J. Math. Anal. Appl. 351 (2009) 163–169

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On a polynomial inequality

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ARTICLE INFO

Article history:
Received 4 February 2008
Available online 8 October 2008
Submitted by D. Khavinson

Keywords:
Polynomials
Visser's inequality

ABSTRACT

Let $p(z) = a_0 + \cdots + a_n z^n$ and $q(z) = b_0 + \cdots$ be polynomials of degree respectively n and less than n such that

$$|p(z)| < |q(z)|, \quad |z| < 1.$$

A result due to Q.I. Rahman states that

$$|a_0| + |a_n| \leqslant |b_0|.$$

In this paper, we slightly improve the above inequality and discuss several sharpness aspects, including all cases of equality.

1. Introduction and statement of the results

Let $\mathcal{P}_n$ be the linear space of polynomials $p(z) = \sum_{k=0}^{n} a_k(p) z^k$ with complex coefficients of degree at most n endowed with the norm

$$|p|_{\mathbb{D}} = \max_{|z| \leqslant 1} |p(z)|, \quad \mathbb{D} = \{z \mid |z| < 1\}.$$

The inequality

$$|a_0(p)| + |a_n(p)| \leqslant |p|_{\mathbb{D}}, \quad p \in \mathcal{P}_n, \tag{1}$$

was obtained by Visser [7] in 1945 and it is well known that equality holds in (1) only for binomials of the type $p(z) = A + Bz^n$. It was later proved by Van der Corput and Visser [6] that

$$|a_0(p)| + |a_k(p)| \leqslant |p|_{\mathbb{D}}, \quad p \in \mathcal{P}_n, \ \frac{n}{2} < k \leqslant n. \tag{2}$$

We refer the reader to the book of Rahman and Schmeisser [4] for a survey of extensions of (1) and (2). We mention however that (here $[\frac{n}{k}]$ means the integer part of $\frac{n}{k}$)

$$|a_0(p)| + \frac{1}{2} \sec\left(\frac{\pi}{[\frac{n}{k}] + 2}\right) |a_k(p)| \leqslant |p|_{\mathbb{D}}, \quad p \in \mathcal{P}_n, \ 1 \leqslant k \leqslant n, \tag{3}$$

has been obtained in [2] where it is also shown that equality holds in (3) only when the polynomial p is constant when $1 \leqslant k \leqslant \frac{n}{2}$. More recently [1] all cases of equality in (3) with $\frac{n}{2} < k < n$ or equivalently all cases of equality in (2) with $\frac{n}{2} < k < n$ were also obtained (and they are surprisingly numerous!).

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.10.005

Let $\mathcal{P}_n^e$ be the subset of $\mathcal{P}_n$ consisting of the polynomials of exact degree n. Let also $p \in \mathcal{P}_n^e$ and $q \in \mathcal{P}_d$ with $d < n$. Rahman [5] proved that under the assumption

$$\left|p(z)\right| < \left|q(z)\right|, \quad |z| < 1,$$

there holds

$$\left|a_0(p)\right| + \left|a_n(p)\right| \leqslant \left|a_0(q)\right|. \tag{4}$$

This clearly is a striking extension of (1). We remark that (4) can be further extended in the following way: let $0 \leqslant k \leqslant n-1$ and

$$\left|p(z)\right| < \left|q(z)\right|, \quad |z| < 1, \text{ where } p \in \mathcal{P}_n^e \text{ and } q \in \mathcal{P}_{n-k-1}. \tag{5}$$

Then for any real θ, the polynomial $f(z) := p(z) - e^{i\theta}q(z)$ does not vanish in $\mathbb{D}$. An application of Grace's theorem [4, pp. 107–116] to f and $g(z) := 1 + 1/\binom{n}{k}z^{n-k}$ yields

$$\sum_{j=0}^{n-k-1} \left(a_j(p) - e^{i\theta}a_j(q)\right)z^j + a_{n-k}(p)\Big/\binom{n}{k}z^{n-k} \neq 0, \quad |z| < 1,$$

and therefore

$$\left|a_0(p) - e^{i\theta}a_0(q)\right| \geqslant \left|a_{n-k}(p)\right|\Big/\binom{n}{k}, \quad \theta \text{ real},$$

i.e.,

$$\left|a_0(p)\right| + \left|a_{n-k}(p)\right|\Big/\binom{n}{k} \leqslant \left|a_0(q)\right|, \quad 0 \leqslant k \leqslant n-1. \tag{6}$$

Before stating our main results, we introduce the following definition: a pair of polynomials (P, Q) will be called an extremal pair if and only if, given a set of nodes $\{z_j\}_{j=1}^n \subset \partial\mathbb{D}$ and $W(z) = \prod_{j=1}^n (1 - z_j z)$ we have

$$Q(z) = a_0(Q)W(z)\sum_{j=1}^n \frac{\ell_j}{1 - z_j z} \quad \text{and} \quad P(z) = \xi\left(Q(z) - ta_0(Q)W(z)\right)$$

where $0 < t \leqslant 1$, $\xi \in \partial\mathbb{D}$, $a_0(Q)$ is a non-zero complex number and each ℓ_j is non-negative with $\sum_{j=1}^n \ell_j = 1$. We clearly have $P \in \mathcal{P}_n^e$, $Q \in \mathcal{P}_{n-1}$ and simple computations show that

$$\left|P(z)\right| < \left|Q(z)\right|, \quad |z| < 1 \quad \text{and} \quad \left|a_0(P)\right| + \left|a_n(P)\right| = \left|a_0(Q)\right|.$$

We shall prove

Theorem 1.1. *Let $k = 0$ and p, q be polynomials as in* (5)*. Then*

$$\left|a_0(p)\right| + \left|a_n(p)\right| = \left|a_0(q)\right|$$

if and only if the pair (p, q) is an extremal pair.

Theorem 1.2. *Let $0 < k \leqslant n-1$ and p, q be polynomials as in* (5)*. Then the inequality* (6) *is always strict.*

2. Proof of Theorem 1.1

Let $p \in \mathcal{P}_n^e$ and $q \in \mathcal{P}_{n-1}$ such that $|p(z)| < |q(z)|$ in $\mathbb{D}$ and $|a_0(p)| + |a_n(p)| = |a_0(q)|$. Our proof of (6) shows there exists a real number θ such that $p - e^{i\theta}q$ has all of its zeros on the unit circle, i.e.,

$$p(z) - e^{i\theta}q(z) = \left(a_0(p) - e^{i\theta}a_0(q)\right)W(z)$$

with $W(z) := \prod_{j=1}^n (1 - z_j z)$ and $\{z_j\}_{j=1}^n \subset \partial\mathbb{D}$. It then follows that $|a_0(p) - e^{i\theta}a_0(q)| = |a_0(p)| + |a_0(q)|$ and by the triangle inequality

$$0 \leqslant \frac{a_0(p)}{e^{i\theta}a_0(q)} := t \leqslant 1.$$

We therefore have

$$p(z) - e^{i\theta}q(z) = e^{i\theta}a_0(q)(t-1)W(z)$$

and we may in the sequel assume without of any loss of generality that $e^{i\theta} = a_0(q) = 1$. It also follows that $0 \leqslant t < 1$ and

$$\left|q(z) - (1-t)W(z)\right| < \left|q(z)\right|, \quad z \in \mathbb{D},$$

or equivalently

$$\inf_{|z|<1} \operatorname{Re}\left(\frac{q(z)}{W(z)}\right) \geqslant \frac{1-t}{2} > 0. \tag{7}$$

Let us now assume that the zeros of $\frac{W}{q}$ are $\{\bar{z}_{j_\nu}\}_{\nu=1}^{N}$ with $1 \leqslant N \leqslant n$. We claim that any such zero is simple; for if for example $\bar{z}_{j_1}$ is not simple we have

$$\frac{q(z)}{W(z)} = \sum_{k=1}^{M} \frac{L_k}{(1 - z_{j_1} z)^k} + G(z)$$

with $M > 1$, $|L_M| > 0$ and G a function holomorphic in a neighbourhood of $\bar{z}_{j_1}$. This however, together with the fact that no polynomial of degree $M > 1$ can have a bounded below real part in the half-plane $\operatorname{Re}(z) > \frac{1}{2}$, contradicts (7). We therefore have

$$\frac{q(z)}{W(z)} = \sum_{\nu=1}^{N} \frac{\ell_\nu}{1 - z_{j_\nu} z}, \quad |\ell_\nu| > 0,$$

and indeed each ℓ_ν is strictly positive; by letting z in $\mathbb{D}$ tend radially to $\bar{z}_{j_\nu}$ in the identity

$$\operatorname{Re}\left(\frac{\ell_\nu}{1 - z_{j_\nu} z}\right) = \operatorname{Re}(\ell_\nu) \operatorname{Re}\left(\frac{1}{1 - z_{j_\nu} z}\right) - \operatorname{Im}(\ell_\nu) \operatorname{Im}\left(\frac{1}{1 - z_{j_\nu} z}\right)$$

we obtain that $\operatorname{Re}(\ell_\nu) \geqslant 0$. Further, by letting z in $\mathbb{D}$ tend to $\bar{z}_{j_\nu}$ on a horocycle in $\mathbb{D}$ tangent to $\bar{z}_{j_\nu}$, we find out that $\operatorname{Im}(\ell_\nu) = 0$ and the pair (p, q) is extremal.

3. Proof of Theorem 1.2

We first shall give another short proof of (6) in the case where $1 \leqslant k \leqslant n-1$. Let p, q be polynomials for which (5) holds. The polynomial

$$\begin{aligned} r(z) &:= z^n p(1/z) - e^{i\theta} z^n q(1/z) \\ &= \sum_{j=0}^{n} a_{n-j}(p) z^j - e^{i\theta} \sum_{j=n-d}^{n} a_{n-j}(q) z^j \end{aligned}$$

(here d equals the degree $\leqslant n-k-1$ of the polynomial q) has all of its zeros in the closed unit disc and according to the theorem of Gauss–Lucas [4, p. 71] the derivative

$$r^{(k)}(z) = a_{n-k}(p) k! z^k + \cdots + \left(a_0(p) - e^{i\theta} a_0(q)\right) k! \binom{n}{k} z^{n-k}$$

also has this property. It follows readily that (6) is valid. Moreover if the equality

$$\left|a_0(p)\right| + \left|a_{n-k}(p)\right| \Big/ \binom{n}{k} = \left|a_0(q)\right| \tag{8}$$

holds, the polynomial $r^{(k)}$ must have all of its zeros on $\partial\mathbb{D}$ and $a_0(p) = t e^{i\theta} a_0(q)$ where θ is real and $0 \leqslant t < 1$. Because the polynomial r is also known to have its zeros on $\partial\mathbb{D}$, it follows again from the theorem of Gauss–Lucas that

$$r(z) = p(z) - e^{i\theta} q(z) = e^{i\theta} a_0(q)(t-1)\left(1 - e^{i\varphi} z\right)^n$$

with φ real and we may now assume that $e^{i\theta} = e^{i\varphi} = a_0(q) = 1$, i.e.,

$$p(z) = q(z) - (1-t)(1-z)^n$$

and as before we obtain

$$\inf_{|z|<1} \operatorname{Re}\left(\frac{q(z)}{(1-z)^n}\right) \geqslant \frac{1-t}{2} > 0. \tag{9}$$

On the other hand the rational function $\frac{q(z)}{(1-z)^n}$ has a pole of order at least 2 at $z = 1$ and admits an expansion as a finite sum

$$\frac{q(z)}{(1-z)^n} = \sum_{j \geqslant 1} \frac{\ell_j}{(1-z)^j} \quad \text{with} \sum_{j \geqslant 2} |\ell_j| > 0.$$

As in the proof of Theorem 1.1 we obtain that $\inf_{|z|<1} \operatorname{Re}(\frac{q(z)}{(1-z)^n}) = -\infty$, thus contradicting (9). The conclusion follows.

4. On the sharpness of Theorem 1.2

Let $1 < k \leqslant n-1$. We recall that the inequality (6) under the assumption (5) is always strict. Given the fact that the proof of (6) was rather short and simple, this raises questions concerning the sharpness of (6).

We first remark that the hypothesis $q \in \mathcal{P}_{n-k-1}$ in (5) is necessary; let us choose a positive but small enough number c such that $\mathrm{Re}(\frac{1}{(1-cz)^k}) > \frac{1}{2}$ if $z \in \mathbb{D}$. Let also

$$q(z) := (1-cz)^{n-k} \quad \text{and} \quad p(z) := (1-cz)^{n-k} - t(1-cz)^n$$

where $0 < t < 1$. It is readily checked that $p \in \mathcal{P}_n^e$, $q \in \mathcal{P}_{n-k}^e$ and $|p(z)| < |q(z)|$ if $|z| < 1$. However

$$\left|a_0(p)\right| + \left|a_{n-k}(p)\right| \Big/ \binom{n}{k} = 1 - t + c^{n-k}\left|1 - t\binom{n}{k}\right|$$

and

$$\lim_{t \to 0^+} \left|a_0(p)\right| + \left|a_{n-k}(p)\right| \Big/ \binom{n}{k} = 1 + c^{n-k} > \left|a_0(q)\right| = 1,$$

i.e., the inequality (6) may not be valid when the degree of q is greater than $n-k-1$.

Next we claim that under the assumption (5) there cannot exist an absolute constant $c = c(k) > 1$, dependent on k but independent of n such that

$$\left|a_0(p)\right| + c|a_{n-k}| \Big/ \binom{n}{k} \leqslant \left|a_0(q)\right|. \tag{10}$$

We only discuss the case $k = 2$ and consider an extremal pair (p, q) with $p \in \mathcal{P}_n^e$, $q \in \mathcal{P}_{n-3}$ and

$$q(z) = \left(\sum_{j=1}^{n} \frac{\ell_j}{1 - \xi_j z}\right) W(z), \qquad p(z) = q(z) - tW(z) \quad \text{with } W(z) = \prod_{j=1}^{n} (1 - \xi_j z).$$

Computations show that for $0 \leqslant \nu < n$

$$a_{n-\nu}(W) = (-1)^{n-\nu} \left(\prod_{j=1}^{n} \xi_j\right) \sum_{1 \leqslant j_1 < j_2 < \cdots < j_\nu \leqslant n} \bar{\xi}_{j_1} \bar{\xi}_{j_2} \cdots \bar{\xi}_{j_\nu},$$

$$a_{n-\nu}(q) = (-1)^{n-\ell} \left(\prod_{j=1}^{n} \xi_j\right) \sum_{1 \leqslant j_1 < j_2 < \cdots < j_\nu \leqslant n} (\ell_{j_1} + \ell_{j_2} + \cdots + \ell_{j_\nu}) \bar{\xi}_{j_1} \bar{\xi}_{j_2} \cdots \bar{\xi}_{j_\nu}$$

while $a_0(q) = a_0(W) = 1$. The constraint $q \in \mathcal{P}_{n-3}$ means that

$$\sum_{1 \leqslant j_1 \leqslant n} \ell_{j_1} \xi_{j_1} = \sum_{1 \leqslant j_1 < j_2 \leqslant n} (\ell_{j_1} + \ell_{j_2}) \xi_{j_1} \xi_{j_2} = 0$$

and this amounts to

$$\sum_{j=1}^{n} \ell_j \xi_j = \sum_{j=1}^{n} \ell_j \xi_j^2 = 0. \tag{11}$$

We choose a set of distinct points $\{e^{i\theta_j}\}_{j=4}^n \subset \partial\mathbb{D}$ closed under conjugation and such that

$$0 \leqslant |\theta_j| < \varepsilon \quad \text{and} \quad \cos(2\theta_j) > 0, \quad j = 4, \ldots, n-3,$$

where $\varepsilon > 0$ is given in advance. Clearly the origin of the complex plane belongs to the convex hull of $\{-1\} \cup \{e^{i\theta_j}\}_{j=4}^n$ and there exists a convex combination

$$L_1(-1) + \sum_{j=4}^{n} L_j e^{i\theta_j} = 0$$

where we may assume that all coefficients L_j are positive, their sum equals unity and $L_{j_1} = L_{j_2}$ if $e^{i\theta_{j_1}} = e^{-i\theta_{j_2}}$. Clearly for any $a \geqslant 0$ we have

$$\frac{L_1}{1+2a}(-1) + \frac{a}{1+2a}(i) + \frac{a}{1+2a}(-i) + \sum_{j=4}^{n} \frac{L_j}{1+2a} e^{i\theta_j} = 0.$$

We set

$$G(a) := \frac{L_1}{1+2a}(-1)^2 + \frac{a}{1+2a}(i)^2 + \frac{a}{1+2a}(-i)^2 + \sum_{j=4}^{n} \frac{L_j}{1+2a}\big(e^{i\theta_j}\big)^2$$
$$= \frac{2a}{1+2a}(-1) + \frac{L_1}{1+2a}(1) + \sum_{j=4}^{n} \frac{L_j}{1+2a}\big(e^{i\theta_j}\big)^2.$$

Then $G(0) = L_1 + \sum_{j=4}^{n} L_j \cos(2\theta_j) > 0$ and $G(\infty) = -1 < 0$; we may therefore choose $a > 0$ such that $G(a) = 0$. We set

$$\ell_1 = \frac{L_1}{1+2a}, \qquad \ell_2 = \ell_3 = \frac{a}{1+2a}, \qquad \ell_j = \frac{L_j}{1+2a}, \quad 4 \leqslant j \leqslant n,$$
$$\xi_1 = -1, \qquad \xi_2 = i, \qquad \xi_3 = -i, \qquad \xi_j = e^{i\theta_j}, \quad 4 \leqslant j \leqslant n.$$

Clearly the parameters $\{\ell_j\}_{j=1}^{n}$ and $\{\xi_j\}_{j=1}^{n}$ satisfy (11) and the extremal pair defined above satisfies $p \in \mathcal{P}_n^e$, $q \in \mathcal{P}_{n-3}$ and if (10) is valid

$$\big|a_0(p)\big| + c\big|a_{n-2}(p)\big| \Big/ \binom{n}{2} = (1-t) + tc\Bigg|\sum_{1\leqslant j_1<j_2\leqslant n} \xi_{j_1}\xi_{j_2}\Bigg|$$

where

$$\sum_{1\leqslant j_1<j_2\leqslant n} \xi_{j_1}\xi_{j_2} = -\sum_{j=4}^{n} e^{i\theta_j} + 1 + \sum_{4\leqslant j_1<j_2\leqslant n} e^{i\theta_{j_1}} e^{i\theta_{j_2}}.$$

By choosing $\varepsilon = \frac{1}{n}$, we see that this last expression equals

$$\binom{n}{2} - n + O(1), \quad n \to \infty,$$

and therefore our claim follows since

$$\lim_{n\to\infty} \big|a_0(p)\big| + c\big|a_{n-2}(p)\big| \Big/ \binom{n}{2} = \lim_{n\to\infty} (1-t) + tc\frac{\binom{n}{2} - n + O(1)}{\binom{n}{2}}$$
$$= 1 - t + tc$$
$$> 1.$$

Let us finally remark that in the construction above

$$\lim_{n\to\infty} \big|a_0(p)\big| + \big|a_{n-2}(p)\big| \Big/ \binom{n}{2} = 1.$$

This shows that in general no inequality of the type

$$\big|a_0(p)\big| + \big|a_{n-k}(p)\big| \Big/ \binom{n}{k} \leqslant d_k\big|a_0(q)\big|$$

shall hold under the hypothesis (5) for a constant $d_k < 1$ independent of n.

5. An application of Laguerre's theorem

We shall obtain a Bernstein type inequality for pairs of polynomial (p, q) which satisfy

$$\big|p(z)\big| < \big|q(z)\big|, \quad |z| < 1 \text{ and } p, q \in \mathcal{P}_n. \tag{12}$$

For such polynomials

$$p(z) - e^{i\theta}q(z) \neq 0, \quad z \in \mathbb{D}, \ \theta \text{ real},$$

and the classical Laguerre's theorem [4, p. 98] yields at once

$$\big|np(z) + (\xi - z)p'(z)\big| \leqslant \big|nq(z) + (\xi - z)p'(z)\big|, \quad |\xi|, |z| \leqslant 1. \tag{13}$$

It is also a consequence of Laguerre's theorem (see [3] for a recent proof and discussion of all cases of equality) that for any $P \in \mathcal{P}_n$

$$\big|P'(z)\big| + \big|zP'(z) - nP(z)\big| \leqslant n|P|_{\mathbb{D}}, \quad |z| \leqslant 1. \tag{14}$$

Let then p, q satisfy (12). It follows from (13) and (14) that $|\xi|, |z| \leqslant 1$,

$$\begin{aligned}\left|p'(z)\right| - \left|np(z) - zp'(z)\right| &\leqslant \left|np(z) + (\xi - z)p'(z)\right| \\ &\leqslant \left|nq(z) + (\xi - z)q'(z)\right|\end{aligned}$$

and

$$\begin{aligned}\left|p'(z)\right| &\leqslant \left|np(z) - zp'(z)\right| + \left|nq(z) + (\xi - z)q'(z)\right| \\ &\leqslant n|p|_{\mathbb{D}} - \left|p'(z)\right| + \left|nq(z) + (\xi - z)q'(z)\right|.\end{aligned} \tag{15}$$

Because ξ in (15) is arbitrary, we obtain

$$\left|p'(z)\right| \leqslant \frac{n}{2}|p|_{\mathbb{D}} + \frac{||nq(z) - zq'(z)| - |q'(z)||}{2}, \quad |z| = 1. \tag{16}$$

By the hypothesis (12) the polynomial q does not vanish in $\mathbb{D}$ and

$$q(z) = a_0(q)\prod_{j=1}^{n}(1 - z_j z) \quad \text{with } |z_j| \leqslant 1,\ j = 1, \dots, n.$$

Then

$$\operatorname{Re}\left(\frac{zq'(z)}{q(z)}\right) = \sum_{j=1}^{n}\operatorname{Re}\left(\frac{-z_j z}{1 - z_j z}\right) \leqslant \frac{n}{2}, \quad |z| \leqslant 1,$$

and

$$\frac{|q'(z)|}{|q(z)|} = \left|\frac{zq'(z)}{q(z)}\right| \leqslant \left|\frac{zq'(z)}{q(z)} - n\right|, \quad |z| = 1.$$

It now follows from (14) and (16) that

$$\begin{aligned}\left|p'(z)\right| &\leqslant \frac{n}{2}|p|_{\mathbb{D}} + \frac{|nq(z) - zq'(z)| - |q'(z)|}{2} \\ &\leqslant \frac{n}{2}|p|_{\mathbb{D}} + \frac{n}{2}|q|_{\mathbb{D}} - \left|q'(z)\right|\end{aligned}$$

i.e.,

$$\left|p'(z)\right| + \left|q'(z)\right| \leqslant \frac{n}{2}\big(|p|_{\mathbb{D}} + |q|_{\mathbb{D}}\big), \quad |z| \leqslant 1. \tag{17}$$

The cases of equality in (17) are most likely numerous and we only mention

- $p(z) \equiv \xi z^n$ and $q(z) \equiv 1$ with $|\xi| = 1$: this is the standard Bernstein inequality!
- $p(z) \equiv 0$ and $q \in \mathcal{P}_n^e$ has all of its zeros on $\partial\mathbb{D}$: it is well known that in that case $|q'|_{\mathbb{D}} = \frac{n}{2}|q|_{\mathbb{D}}$.
- The extremal pairs (p, q) where $p(z) = z^k + z^n$, $q(z) = 1 + z^{n-k}$, $1 \leqslant k \leqslant n-1$ also transform (17) into an inequality.

We shall end this paper by a further application of Laguerre's theorem to our ideas. Let $1 \leqslant k \leqslant n-1$ and p, q polynomials which satisfy (5). Then

$$\left|p(z) + \frac{(\xi - z)}{n}p'(z)\right| < \left|q(z) + \frac{(\xi - z)}{n}q'(z)\right|, \quad \xi, z \in \mathbb{D},$$

and we may now apply (6) to

$$P(z) := p(z) + \frac{\xi - z}{n}p'(z) = \sum_{j=0}^{n-1}\left(\left(1 - \frac{j}{n}\right)a_j(p) + \frac{\xi}{n}(j+1)a_{j+1}(p)\right)z^j$$

and

$$Q(z) := q(z) + \frac{\xi - z}{n}q'(z).$$

Note that the hypotheses (5) imply that $P \in \mathcal{P}_{n-1}^e$ for all $\xi \in \mathbb{D}$ (except for a finite numbers of those!) and $Q \in \mathcal{P}_{n-k-1} = \mathcal{P}_{(n-1)-(k-1)-1}$. We obtain

$$\left|a_0(p) + \frac{\xi}{n}a_1(p)\right| + \left|a_{n-k}(p)\Big/\binom{n}{k} + \frac{\xi}{n}a_{n-k+1}\Big/\binom{n}{k-1}\right| \leqslant \left|a_0(q) + \frac{\xi}{n}a_1(q)\right|$$

for all ξ, z with $|\xi|, |z| \leqslant 1$. This may be the best explanation concerning the fact that the inequality (6) is always strict when $1 \leqslant k \leqslant n-1$.

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