 = \emptyset$, $E(Z_3, Z_6) = \emptyset$ and $E(\{v_4\}, Z_3 \cup Z_6) = \emptyset$. Thus, we have $\alpha(\{v_4\} \cup Z_3 \cup Z_6) \geqslant 7$, a contradiction. Hence, we have $U_1 \cap (U_2 \cup U_4) = \emptyset$.

By the arguments above, we have $U_1 \cap U_i = \emptyset$ for $i \neq 1$. By Claim 2.5, $U_3 \cap (U_2 \cup U_4) = \emptyset$. By Claim 2.6, $U_3 \cap U_5 = \emptyset$. Thus we have $U_3 \cap U_i = \emptyset$ for $i \neq 3$. Let $u_i \in U_i$ and $V_i = N_U(u_i)$ for $i = 1, 3$. Then $|V_i| \geqslant 5$. By Claims 2.1, 2.3 and 2.4, we have $\alpha(V_i) \geqslant 3$ for $i = 1, 3$. Note that G contains no C_7, we have $V_1 \cap V_3 = \emptyset$, $E(V_1, V_3) = \emptyset$ and $E(\{v_4\}, V_1 \cup V_3) = \emptyset$. This implies that $\alpha(\{v_4\} \cup V_1 \cup V_3) \geqslant 7$, a contradiction. □

We now begin to prove Theorem 5.

By Lemma 3, G contains a B_2. Let $v_1v_2v_3v_4$ be a cycle with diagonal v_2v_4. Set $U = V(G) - \{v_1, v_2, v_3, v_4\}$ and $U_i = N_U(v_i)$ for $1 \leqslant i \leqslant 4$.

We first show that $E(U_1, U_3) = \emptyset$. Otherwise, we let $v_5 \in U_1$, $v_6 \in U_3$ and $v_5v_6 \in E(G)$. Let $X = \{v_i | 1 \leqslant i \leqslant 6\}$, $Y = V(G) - X$ and $Y_i = N_Y(v_i)$ for $1 \leqslant i \leqslant 6$. Since G contains no C_7, it is easy to see that $Y_i \cap Y_j = \emptyset$ for $i = 2, 4$ and $j \neq i$, and $Y_5 \cap (Y_1 \cup Y_6) = \emptyset$. Thus, let $z_i \in Y_i$ and $Z_i = N_Y(z_i)$ for $i = 2, 5$, we have $|Z_2| \geqslant 5$ and $|Z_5| \geqslant 4$. By Claims 2.1, 2.3 and 2.4, we have $\alpha(Z_2) \geqslant 3$ and $\alpha(Z_5) \geqslant 2$. Noting that G contains no C_7, we see that $E(\{v_1, v_3\}, Z_2 \cup Z_5) = \emptyset$, $Z_2 \cap Z_5 = \emptyset$ and $E(Z_2, Z_5) = \emptyset$. By Claim 2.4, $v_1v_3 \notin E(G)$. Thus, we have $\alpha(\{v_1, v_3\} \cup Z_2 \cup Z_5) \geqslant 7$, a contradiction. Hence, we have $E(U_1, U_3) = \emptyset$.

Next, we show that $E(U_1 \cup U_3, U_2 \cup U_4) = \emptyset$. By symmetry, we need only to show that $E(U_3, U_4) = \emptyset$. If not, we let $v_5 \in U_3$, $v_6 \in U_4$ and $v_5v_6 \in E(G)$. Let $X = \{v_i | 1 \leqslant i \leqslant 6\}$, $Y = V(G) - X$ and $Y_i = N_Y(v_i)$ for $1 \leqslant i \leqslant 6$. Since G contains no C_7, we have $Y_1 \cap Y_i = \emptyset$ for $i \neq 1$, $Y_3 \cap (Y_2 \cup Y_5) = \emptyset$ and $Y_6 \cap (Y_2 \cup Y_4 \cup Y_5) = \emptyset$. By Claim 2.5, $Y_3 \cap Y_4 = \emptyset$. Let $z_i \in Y_i$ and $Z_i = N_Y(z_i)$ for $i = 1, 3, 6$. Since $v_3v_6 \notin E(G)$ by Claim 2.5 and $\delta(G) \geqslant 6$, we have $|Y_i| \geqslant 2$ for $i = 3, 6$. Thus, we may assume $z_3 \neq z_6$. By the arguments above, we have $|Z_1| \geqslant 5$ and $|Z_i| \geqslant 4$ for $i = 3, 6$. By Claims 2.1, 2.3 and 2.4, we have $\alpha(Z_1) \geqslant 3$ and $\alpha(Z_i) \geqslant 2$ for $i = 3, 6$. Noting that G contains no C_7, we see that Z_1, Z_3, Z_6 are pairwise disjoint and there is no edge between any two of them. This implies that $\alpha(Z_1 \cup Z_3 \cup Z_6) \geqslant 7$, and hence we have $E(U_1 \cup U_3, U_2 \cup U_4) = \emptyset$.

By Claims 2.5–2.7, we have $U_i \cap U_j = \emptyset$ for $1 \leqslant i < j \leqslant 4$. Since $\delta(G) \geqslant 6$, we have $|U_i| \geqslant 3$ for $1 \leqslant i \leqslant 3$. By Claim 2.4, we have $\alpha(U_i) \geqslant 2$ for $1 \leqslant i \leqslant 3$. By Claims 2.5 and 2.6, $E\left(\{v_4\}, \bigcup_{i=1}^{3} U_i\right) = \emptyset$. Thus, noting that $E(U_1, U_3) = \emptyset$ and $E(U_2, U_1 \cup U_3) = \emptyset$, we have $\alpha\left(\{v_4\} \cup \bigcup_{i=1}^{3} U_i\right) \geqslant 7$, a contradiction.

By the arguments above, we have $R(C_7, K_7) \leqslant 37$. On the other hand, since $6K_6$ contains no C_7 and its complement contains no K_7, we have $R(C_7, K_7) \geqslant 37$, and hence $R(C_7, K_7) = 37$. □

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