Stochastic Calculus of Variations on complex line bundle and construction of unitarizing measures for the Poincaré disk

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Abstract

Holomorphic representation of Lie algebra can be realized through Kählerian symplectic formalism; underlying holomorphic convexity requires then the introduction of elliptic operators with complex coefficients. We construct the Stochastic Calculus of Variations for those elliptic operators; remote past vanishing of projections of the underlying process implies convergence in law; then limit laws lead to the unitarizing measure of the given representation; this general approach is developed in full details on Poincaré disk.

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1. Transfer principle, projected transfer principle

In this paper we develop a new appearance of the following general mathematical principle: remote past vanishing implies ergodicity. This principle has already given rise to theorems in the context of Zakai filtering \[5\]. In \[4\] the same principle gives rise to a general criterium, confinement by curvature positivity, for the existence of an invariant measure of a diffusion. Let us recall below shortly this last result.

Let \( M \) be a Riemannian manifold; let \( \Delta \) its Laplace–Beltrami operator; let \( \rho \) be a fixed 1-differential form on \( M \): consider the elliptic operator defined by

\[
\mathcal{L}_0 f := \frac{1}{2} \Delta f - (\rho \mid df).
\]

Let \( x \) the Brownian motion driving the diffusion associated to \( \mathcal{L}_0 \): set \( \Phi_{x,t} \) the stochastic flow associated to \( \mathcal{L}_0 \); set \( m_0 \) be a well-chosen point of \( M \) such that \( \rho_0 = 0 \); in the normal chart at 0 the \( \mathcal{L}_0 \)-process is asymptotic to an Euclidean Brownian motion \( y \). Consider the semi-martingale \( \zeta_{x,t}(\epsilon) = \Phi_{x,t}(y(\epsilon)) \); by Itô Calculus its stochastic differential is given, for \( \epsilon \) small enough, by

\[
d\zeta_{x,t} = \Phi'_{x,t}(m_0)(dy) + \frac{1}{2} \Phi''_{x,t}(m_0)(dy,dy).
\]

We have made this calculus in the normal chart at the end point \( \Phi_{x,t}(m_0) \): then define

\[
Q(t,m) := E_{\Phi_{x,t}(m_0)=m}\Phi'_{x,t}(m_0),
\]

\[
q(t,m) := \frac{1}{2}\text{trace}\left(E_{\Phi_{x,t}(m_0)=m}\Phi''_{x,t}(m_0)(\ast, \ast)\right).
\]

Consider the process driven by the following Itô SDE:

\[
dl_{y,t} = Q(t,l_{y,t})dy(t) + q(t,l_{y,t})dt.
\]

Then in \[4\] the following results are proved.

1.2. Theorem (Transfer from the initial to the final value). Set \( \mu_t \), the heat measure which is the law of \( \Phi_{x,t}(m_0) \); start the process (1.1) at time \( t_0 \) taking for starting measure \( \mu_{t_0} \); then the law of \( l_{t,y} \) is equal to \( \mu_t \), \( t > t_0 \).

1.3. Theorem (Remote past vanishing). Assume that \( Q_t \), \( q_t \) decay exponentially in time, then \( \lim_{t \to \infty} \mu_t \) exists \( =: \mu_{\infty} \) and is an invariant measure for \( m_{x,t} \).

1.4. Projected transfer

Suppose that a map \( H : M \mapsto R^d \) is given; set \( \tilde{\mu}_t \) the law of \( H \circ \Phi_{x,t} \) that is the image measure
\( \tilde{\mu}_t := H_n(\mu_t) \). Set \( \tilde{\Phi}_{x,t} = H \circ \Phi_{x,t} \). Using Itô Calculus compute the stochastic differential of \( \tilde{\zeta}_{x,t} := H(\zeta_{x,t}) \):

\[
d\tilde{\zeta}_{x,t} = \tilde{\Phi}'_{x,t}(m_0)(dy) + \frac{1}{2} \tilde{\Phi}''_{x,t}(m_0)(dy,dy). \tag{1.4}_a
\]

Then define

\[
\tilde{Q}(t,h) := \mathbb{E}_{\tilde{\Phi}_{x,t}(m_0) = h}(\tilde{\Phi}'_{x,t}(m_0)),
\]

\[
\tilde{q}(t,h) := \frac{1}{2} \text{trace}(\mathbb{E}_{\tilde{\Phi}_{x,t}(m_0) = h}(\tilde{\Phi}''_{x,t}(m_0)(\ast,\ast))). \tag{1.4}_b
\]

Consider the process driven by the following Itô SDE:

\[
d\tilde{\gamma}_{y,t} = \tilde{Q}(t,\tilde{\gamma}_{y,t})dy(t) + \tilde{q}(\tilde{\gamma}_{y,t},t)dt. \tag{1.4}_c
\]

Then

1.5. **Theorem (Projected transfer).** Start the process (1.4)_c at time \( t_0 \) taking for starting measure \( \tilde{\mu}_{t_0} \); then the law of \( \tilde{\gamma}_{y,t} \) is equal to \( \tilde{\mu}_t \), \( t > t_0 \).

Assume that \( \tilde{Q}_t, \tilde{q}_t \) decay exponentially in time, then the random variable \( (H \circ \Phi_{x,t})(m_0) \) converges in law when \( t \to +\infty \).

**Proof.** By Itô Calculus the image \( \xi_{y,t} := H(\tilde{\gamma}_{y,t}) \) is a semi-martingale

\[
d\xi_{y,t} = \alpha dy + \beta dt, \quad \text{where} \quad \mathbb{E}^{H(\tilde{\gamma}_{y,t}) = h}(\xi_{y,t}) = \tilde{Q}(t,h), \quad \mathbb{E}^{H(\tilde{\gamma}_{y,t}) = h}(\xi_{y,t}) = \tilde{q}(t,h). \tag{1.5}_a
\]

following [1], to \( \phi \) any \( C^2 \)-function defined on the range of \( H \) associate the semi-martingale

\[
F(y,t) = (\phi \circ H)(\tilde{\gamma}_{y,t}).
\]

Then, similarly to Theorem 1.3. the following limit exists

\[
\lim_{t \to +\infty} \mathbb{E}(F(y,t)) = \lim_{t \to +\infty} \int \phi(h)\tilde{\mu}_t(dh). \quad \Box
\]

2. **Stochastic Calculus of Variations on complex line bundle**

In the geometry of Kähler manifolds the de Rham cohomology must be replaced by the Hodge cohomology; the corresponding Weitzenböck formula involves derivation along complex vector fields. Probabilistic representation of the corresponding semi-group, together with its associated Stochastic Calculus of Variations, is the object of this section.
2.1. Complex derivatives on a Kähler manifold

Following [7], we shall set well-known facts into the framework of orthonormal frame bundle. Denote by \( M \) a Kähler manifold, that is a Riemannian manifold of dimension \( 2n \) such that there exists an atlas where change of local coordinates are given by holomorphic diffeomorphisms and such that in each local chart the Riemannian metric can be expressed as the \((1, 1)\) Hessian of a real function.

An orthonormal frame \( \{e_q\} \) of \( M \) is compatible with the complex structure \( J \) if \( J(e_{2s}) = -e_{2s-1} \). Denote by \( U(M) \) the set of all orthonormal frames which are compatible with the complex structure; set \( p \) the natural projection \( p : U(M) \mapsto M \). Then \( U(M) \) is a principal bundle having for structural group the \( n \)-dimensional unitary group. Set \( O(M) \) the bundle of all orthonormal frames; then \( U(M) \subset O(M) \).

Given \( r_0 \in U(M) \) denote by \( A_k(r_0), k \in [1, 2n] \), the canonical horizontal vector fields of \( O(M) \) at the points \( r_0 \) then \( A_k(r_0) \) are tangent at the submanifold \( U(n) \subset O(M) \); therefore by parallel transport a frame which is compatible with the complex structure stays compatible with the complex structure: the Levi-Civita parallel transport preserves \( U(M) \).

Introduce on \( U(M) \) the following differential operators with complex coefficients:

\[
\partial_q = \frac{1}{2} (\partial A_{2q-1} - i \partial A_{2q}), \quad \bar{\partial}_q = \frac{1}{2} (\partial A_{2q-1} + i \partial A_{2q}), \quad q \in [1, n].
\]  

The horizontal Laplacian is defined as

\[
\Delta = \sum_q \partial_q \bar{\partial}_q + \bar{\partial}_q \partial_q \simeq 2 \sum_q \partial_q \bar{\partial}_q,
\]

where the sign \( \simeq \) means that the two operators take the same values on functions \( F \) of the form \( F = f \circ p \) (basic functions): in fact \( \{\partial_q, \bar{\partial}_q\} \) are vertical vector fields.

If we suppose that \( M \) is compact we have a natural volume measure \( dr \) of finite total mass on \( U(M) \). Given a real function \( K \) denote \( L^2_K \) the Hilbert space of complex-valued functions which are square integrable for the measure \( \exp(-K) dr \):

\[
(h_1 \mid h_2)_{L^2_K} = \int_{U(M)} h_1 \bar{h}_2 \exp(-K) dr.
\]

The adjoint in \( L^2_K \) of \( \partial_q \) is \( \partial_q^* = -\bar{\partial}_q + \bar{\partial} K \).

The operator

\[
\mathcal{L} := \sum_{q=1}^{n} \partial_q \bar{\partial}_q - \bar{\partial}_q K \times \partial_q \text{ is autoadjoint in } L^2_K.
\]

This results from the equality
\begin{equation}
(\mathcal{L} f_1 | f_2)_{L^2_K} = \sum_{q=1}^{n} (\bar{\partial}_q - \partial_q K)(\partial_q f_1) f_2 \exp(-K) dr
= - \sum_{q=1}^{n} (\partial_q f_1 | \partial_q f_2)_{L^2_K}.
\end{equation}

If \( F = f \circ p \) is basic then \( \mathcal{L} f \) is basic and \( \exp(t \mathcal{L}) \) defines a semi-group on \( M \).

In fact the expression \( \sum_q \bar{\partial}_q K \times \partial_q \) is invariant under the action of an unitary operator on \( U(M) \).

Consider \( x_s, s \in [1, 2n] \), independent \( R \)-valued Brownian motions; then the horizontal lift to \( U(M) \) of the Brownian motion on the Riemannian manifold \( M \) is given by the following Stratonovitch SDE:

\begin{equation}
drx(t) = \sum_{s=1}^{2n} A_s(rx(t)) \circ dx_s(t).
\end{equation}

**Theorem.** Assume

\begin{equation}
E \left( \exp \left( \int_0^t \sum_{q=1}^{n} \bar{\partial}_q K \left( r_s(s) \right) ds \right) \right) < \infty \ \forall t.
\end{equation}

Then the semi-group associated to \( \mathcal{L} \) has the following Girsanov type expression:

\begin{equation}
\left[ \exp(t \mathcal{L}) F \right](r_0) = E_{r_0} \left( \exp \left( - \int_0^t \sum_{q=1}^{n} \bar{\partial}_q K (dx_{2q-1} + i dx_{2q}) \right) \times F(r_x(t)) \right),
\end{equation}

where the integral appearing inside the exponential is an Itô integral and where \( F = f \circ p \) is basic.

**Proof.** Set \( * \) the Itô contraction between stochastic differentials, then we have:

\begin{equation}
(dx_{2q-1} + i dx_{2q}) * (dx_{2q-1} + i dx_{2q}) = 0,
\end{equation}

\begin{equation}
(dx_{2q-1} + i dx_{2q}) * (dx_{2q-1} - i dx_{2q}) = 2 dt.
\end{equation}

2.2. Weitzenböck formula

Our objective is to construct a Stochastic Calculus of Variations (SCV) associated to the semi-group \( \exp(t \mathcal{L}) \). The infinitesimal form of the SCV will correspond to Weitzenböck–Kodaira formula associated to \( \mathcal{L} \); we rewrite below these classical results translated into frame bundle terminology.

Set \( \xi_q \) the real valued 1-differential form on \( U(M) \) which vanish on the fibers \( p^{-1}(m) \) and which constitute a dual basis of the horizontal vector fields \( A_s \). Introduce the following complex-valued differentials forms

\begin{equation}
\eta_q = \xi_{2q-1} + i \xi_{2q}, \quad \bar{\eta}_q = \xi_{2q-1} - i \xi_{2q}.
\end{equation}
Then $\eta_q$, $\bar{\eta}_q$, $q \in [1, n]$, constitute a C-basis of complex coefficients basic differential forms. Introduce the operators going from functions to differential forms defined by

$$\partial F := \sum_{q=1}^{n} \partial_q F \times \eta_q, \quad \bar{\partial} F := \sum_{q=1}^{n} \bar{\partial}_q F \times \bar{\eta}_q.$$ (2.2)b

Then if $F$ is basic, $\partial F = p^* \sigma$ where $\sigma$ is a 1-differential form on $M$.

**Theorem.** For all basic smooth function $F = f \circ p$ we have

$$\partial(\mathcal{L} F) = \mathcal{L}(\partial F) - \frac{1}{2} \text{Ricci}(\partial F) - \sum_q \eta_q \times \sum_j (\partial_q \bar{\partial}_j K) \times \partial_j F =: \mathcal{L}_1(\partial F),$$ (2.2)c

where the operator $\mathcal{L}$ does not operate in the first term of the right-hand side on the $\eta_q$.

**Proof.** See for instance [8]. \(\square\)

### 2.3. Line bundle formalism

The line bundle machinery will consist in adding two auxiliary dimensions to $M$; on the extended space all complex coefficients first order differential operators on $M$ will appear as real coefficients second order differential operators and therefore could be integrated through suitable SDE.

The line bundle above $M$ will be the product

$$L = M \times \mathbb{C}.$$ (2.3)a

As we want to make $\sqrt{-1}$ disappear from our formalism, we denote the $R$-basis of $\mathbb{C}$ by the two letters $\epsilon_0 = 1$, $\epsilon_1 = \sqrt{-1}$. Then the line bundle appear as a fiber bundle above $M$ the fibers, being two-dimensional Euclidean spaces.

We associate to a complex-valued function $f = u + iv$ the complex-valued function $\sigma_f$ defined on $L$ by

$$\sigma_f$$ is linear on each fiber,

$$\sigma_f(\epsilon_0) := u + iv,$$

$$\sigma_f(\epsilon_1) := \sigma_i \times f(\epsilon_0) = -v + iu.$$ (2.3)b

On the Euclidean space $L_{\epsilon_0}$ we have two natural vector fields: the vector field of infinitesimal homotheties and the vector field of infinitesimal rotations; they prolongate into two vertical vector fields on $L$ denoted $\partial_\alpha$, $\partial_\beta$. Then we have

$$\partial_\alpha \sigma_f = \sigma_f, \quad \partial_\beta \sigma_f = \sigma_i \times f.$$ (2.3)c

The multiplicative group $\mathbb{C}^*$ of complex numbers acts on $\mathbb{C}$ as the group of similitudes constituted by homotheties and rotations. We denote by $S$ this group and by $S$ its Lie algebra; set $\hat{\partial}_\alpha$, $\hat{\partial}_\beta$ the natural basis of $S$. Then

$$[\partial_\alpha]_l = \exp(\epsilon \hat{\partial}_\alpha)l, \quad [\partial_\beta]_l = \exp(\epsilon \hat{\partial}_\beta)l, \quad \epsilon \to 0.$$
Given two vector fields $B, C$ on $M$, consider the complex coefficients first order operator $Z = \partial_B + i \partial_C$. We have

$$\sigma_{Zf} = \partial_\alpha \partial_{B'} \sigma_f + \partial_\beta \partial_{C'} \sigma_f,$$

where $B', C'$ are the lift of $B, C$ to $L$ through the trivial connection.

A complex coefficients first order operator on $M$ can be looked upon as a real coefficients second order operator on $L$. From this fact it is clear that the operator $L$ can be written as a real second order operator on $L$; we shall explicit in (2.6) this fact.

2.4. A connection on the line bundle

Introduce on $U(M)$ the following real operators sending a real function into differential forms with real coefficients:

$$d = \partial + \bar{\partial}, \quad d^c = -i \times (\partial - \bar{\partial}).$$

Then $2\partial = d + id^c$, $2\bar{\partial} = d - id^c$.

The operators $d$ and $d^c$ have the following explicit expressions:

$$dF = \sum_{s=1}^{2n} (\partial_{A_s} F) \times \xi_s, \quad d^c F = \sum_{q=1}^{n} (\partial_{A_{2q-1}} F) \times \xi_{2q} - (\partial_{A_{2q}} F) \times \xi_{2q-1}.$$  \hspace{1cm} (2.4)\hspace{1cm} a

We have

$$2(\partial + \bar{\partial}) F = \sum_{q=1}^{n} (\partial_{A_{2q-1}} - i \partial_{A_{2q}}) F \times (\xi_{2q} - i \xi_{2q-1}) + (\partial_{A_{2q-1}} + i \partial_{A_{2q}}) F \times (\xi_{2q-1} + i \xi_{2q})$$

$$= 2 \sum_{s=1}^{2n} (\partial_{A_s} F) \times \xi_s.$$

We deduce the expression for $d^c$ in an analogous way.

The first formula in (2.4)\hspace{1cm} b is the usual expression of the differential of a function on frame bundle, fact which justifies a posteriori our notation. Remark that if $F$ is basic function then $dF$ and $d^c F$ are basic differential forms.

Define a differential form $\omega$ on $M$ with value in the Lie algebra $S$ of the group of similitudes $S$ of $R^2$ by the formula

$$2\omega = -\partial_\alpha \times dK + \partial_\beta \times d^c K.$$  \hspace{1cm} (2.4)\hspace{1cm} c

Then $\omega$ defines a connection on the principal bundle $S \times M =: P$.

On each fiber $L_m$ of the line bundle $L$ define the Euclidean metric

$$(\xi_1 | \xi_2)_m := \exp(-K(m))\overline{\xi_1} \xi_2.$$  \hspace{1cm} (2.4)\hspace{1cm} d
Set \( \sigma \) a section of \( L \) and set \( \nabla^\omega \) the covariant derivative on sections associated to the connection \( \omega \); then for any vector field \( z \) on \( M \) the connection preserves the metric:

\[
\frac{d}{d\epsilon} \left[ (\sigma_1 \mid \sigma_2)_{m+\epsilon z_m} \right] = (\nabla^\omega_z \sigma_1 \mid \sigma_2) + (\sigma_1 \mid \nabla^\omega_z \sigma_2). \tag{2.4}_c
\]

In fact,

\[
\frac{d}{d\epsilon} \left[ (\sigma_1 \mid \sigma_2)_{m+\epsilon z_m} \right] = (\partial_z \sigma_1 \mid \sigma_2) + (\sigma_1 \mid \partial_z \sigma_2) - \langle z, dK \rangle \times (\sigma_1 \mid \sigma_2).
\]

On the other hand,

\[
2\langle z, dK \rangle \times (\sigma_1 \mid \sigma_2) = \left( \left[ \langle z, dK \rangle - i\langle z, d^cK \rangle \right] \times \sigma_1 \mid \sigma_2 \right) + (\sigma_1 \mid \left[ \langle z, dK \rangle - i\langle z, d^cK \rangle \right] \times \sigma_2).
\]

2.5. An hypoelliptic operator

Set \( \tilde{A}_k \) the lift to \( U(L) \) of the horizontal vector field \( A_k \) defined in (2.1) through the connection \( \omega \) defined in (2.4)_c,

\[
\tilde{A}_k = \left( A_k, \omega(A_k) \right).
\tag{2.5}_a
\]

Let \( \Delta_L \) be the infinitesimal generator associated to the lift to \( L \) of the horizontal Brownian motion on \( U(M) \):

\[
\Delta_L = \frac{1}{2} \sum_{s=1}^{2n} \partial^2_{\tilde{A}_k}.
\tag{2.5}_b
\]

**Theorem.** For every complex-valued function \( f \) defined on \( M \) we have

\[
\Delta_L \sigma f = \sigma \mathcal{L}_2 f, \quad \text{where} \quad \mathcal{L}_2 f = \mathcal{L} f - (\Delta K) \times f. \tag{2.6}_a
\]

**Proof.** We have

\[
\partial^2_{\tilde{A}_s} = \left( \partial_{A_s} - \frac{1}{2} \langle A_s, dK \rangle \partial_\alpha + \frac{1}{2} \langle A_s, d^cK \rangle \partial_\beta \right)^2. \tag{2.6}_b
\]

Using the commutation \([\partial_{A_s}, \partial_\alpha] = 0, [\partial_{A_s}, \partial_\beta] = 0,\)

\[
\partial^2_{\tilde{A}_s} = \partial^2_{A_s} - \langle A_s, dK \rangle \partial_\alpha \partial_{A_s} + \langle A_s, d^cK \rangle \partial_\beta \partial_{A_s} + \frac{1}{2} Q_s + \frac{1}{4} R_s, \tag{2.6}_c
\]

where

\[
\sum_s Q_s = \sum \left( -\partial_{A_s} \langle A_s, dK \rangle \partial_\alpha + \partial_{A_s} \left( \langle A_s, d^cK \rangle \partial_\beta \right) \right) = 2\Delta K \partial_\alpha.
\]
granted that the second term vanishes. On the other hand,

\[ R_s := ((A_s, dK) \partial_\alpha - \langle A_s, d^c K \rangle \partial_\beta)^2. \]

Using the fact that \( \partial_\beta \sigma_f = i \sigma_f \), \( \partial_\alpha \sigma_f = \sigma_f \) we get

\[ R_s \sigma_f = \{ (\| dK \|^2 - \| d^c K \|^2) - 2i (\langle A_s, dK \rangle \langle A_s, d^c K \rangle) \} \sigma_f. \]

**Lemma.** For every real function \( K \), we have

\[ \| dK \|^2 = \| d^c K \|^2, \quad (dK \mid d^c K) = 0. \quad (2.6)_d \]

**Proof.**

\[ \partial_q = \frac{1}{2} (\partial A_{2q-1} - i \partial A_{2q}), \quad \tilde{\partial}_q = \frac{1}{2} (\partial A_{2q-1} + i \partial A_{2q}), \quad q \in [1, n], \]

\[ \eta_q = \xi_{2q-1} + i \xi_{2q}. \]

\[ \partial K = \sum_{q=1}^{n} \partial K \times \eta_q, \quad \tilde{\partial} K = \sum_{q=1}^{n} \tilde{\partial} K \times \eta_q, \]

\[ dK = 2\Re \partial K; \quad d^c K = 2\Im \partial K, \]

\[ dK = \sum_q \partial A_{2q-1} K \times \xi_{2q-1} + \partial A_{2q} K \times \xi_{2q}, \]

\[ d^c K = \sum_q \partial A_{2q-1} K \times \xi_{2q} - \partial A_{2q} K \times \xi_{2q-1}, \]

\[ \| dK \|^2 = \sum_q [\partial A_{2q-1} K]^2 + [\partial A_{2q} K]^2, \]

\[ \| d^c K \|^2 = \sum_q [\partial A_{2q-1} K]^2 + [\partial A_{2q} K]^2, \]

\[ (dK \mid d^c K) = \sum_q [-\partial A_{2q-1} K \times \partial A_{2q} K + \partial A_{2q} K \times \partial A_{2q-1} K] = 0. \]

Applying the lemma we get

\[ \left( \sum_{s=1}^{2n} R_s \right) \sigma_f = \left( (\| dK \|^2 - \| d^c K \|^2) - 2i (dK \mid d^c K) \right) \sigma_f = 0. \quad (2.6)_e \]

Using (2.3)_d,

\[ - \left( \sum_{s=1}^{2n} \langle A_s, dK \rangle \partial_\alpha \partial A_s - \langle A_s, d^c K \rangle \partial_\beta \partial A_s \right) \sigma_f \]
\[
= -\left( \sum_{s=1}^{2n} \langle A_s, dK \rangle \partial A_s - i \times \langle A_s, d\bar{K} \rangle \partial A_s \right) \sigma_f.
\]

We use now (2.4)\textsubscript{a},
\[
= -2 \left( \sum_{s=1}^{2n} \langle A_s, \bar{\partial} K \rangle \partial A_s \right) \sigma_f = -\sigma_{\bar{\partial} K \partial f}
\]
granted the following identity,
\[
\sum_{s=1}^{n} \left\{ (A_{2s-1}, \eta_s) \partial A_{2s-1} f + \langle A_{2s}, \bar{\eta}_s \partial A_{2s}, f \rangle (\partial A_{2s-1} K + i \partial A_{2s} K) \right\} = \sum_{s=1}^{n} \partial_s f \times \bar{\partial}_s K.
\]

**Corollary.** Assume
\[
E \left( \exp \int_0^t (\Delta K)(r_{x,s}) ds \right) < \infty \quad \forall t.
\]

Then the process associated to \( \Delta_L \) is given by
\[
(r_{x,t}, \gamma_{x,t}) \quad \text{where} \quad \gamma_{x,t} = \exp \left( -\int_0^t \sum_{q=1}^{n} \bar{\partial}_q K (dx_{2q-1} + i dx_{2q}) - \Delta K dt \right). \quad (2.6)\textsubscript{f}
\]

The lift to \( L \) of the Brownian motion \( m_{x,t} \) on \( M \) is given by
\[
(m_{x,t}, \xi_{x,t}) \quad \text{where} \quad \xi_{x,t} := \gamma_{x,t}(\xi_0), \quad \xi_0 \in L_{m_{x,0}}. \quad (2.6)\textsubscript{g}
\]

The process \( \xi_{x,t} \) lives on the circle bundle which means that
\[
(\xi_{x,t} | \xi_{x,t})_{m_{x,t}} = \text{const.} \quad (2.6)\textsubscript{h}
\]

**Proof.** The result follows from the stochastic representation for the semigroup associated to \( L \) given by (2.1)\textsubscript{g} together with a Feynman–Kac formula (cf. [7]) for the zero order term \( -\Delta K \times f \) in \( L_2 \).

Finally \( (2.6)\textsubscript{h} \) results from (2.4)\textsubscript{c}. \( \square \)

**Corollary.** Consider the subspace \( \mathcal{W} \) of functions on \( L \) which are linear on each fiber. Then
\[
\Delta_L(\mathcal{W}) \subset \mathcal{W}, \quad \exp(t\Delta_L)(\mathcal{W}) \subset \mathcal{W}. \quad (2.6)\textsubscript{i}
\]

**Proof.** Any element of \( \mathcal{W} \) is of the form \( \sigma_f \). According to (2.6)\textsubscript{a} we have
\[
\sigma_{\exp(t\mathcal{L}_2)}f(m_0) = E_{(m_0, \epsilon_0)}(f(m_{x,t})\xi_{x,t}). \quad (2.6)\textsubscript{j}
\]
2.7. Stochastic Calculus of Variation for $\Delta_L$

Set $\Phi_{x,t}$ the flow of diffeomorphisms on $U(M) \times C$ generated by the SDE associated to $\Delta_L$:

$$d\Phi_{x,t}(l_0) = \sum_{k=1}^{2n} \tilde{A}_k(\Phi_{x,t}(l_0)) \circ dx_k, \quad \Phi_{x,0}(l_0) = l_0. \quad (2.7)_a$$

Then

$$\Phi_{x,t}(\{r_0, \zeta\}) = \{\Psi_{x}(t), \gamma_{x,t} \times \zeta\},$$

$$\gamma_{x,t} := \exp\left( -\int_0^t \sum_{q=1}^n \tilde{\partial}_q K (dx_{2q-1} + i dx_{2q}) - \Delta K dt \right), \quad (2.7)_b$$

where $\Psi_{x,t}$ is the horizontal Brownian flow on $U(M)$.

The construction of a SCV for $\Phi_{x,t}$ will be done by lifting the SCV for $\Psi_{x,t}$ through $(2.7)_b$.

We recall the Bismut formula [2] governing the SCV on $\Psi_{x,t}$ following the notation of [6] and [3].

A *tangent process* is the infinitesimal transformation of the Brownian motion $x$ defined, as $\epsilon \to 0$, by

$$x(t) \mapsto x_\epsilon := x + \epsilon \int_0^t \rho(s) \, dx(s) + \epsilon h(t), \quad (2.7)_c$$

where $\rho(s)$ is an adapted process taking its values in the antisymmetric matrices. Given a smooth functional $\Theta$ on the Wiener space, its derivative will be defined as

$$(D(\rho,h)\Theta)(x) := \frac{d}{d\epsilon=0} \Theta(x_\epsilon).$$

We propagate a variation of the initial condition through the variation of the path on $U(M)$ given by

$$\dot{h} + \frac{1}{2} \text{Ricci} \, h = 0, \quad d\rho = -\Omega(\od x, h), \quad (2.7)_d$$

where $\rho(0) = 0$ and $h(0)$ is given and where $\Omega$ denotes the curvature tensor. We have

$$D_{\rho,h}(\gamma_{x,t}) = -\gamma_{x,t} \left( \int_0^t \left\{ \frac{d}{d\epsilon=0} \tilde{\partial}_q K \left( x(\ast) + \epsilon h(\ast) \right) \right\} \times (dx_{2q-1} + i dx_{2q}) \right.$$

$$\left. + \left\{ \frac{d}{d\epsilon=0} \Delta K \left( x(\ast) + \epsilon h(\ast) \right) \right\} \times dt + \int_0^t \left[ \tilde{\partial}_q K \right] \times \left[ \rho \ast (dx_{2q-1} + i dx_{2q}) \right]_q \right). \quad (2.7)_e$$
2.8. Stochastic Calculus of Variations for $L$

We have interpreted the semi-group associated to the complex coefficients differential operator $L_2$ defined in (2.6)\textsubscript{a}, (2.1)\textsubscript{d} in terms of the stochastic flow $\Phi_{x,t}$ defined on the line bundle. In this section we make the basic hypothesis

$$\Delta K = \text{const} =: c. \quad (2.8)$$

Then

$$L = L_2 + c.$$ We want to recapture the stochastic theory corresponding to the formula of differential geometry (2.2)\textsubscript{c}.

The tangent space $T_*(U(M))$ at a generic point of $U(M)$ is identified through the horizontal parallelism to $C^n \times U(n)$ where $U(n)$ is the Lie algebra of the unitary group of dimension $n$.

The tangent space $T_*(L(M))$ at a generic point $\ast$ of $L(M)$ is identified to $T_*(U(M)) \times S$ then $\Phi'_{x,t} : T_{\Phi_{x,0}}(L(M)) \mapsto T_{\Phi_{x,t}}$ is given by

$$\Phi'_{x,t}(h_0, \rho_0, \zeta_0) = (h_t, \rho_t, \zeta_t), \quad (2.8)\textsubscript{a}$$

where $(h_t, \rho_t)$ are given in (2.7)\textsubscript{d} and where

$$\zeta_t = \zeta_0 - \int_0^t \{D_h \tilde{\partial}_q K} \times (dx_{2q-1} + i \, dx_{2q}) - \int_0^t \{\tilde{\partial}_q K} \times [\rho(dx_{2q-1} + i \, dx_{2q})]_q. \quad (2.8)\textsubscript{b}$$

We say that a first order differential operator is of type (1.0) if it is a linear combination with complex coefficients of the differential operators $\partial_q$, $q \in [1, n]$.

If $h(0)$ is of type (1, 0) then $h(t)$ will be of type (1, 0). \quad (2.8)\textsubscript{c}

In fact, as Ricci tensor preserves the complex structure, this results from (2.7)\textsubscript{d}.

If $h$ is of type (1, 0) we have $D_{0,h}(\gamma_{x,t}) = -\gamma_{x,t} \int_0^t \partial_h \tilde{\partial}_q K \times (dx_{2q-1} + i \, dx_{2q}). \quad (2.8)\textsubscript{d}$

The stochastic flow $\Phi_{x,t}$ preserves the vector fields $\partial_{\alpha}$, $\partial_{\beta}$. \quad (2.8)\textsubscript{e}

In fact a variation of $\zeta_0$ in (2.8)\textsubscript{b} leads to the same variation of $\zeta_t$.

Operator $L_1$ defined in (2.2)\textsubscript{c} can be extended as operating on (1, 0) differential forms and generates a semi-group $\exp(t L_1)$ operating on (1, 0) differential forms on $M$.

**Theorem.** For every (1, 0) differential form $\omega$ defined on $M$ we have

$$\langle \exp(t L_1)(\omega), \eta^*_q \rangle = E (\langle \omega, \Phi'_{x,t}(\eta_q^*) \rangle), \quad (2.9)\textsubscript{a}$$

where $\eta^*_q$ denotes the basis of (1, 0) vector fields which is dual of the basis $\eta_q$. 

Proof. Remark that the right-hand side of (2.9)_a is a basic (1, 0) differential form; it defines a semi-group on (1, 0) differential forms. We shall prove identity (2.9)_a by showing the equality of the two infinitesimal generators. Let us compute

\[ Q := \lim_{t \to 0} \frac{1}{t} E \left( \langle \omega, \Phi'_{x,t}(\eta^*_q) \rangle - \langle \omega, \Phi_{x,t}(m), (h_t, \rho_t, \zeta_t) \rangle \right), \]

where \((h_t, \rho_t, \zeta_t)\) has been defined in (2.8)_a. As we shall take in (2.9)_a an expectation we can forget the contribution of \(D_{\rho,0}\). Then we get

\[ Q = \lim_{t \to 0} \frac{1}{t} E \left( \langle \omega, \Phi_{x,t}(m), h_0 \rangle - \frac{t}{2} \langle \omega_m, \text{Ricci}(h_0) \rangle - \langle \omega, \Phi_{x,t}(m), h_0 \rangle \times G \right), \]

where \(G\) is the horizontal derivative of the Girsanov factor which has been defined in (2.8)_d; by inspection of the last formula we get \(Q = L_1(\omega)\). □

3. Commutation of differential operators on Poincaré disk

Let \(D\) be the Poincaré disk \(D = \{ z \in \mathbb{C} : |z| < 1 \}\) with the Kähler metric \(ds^2 = \frac{dz \overline{dz}}{(1-|z|^2)^2}\). Consider the transformation \(D \to D\) defined by \(z \to az + b\overline{bz} + \overline{a}\) where \(|a|^2 - |b|^2 = 1\).

Denote by \(G\) the group SU(1, 1) constituted by matrices \(g = \left( \begin{array}{cc} a & b \\ b & \overline{a} \end{array} \right) \) with \(|a|^2 - |b|^2 = 1\). Its Lie algebra su(1, 1) is identified with \(\left( \begin{array}{cc} i & \beta \\ \beta & -i \alpha \end{array} \right)\), where \(\alpha\) is real. The Poincaré disk \(D\) appears as \(D = \text{SU}(1,1)/S^1\), where \(S^1\) is the subgroup of \(G\) defined by the equation \(b = 0\). The Kähler potential

\[ K(z) := -\log(1 - |z|^2) \]

satisfies

\[ K(gz) = 2\Re \log |\overline{bz} + \overline{a}| + K(z), \]

(3.1)_a

identity which implies that the Kählerian metric \(\partial \overline{\partial} K\) is invariant under the action of \(G\). Indeed, we have

\[ K(gz) - K(z) = -\log \left( 1 - \frac{|az + b|^2}{|bz + \overline{a}|^2} \right) + \log(1 - |z|^2) \]

\[ = \log \left( \frac{|\overline{bz} + \overline{a}|^2(1 - |z|^2)}{|bz + \overline{a}|^2 - |az + b|^2} \right) = \log |\overline{bz} + \overline{a}|^2. \]

Theorem. Let \(\mathcal{H}(D)\) be the space of holomorphic functions on the Poincaré disk \(D\). Let \(\mathcal{H}_\gamma(D)\) be the subspace of holomorphic functions for which the following Hilbertian norm is finite

\[ \| f \|_\gamma^2 := \int_{D} |f(z)|^2 \exp(-\gamma K(z)) \, dv(z), \]

(3.1)_b
where $dv$ is the hyperbolic volume measure on $D$: $dv := (1 - |z|^2)^{-2} dz \wedge d\bar{z}$.

Define an action of $G$ on $\mathcal{H}(D)$ by

$$
(T_{a,b}^\gamma f)(z) = \frac{1}{(bz + \bar{a})^\gamma} f\left(\frac{az + b}{bz + \bar{a}}\right).
$$

(3.1)

Then $T_{a,b}^\gamma$ induces on $\mathcal{H}_\gamma(D)$ an unitary action.

**Proof.** We have

$$
\|T_{a,b}^\gamma(f)\|_\gamma^2 = \int_D \frac{1}{|bz + \bar{a}|^{2\gamma}} |f(g(z))|^2 \exp(-\gamma K(z)) \, dv(z)
$$

$$
= \int_D |f(g(z))|^2 \exp(-\gamma K(g(z))) \, dv(z).
$$

equality resulting from (3.1.a). Make the change of variable $z = g^{-1}(z')$; we get by the invariance of the hyperbolic measure:

$$
\|T_{a,b}^\gamma(f)\|_\gamma^2 = \int_D \exp(-\gamma K(z')) \times |f(z')|^2 \, dv(z'). \quad \square
$$

**Definition.** We shall call the finite mass measure

$$
\mu_\gamma = (1 - |z|^2)^\gamma \, dv(z), \quad \gamma > 1,
$$

(3.1.d)

the unitarizing measure associated to the representation $T$.

Let $\mathcal{G}$ be the Lie algebra of $G$; take for orthonormal basis of $\mathcal{G}$:

$$
e_1 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \frac{1}{2} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.
$$

Then

$$
[e_1, e_2] = e_3, \quad [e_1, e_3] = -e_2, \quad [e_2, e_3] = -e_1.
$$

(3.1.e)

We consider $\mathcal{G}$ as a real Lie algebra defined by the above bracket relations. The vector $e_1$ generates a group which is isomorphic to the circle $S^1$. The quotient $\pi: G \mapsto G/S^1$ is the Poincaré disk $D$, which is a Riemannian manifold, homogeneous under the left action of $G$. The transform of the base point $0 \in D$ by $g \in G$ is $g(0) = b(\bar{a})^{-1}$. The Kähler potential is defined on $D$ by $-\log(1 - |z|^2)$; it lifts to $G$ as

$$
K(g) := -\log(1 - |b(\bar{a})^{-1}|^2) = \log(1 + |b|^2).
$$

(3.1.f)
Remark that if
\[ g = \begin{pmatrix} a & b \\ b & \bar{a} \end{pmatrix}, \]
then
\[ g^{-1} = \begin{pmatrix} \bar{a} & -b \\ -b & a \end{pmatrix}; \]
therefore
\[ K(g^{-1}) = K(g). \quad (3.1) \]

We define the following left (respectively right) derivatives on \( G, g = \begin{pmatrix} a & b \\ b & \bar{a} \end{pmatrix}, \)
\[ (\partial^l_{e_1} F)(g) := \frac{d}{d\epsilon=0} F(\exp(\epsilon e_1)g), \quad (\partial^r_{e_1} F)(g) := \frac{d}{d\epsilon=0} F(g \exp(\epsilon e_1)) \]
then

**Proposition.** Consider the functions \( a, b \) defined as the first line of the matrix \( g \in SU(1, 1) \). Then
\[ 2\partial^r_{e_2}(a) = b, \quad 2\partial^r_{e_2}(b) = a, \quad 2\partial^r_{e_3}(a) = -ib, \quad 2\partial^r_{e_3}(b) = ia, \]
\[ 2\partial^l_{e_2}(a) = \bar{b}, \quad 2\partial^l_{e_2}(b) = \bar{a}, \quad 2\partial^l_{e_3}(a) = i\bar{b}, \quad 2\partial^l_{e_3}(b) = i\bar{a}. \quad (3.1)_h \]

**Proof.**
\[ 2\partial^r_{e_2} \begin{pmatrix} a & b \\ b & \bar{a} \end{pmatrix} = \frac{d}{d\epsilon=0} \left( \begin{pmatrix} a & b \\ b & \bar{a} \end{pmatrix} \exp(e_2) \right) = \frac{d}{d\epsilon=0} \begin{pmatrix} a & b \\ b & \bar{a} \end{pmatrix} \left( \begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix} \right) = \frac{d}{d\epsilon=0} \left( \begin{pmatrix} a & b \\ b & \bar{a} \end{pmatrix} + \epsilon \begin{pmatrix} b & a \\ \bar{a} & \bar{b} \end{pmatrix} \right). \]
The derivative with respect to \( e_3 \) gives
\[ \frac{d}{d\epsilon=0} \begin{pmatrix} a & b \\ b & \bar{a} \end{pmatrix} \left( \begin{pmatrix} 1 & i\epsilon \\ -i\epsilon & 1 \end{pmatrix} \right) = \frac{d}{d\epsilon=0} \left( \begin{pmatrix} a & b \\ b & \bar{a} \end{pmatrix} + i\epsilon \begin{pmatrix} -b & a \\ -\bar{a} & \bar{b} \end{pmatrix} \right). \quad \square \]
The computation of the left derivatives is analogous.

**Corollary.** Let \( \partial (g) := g^{-1}(0) = -b(a)^{-1}. \) Then
\[ (\partial^r_{e_2} K)(g) = -\Re(\partial (g)), \quad (\partial^r_{e_3} K)(g) = -\Im(\partial (g)). \quad (3.1)_i \]
3.2. Horizontal Brownian motion

Consider the $G$-valued process defined by the Stratonovitch SDE,

$$
\begin{equation}
\frac{dg_{x,t}}{t} = g_{x,t}(e_2 \circ dx_2(t) + e_3 \circ dx_3(t)),
\end{equation}
$$

where $x$ is an $R^2$-valued Brownian motion. The infinitesimal generator of this process is

$$
\Delta = \frac{1}{2} \left( [\partial_r e_2]^2 + [\partial_r e_3]^2 \right). \tag{3.2b}
$$

Write (3.2)a in Itô formalism, then appears the Itô contraction

$$
\frac{1}{2} \left( [e_2]^2 + [e_3]^2 \right) = \frac{1}{4} \text{Identity} \tag{3.2c}
$$

and (3.2)a becomes

$$
\begin{equation}
\frac{dg_{x,t}}{t} = g_{x,t} \left( e_2 dx_2(t) + e_3 dx_3(t) + \frac{1}{4} dt \right). \tag{3.2d}
\end{equation}
$$

3.3. Theorem. The projection of $g_{x,t}$ on the Poincaré disk $D$ is the Brownian motion of the Riemannian manifold $D$.

The bundle of orthonormal frame $O(D)$ of $D$ can be identified to $G$, the canonical horizontal vector fields being $\partial_r e_2$, $\partial_r e_3$.

The process $t \mapsto g_{x,t}$ is the horizontal lift of the Brownian motion on $D$ to $O(D)$.

Proof. Set $p$ the projection $G \mapsto G/S^1 = D$ defined by

$$
p(g) = g(0) = \frac{b}{\bar{a}}.
$$

By homogeneity it is sufficient to verify the statement nearby zero; there, for $t$ small, and writing the first line of the matrix $g(t)$, we obtain

$$
\begin{align*}
b(t) & \simeq \frac{1}{2} (x_2(t) + ix_3(t)) + o(t), \\
\bar{a}(t) & = 1 + \frac{1}{4} t + o(t), \\
\frac{b(t)}{\bar{a}(t)} & = \frac{1}{2} (x_2(t) + ix_3(t)) + o(t). \tag{3.3a}
\end{align*}
$$

The normal chart at zero coincides with the embedding of $D$ into $C$. Therefore straight lines $\tau \mapsto \tau \in C$ and $\tau \mapsto i\tau$ are osculatrix to geodesics at 0. Compare these straight lines to the exponentials

$$
\tau \mapsto \exp(\tau e_2) = 1 + \tau e_2 + \frac{1}{2} \tau^2 + o(\tau^2).
$$
Then
\[ \frac{b}{\bar{a}} = \frac{\tau}{1 + \frac{\tau^2}{2}} = \tau + o(\tau^2) \]
and up to the second order the exponential map on $G$ coincides to the exponential map of the normal chart.  \( \Box \)

3.4. First order complex differential operators

By (3.3)\textsubscript{a} the complex structure on the Poincaré disk sends $e_2 \mapsto e_3$ and $e_3 \mapsto -e_2$ and is therefore realized by $\text{ad}(e_1)$, the adjoint action of $e_1$. Introduce the operators
\[ \partial^r = \frac{1}{2}(\partial^r_{e_2} - i \partial^r_{e_3}), \quad \bar{\partial}^r = \frac{1}{2}(\partial^r_{e_2} + i \partial^r_{e_3}), \quad \partial^l = \frac{1}{2}(\partial^l_{e_2} - i \partial^l_{e_3}), \quad \bar{\partial}^l = \frac{1}{2}(\partial^l_{e_2} + i \partial^l_{e_3}). \]

(3.4)\textsubscript{a} A function $f$ on $G$ is said right-$G$-holomorphic if it satisfies the equation
\[ \bar{\partial}^r f = 0. \]

(3.4)\textsubscript{b} A function $f$ on $G$ is said basic-holomorphic if there exists an holomorphic function $\tilde{f}$ on $D$ such that $f = \tilde{f} \circ p$.

A function $f$ on $G$ is basic-holomorphic if and only if it is invariant under the right action of $S^1$ and satisfies Eq. (3.4)\textsubscript{b}.

(3.4)\textsubscript{c}

Indeed the operator $\bar{\partial}^r$ is the horizontal lift of the Cauchy–Riemann operator.

3.5. Proposition. Define $\varphi(g) = g(0)$. Then $\varphi$ is right-holomorphic.
Define $\vartheta(g) = g^{-1}(0) = \varphi(g^{-1})$. Then $\vartheta$ is left-holomorphic.

Proof. As $\vartheta(g) = \varphi(g^{-1})$ it is sufficient to prove proposition for $\varphi$. We have
\[ \varphi = b(\bar{a})^{-1}. \]
Using (3.1)\textsubscript{h},
\[ 2\partial^r_{e_2}(\varphi) = \frac{|a|^2 - |b|^2}{\bar{a}^2} = \frac{1}{\bar{a}^2}, \]
\[ 2\partial^r_{e_3}(\varphi) = \frac{i|a|^2 - i|b|^2}{\bar{a}^2} = \frac{i}{\bar{a}^2}, \]
\[ \frac{1}{2}(\partial^r_{e_2} + i \partial^r_{e_3})(\varphi) = 0. \quad \Box \]

3.6. Corollary. The function $\bar{\partial}^r K$ is left holomorphic.
Proof. Reading (3.1) we have $(\partial_{e_2}^r K)(g) = -2\Re(\vartheta(g))$, $(\partial_{e_3}^r K)(g) = -2\Im(\vartheta(g))$; therefore $\bar{\partial}^r K = -2\vartheta$ then Proposition 3.5 proves the corollary, since

$$\left(\partial_{e_2}^l + i\partial_{e_3}^l\right)(b(a)^{-1}) = \frac{1}{a^2} - \frac{1}{a^2} = 0.$$  

Remark. We have

$$\bar{\partial}_{e_2}^r (\partial_{e_2}^r K(g)) = 1 - \Re\frac{b^2}{a^2}, \quad \bar{\partial}_{e_3}^r (\partial_{e_2}^r K(g)) = -\Im\frac{b^2}{a^2},$$

$$\bar{\partial}_{e_2}^r (\partial_{e_3}^r K(g)) = -\Im\frac{b^2}{a^2}, \quad \bar{\partial}_{e_3}^r (\partial_{e_3}^r K(g)) = 1 + \Re\frac{b^2}{a^2}.$$  

We can check that the Hessian is always a positive matrix, but has an arbitrary small eigenvalue for some parameters $a, b$. The degeneracy of this Hessian forbids the direct use of [4].

3.7. A complex coefficient elliptic operator

Define on $G$ the following differential operator:

$$\mathcal{L} f = \Delta - \gamma \bar{\partial}^r \times \partial^r f,$$  

(3.7)\text{a}

where $\gamma$ is a positive constant.

Introduce the Girsanov-type functional

$$A_{x,g_0}(t) := \exp\left(-\gamma \int_0^t \bar{\partial}^r K(g_x(t)) \times (dx^2(s) + i dx^3(s))\right),$$  

(3.7)\text{b}

where the integral appearing in this formula is an Itô stochastic integral.

The process $A_{x,g_0}(\cdot)$ is a martingale.  

(3.7)\text{c}

In fact the following Itô contraction vanishes:

$$(dx^2(s) + i dx^3(s)) \ast (dx^2(s) + i dx^3(s)) \equiv (1 + i^2) ds = 0.$$  

Theorem. The semi-group associated to the elliptic operator $\mathcal{L}$ has the following representation

$$\exp(t \mathcal{L}) f(g_0) = E(A_{x,g_0}(t) f(g_0 g_x(t))).$$  

(3.7)\text{d}

Proof. Itô Calculus.  

(3.7)\text{e}

Theorem (Algebraic commutations). We have

$$\partial^r \mathcal{L} - \mathcal{L} \partial^r = (1 - \gamma),$$  

(3.7)\text{f}

$$\bar{\partial}^l \mathcal{L} = \mathcal{L} \bar{\partial}^l,$$  

(3.7)\text{g}

$$\partial_{e_1}^l \mathcal{L} = \mathcal{L} \partial_{e_1}^l.$$  

(3.7)\text{h}
Proof. We have
\[
\partial \Delta = \Delta \partial - \frac{1}{2} \text{Ricci} \partial
\]
and, from [8, Chapter 3, Proposition 6.4], we have Ricci = -2Id since
\[
\bar{\partial} \partial \log \frac{1}{(1-|z|^2)^2} = \frac{2}{(1-|z|^2)^2},
\]
\[
\bar{\partial} \partial (\bar{\partial} \partial f) = (\bar{\partial} \partial \bar{\partial} f) = \bar{\partial} \partial f + \bar{\partial} \partial \partial f = \bar{\partial} \partial f + \bar{\partial} \partial (\bar{\partial} \partial f)
\]
since, from (3.1)\text{a},
\[
(\bar{\partial} \partial \bar{\partial} f)(g) = (\bar{\partial} \partial \bar{\partial} f)(0) = -\{\bar{\partial} \partial \partial \log(1 - z \bar{z})\}_{z=0} = 1.
\]
We have
\[
\bar{\partial}^l = \partial^l_{e_2} + i \partial^l_{e_3}, \quad [\partial^l_{e_j}, \partial^l_{e_k}] = 0.
\]
Therefore
\[
[\bar{\partial}^l, \mathcal{L}] = -\gamma \times \bar{\partial}^l \bar{\partial} \partial f \times \partial \partial f.
\]
Using Corollary 3.6, \(\bar{\partial}^l \bar{\partial} \partial f = 0\). On the other hand,
\[
\partial^l_{e_1} \bar{\partial} \partial f = \bar{\partial} \partial \partial^l_{e_1} f.
\]
Using (3.1)\text{g} we get
\[
\bar{\partial} \partial ^l_{e_1} f = \bar{\partial} \partial (\partial^l_{e_1} f) = 0. \quad \square
\]
Corollary.
\[
\bar{\partial} \partial \exp(t \mathcal{L}) = \exp((1 - \gamma) t) \exp(t \mathcal{L}) \bar{\partial} \partial f.
\]
(3.7)\text{h}

3.8. The space \(\mathcal{H}\)

Consider the vector space \(\mathcal{H}\) of functions on \(G\) which are of the form
\[
\phi(g^{-1}(0)), \quad \text{where } \phi \text{ is a bounded holomorphic function on } D.
\]
(3.8)\text{a}

Obviously,
\[
\mathcal{H} \text{ is an algebra for the pointwise multiplication.} \quad (3.8)\text{b}
\]

Theorem. A function \(\phi\) defined on \(G\) belongs to \(\mathcal{H}\) if and only if it satisfies the following two first order differential equations
\[
\partial^l_{e_1} f = 0, \quad \bar{\partial}^l f = 0. \quad (3.8)\text{c}
\]
Proof. Define \( F \) by \( F(g) = f(g^{-1}). \) Then
\[
\partial_{e_1}^l F = 0, \quad \bar{\partial}^r F = 0.
\]
The first equation means that there exists a function \( \varphi \) defined on \( D \) such that \( F = \varphi \circ p; \) the second equation (3.8)\(_c\) means that \( \varphi \) is holomorphic. \( \square \)

3.9. Theorem. The semi-group \( \exp(tL) \) conserves the space \( \mathcal{H} \).

Proof. The commutation formulae (3.7)\(_f\), (3.7)\(_g\) imply that
\[
\bar{\partial}^l \exp(tL) = \exp(tL) \bar{\partial}^l, \quad \partial_{e_1}^l \exp(tL) = \exp(tL) \partial_{e_1}^l,
\]
relations which imply that the kernels of the two differential operators \( \bar{\partial}^l, \partial_{e_1}^l \) are preserved by \( \exp(tL) \). \( \square \)

4. Remote past vanishing, convergence in law and unitarizing measure

Remark that in the case of the Poincaré disk, we can apply (2.8) according the fact that
\[
\Delta K = 1, \quad \text{therefore} \; \mathcal{L} = \mathcal{L}_2 + 1,
\]
and the Stochastic Calculus of Variations for \( \mathcal{L} \) and \( \mathcal{L}_2 \) are equivalent. We have already in (2.9)\(_a\) transferred the commutation formula (3.7)\(_e\) in terms of semi-group: it remains to transfer (3.7)\(_f\).

Set
\[
A_{x,t} = \exp \left( \int_0^t \left( \bar{\partial}^r K(x_{x,t}) \times (dx_2 + idx_3) \right) \right).
\]  \( (4.1)_a \)

Theorem.
\[
\bar{\partial}^l E(\{\sigma_f, \Phi_{x,t}(l)\}) = E(\{\sigma_{\bar{\partial}^l f}, \Phi_{x,t}(l)\}). \quad (4.1)_b
\]

Proof. The operator \( \Delta \) defined in (2.1)\(_b\) commutes with the left translation. On the other hand, from (3.6),
\[
\bar{\partial}^l A_{x,t} = -A_{x,t} \times \int_0^t \bar{\partial}^l \bar{\partial}^r K(dx_2 + i dx_3) = 0. \quad \square
\]

Let \( \mathcal{H}^\infty(D) \) be the Banach space of bounded holomorphic functions on the disk \( D \). To \( \varphi \in \mathcal{H}^\infty(D) \) let us associate
\[
f_\varphi(g) := \varphi(g^{-1}(0)) = \varphi \left( \frac{b}{a} \right). \quad (4.2)_a
\]
Set $\delta$ is the initial point of $L$ defined as

$$\delta = (\text{Identity}, \epsilon_0, 0) \quad (4.2)_b$$

and define

$$B_t(g) = E_{g}^{\delta_{\epsilon_0,0}}(A_{x,t}). \quad (4.2)_c$$

The notion of measure with values in a line bundle it is given by a probability measure on $M$ together with a section of $L$; set $p_t(dg)$ the probability measure describing the law of the Brownian motion starting from the identity; then the couple $(B_t(g), p_t(dg))$ defines a measure with values in $L$ which can be considered as the heat measure associated to $L$.

**Definition.** The heat measure defined on $G$ will be the complex-valued measure

$$\nu_t := B_t(g) \times p_t(dg). \quad (4.2)_d$$

**Theorem.** Assume $\gamma > 1$.

*When* $t \to \infty$, *the heat measures* $\nu_t$ *converge weakly to a finite mass measure* $\nu_\infty$. $(4.2)_e$

**Proof.** Firstly recall the notion of image measure of $\nu_t$; given $\Psi$ a $R^d$-valued function on $G$, define the image measure $\Psi\ast(\nu_t)$ as the complex-valued measure on $R^d$, associating to a Borelian $A \subset R^d$ the complex number

$$(\Psi\ast\nu_t)(A) := \nu_t(\Psi^{-1}(A)). \quad (4.2)_f$$

**Main lemma.** Given $\varphi := \varphi_1, \ldots, \varphi_d \in H^\infty(D)$, associate $\Psi^{\varphi} : G \to C^d$ defined by $\Psi(g) := (\varphi_1(g), \ldots, \varphi_d(g))$; then, when $t \to \infty$,

$$\Psi^{\varphi}_t \ast \nu_t \text{ converges weakly to } \nu_\infty^{\varphi}, \quad (4.3)_a$$

$$\nu_\infty^{\varphi} \text{ has all its moments finite.} \quad (4.3)_b$$

**Proof.** The proof is written for $d = 2$.

Set $\Phi_{x,t}$ the stochastic flow on $L(M)$ defined in $(2.7)_a$. Define a $C^2$-valued function $H$ on $L(M)$ by

$$H(l) = (l\sigma_{f_1}, l\sigma_{f_2}).$$

Set

$$\tilde{\Phi}_{x,t} := H \circ \Phi_{x,t}.$$

In order to apply the projected transfer principle $(1.5)$ we must compute

$$E^{\Phi_{x,t}(\delta)}z(\tilde{\Phi}'_{x,t}(\delta)), \quad E^{\Phi_{x,t}(\delta)}z(\tilde{\Phi}''_{x,t}(\delta)). \quad (4.3)_c$$
where $\delta$ has been defined in (4.2)$_b$. Set $y(\epsilon) = e_2 y_1(\epsilon) + e_3 y_2(\epsilon)$ the Brownian motion on $D$ around 0 read in the normal chart at 0.

We have the following expression of a stochastic differential of a function $u$ defined in a neighborhood of 0:

$$u = \partial^r u(y_2(\epsilon) + i y_3(\epsilon)) + \bar{\partial}^l u(y_2(\epsilon) - i y_3(\epsilon)) + \frac{\epsilon}{2} \Delta u + o(\epsilon).$$  \hfill (4.3)$_d$

We apply this identity to

$$u := \tilde{\Phi}_{x,t}(y(\epsilon)).$$  \hfill (4.3)$_e$

Then according to (4.1)$_b$, 

$$\bar{\partial}^l \tilde{\Phi}_{x,t} = \sigma_{\bar{\partial}^l f_{\varphi_j}} \circ \Phi_{x,t}(\delta) = 0, \quad j = 1, 2.$$  \hfill (4.3)$_f$

The Laplacian at the origin of the local chart has the following expression:

$$\Delta = \bar{\partial}^l \partial^r = \partial^r \bar{\partial}^l.$$  

By the last equality and by (4.3)$_d$, 

$$(\Delta \tilde{\Phi}_{x,t})(\delta) = 0,$$  \hfill (4.3)$_g$

$$\tilde{\Phi}_{x,t}(y(\epsilon)) = H' \circ (\partial^r \Phi_{x,t})(\delta) \times (y_2(\epsilon) + i y_3(\epsilon)) + o(\epsilon).$$  \hfill (4.3)$_h$

Using (2.9)$_a$ and (2.2)$_c$ we get

$$\tilde{\Phi}_{x,t}(y(\epsilon)) = \{\exp(i(1 - \gamma))\sigma_{\partial^r f_{\varphi_j}}(\Phi_{x,t}(\delta)) \times (y_2(\epsilon) + i y_3(\epsilon))\}_{j=1,2} + o(\epsilon).$$  \hfill (4.3)$_i$

We conclude by proving

$$\left| (\partial^r f_{\varphi}) (g) \right| = \left| \left( \frac{d}{dz} \varphi^g \right)(0) \right| \leq \| \varphi \|_{H^\infty},$$  \hfill (4.3)$_j$

where $\varphi^g(z) := \varphi(g^{-1}(z))$. Indeed $\varphi^g$ is a bounded holomorphic function in the unit disk and its derivative at 0, by the Schwarz lemma, is bounded by $\| \varphi^g \|_{H^\infty}$. \hfill $\square$

**Proof of (4.2)$_e$.** Denote by $V$ the complex vector space generated by the $f_{\varphi_1}, \tilde{f}_{\varphi_2}, \varphi_1, \varphi_2 \in H^\infty(D)$. It separates points; it constitutes an algebra; it is stable by conjugation. By Stone–Weierstrass it is dense in the space of continuous functions. Applying (4.3)$_a$, with $d = 2$, we obtain that

$$\lim_{t \to \infty} \int_G f_{\varphi_1} \tilde{f}_{\varphi_2} d\nu_t \quad \text{exists.}$$  \hfill (4.4)$_a$
To conclude we need to have a uniform estimate of the total variation $\|\nu_t\|$ of $\nu_t$. We pick a function $\varphi_0$ which is injective on the unit disk, then
\[
\|\nu_t\| = \|\nu_t^{\varphi_0}\|,
\]
the right-hand side being uniformly bounded in virtue of the Main lemma.

**Theorem.** Denote by $\nu_t^1$ the heat measure for the real elliptic operator
\[
\mathcal{L}_1 := \Delta - \gamma \nabla K * \nabla
\]
corresponding to the source at time 0 being the identity. Then
\[
\nu_t = \nu_t^1 \quad \forall t > 0.
\]

**Proof.** Set $R_\theta$ the action of $\theta \in S^1$ and $R_\theta^*$ the action of $R_\theta$ by inverse image on functions; then we have
\[
[\mathcal{L}, R_\theta^*] = 0, \quad [\mathcal{L}_1, R_\theta^*] = 0.
\]
In fact we have that $R_\theta^* K = K$ and $R_\theta$ preserves the Riemannian structure and the complex structure. As the source identity is preserved by $R_\theta$ we get
\[
(R_\theta)_* \nu_t = \nu_t,
\]
\[
(L - \mathcal{L}_1) f = i \gamma (d^c K \mid df),
\]
\[
\frac{d}{dt} \int f \, d\nu_t = \int \mathcal{L}_1 f \, d\nu_t + i \int (d^c K \mid df) \, d\nu_t.
\]
Make the holomorphic change of coordinates
\[
\log z = \zeta = \xi + i \eta.
\]
As $K(z) = -\log(1 - \exp(2\xi))$ we have
\[
(d^c K \mid df) = 2 \frac{1}{\sinh(\xi)} \times \partial_{e_1}^r f,
\]
\[
\int (d^c K \mid df) \, d\nu_t = -2 \int f \times \text{div}_{\nu_t} \left( \frac{1}{\sinh(\xi)} \partial_{e_1}^r \right).
\]
We have,
\[
\text{div}_{\nu_t} \left( \frac{1}{\sinh(\xi)} \partial_{e_1}^r \right) = -\partial_{e_1}^r \left( \frac{1}{\sinh(\xi)} \right) + \frac{1}{\sinh(\xi)} \text{div}_{\nu_t} (\partial_{e_1}^r) = 0
\]
granted (4.5)b. Therefore,
\[
\frac{d}{dt} \int f \, d\nu_t = \int \mathcal{L}_1 f \, d\nu_t.
As by definition
\[
\frac{d}{dt} \int f \, d\nu_t^1 = \int \mathcal{L}_1 f \, d\nu_t^1,
\]
we deduce the theorem by transposition. ☐

4.6. **Corollary.** When \( t \to \infty \) the measure \( \nu_t^1 \) converges towards a probability measure.

**Remark.** The result (4.6) can be obtained by much shorter approaches as, for instance, by Lyapounov function methodology. Our purpose was to prove it from infinitesimal considerations, in the spirit of [4], managing to take in account the lack of strong positivity for the full Hessian of \( K \).

**References**


