Flat connections on punctured surfaces and geodesic polygons in a Lie group

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Abstract

Let \( S \) be a subset of \( n \) points on a compact connected oriented surface \( M \) of genus \( g \), and let \( G \) be a compact semisimple Lie group. The space of isomorphism classes of flat \( G \)-connections on \( P := M \setminus S \) with fixed conjugacy class of monodromy around each point of \( S \) will be denoted by \( \mathcal{R} \). It is known that \( \mathcal{R} \) has a natural symplectic structure. We relate \( \mathcal{R} \) with the space of geodesic \((4g + n)\)-gons in \( G \). A natural 2-form on the space of geodesic \((4g + n)\)-gons in \( G \) is constructed using the Killing form on \( \text{Lie}(G) \). We establish an identity between the symplectic form on \( \mathcal{R} \) and this 2-form on geodesic \((4g + n)\)-gons in \( G \).

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1. introduction

Let \( M \) be a compact connected oriented surface of genus \( g \). Fix a finite subset

\[ S := \{s_1, s_2, \ldots, s_n\} \subset M. \]

The complement \( M \setminus S \) will be denoted by \( P \). Fix a compact connected semisimple Lie group \( G \). By a conjugacy class in \( G \) we will mean an orbit for the adjoint action of \( G \) on itself. For each point \( s_i \in S \), fix a conjugacy class \( C_{s_i} \subset G \). The adjoint action of \( G \) on itself produces an action of \( G \) on the space of homomorphisms from \( \pi_1(P) \) to \( G \). Let

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be the subspace consisting of all homomorphisms \( \rho \) such that \( \rho \) takes the oriented loop around \( s_i \) into \( C_{s_i} \) for every \( i \). This \( \mathcal{R} \) has a natural symplectic structure (see \([3–6,1]\)).

Let \( B \) be the Killing form on \( g := \text{Lie}(G) \). A Riemannian metric on \( G \) is constructed by taking \( G \)-translations of \( B \). A geodesic \( N \)-gon in \( G \) is a continuous mapping \( \tau : [1, N] \rightarrow G \) such that \( \tau(0) = \tau(N) = e \) and for each integer \( 1 \leq i \leq N \), the restriction of \( \tau \) to \([i-1, i] \) is a geodesic in \( G \). \( \mathcal{D}(N) \) denote the space of all geodesic \( N \)-gons in \( G \). There is a natural \( \mathcal{C}^\infty \) two-form on \( \mathcal{D}(N) \) defined by the trilinear form \( (a,b,c) \mapsto -\rightarrow B(a, [b,c]) \) on \( g \). This two-form on \( \mathcal{D}(N) \) will be denoted by \( \theta \).

Our aim here is to establish a relationship between this two-form \( \theta \) on \( \mathcal{D}(4g+n) \) and the symplectic form on \( \mathcal{R} \). This was done earlier in \([2]\) for the special of \( g = 0 \). We recall this result of \([2]\).

Let \( M = S^2 \). Let \( \widetilde{\Omega}_0 \) be the symplectic form on \( \mathcal{D}(4g+n) \) and the symplectic form on \( \mathcal{R} \). This was done earlier in \([2]\) for the special of \( g = 0 \). We recall this result of \([2]\).

Let \( M \) be a compact connected oriented surface of genus \( g \). We fix \( n \) points \( S := \{s_1, s_2, \ldots, s_n\} \) of \( M \). Let \( P := M \setminus S \) be the punctured surface. Fix a base point \( p \in P \). The fundamental group \( \pi = \pi_1(P, p) \) is generated by \( 2g + n \) elements

\[ \Gamma := \{a_j, b_j, m_i \mid 1 \leq j \leq g, \; 1 \leq i \leq n\} \]
admitting a single relation
\[
\left( \prod_{i=1}^{g} a_j b_j a_j^{-1} b_j^{-1} \right) \prod_{j=1}^{n} m_i = 1.
\]

Let \( G \) be a connected compact semisimple Lie group; its Lie algebra will be denoted by \( \mathfrak{g} \).

Fix a conjugacy class \( \tilde{C}_{s_i} \subset \mathfrak{g} \) for each \( i \in [1, n] \). Let \( A \) be the space of all maps \( f : \Gamma \rightarrow \mathfrak{g} \) such that
\[
f(m_i) \in \tilde{C}_{s_i}, \quad f(a_j^{-1}) = -f(a_j), \quad f(b_j^{-1}) = -f(b_j), \quad (2.1)
\]
and the following condition holds:
\[
\prod_{i=1}^{g} \exp(f(a_i)) \exp(f(b_i)) \exp(-f(a_i)) \exp(-f(b_i)) \prod_{j=1}^{n} \exp(f(m_j)) = e. \quad (2.2)
\]

We will construct an action of \( G \) on \( A \). For any \( h \in G \) and \( f \in A \), define
\[
(h \circ f)(a_j) := \text{Ad}_h(f(a_j)), \quad (h \circ f)(b_j) := \text{Ad}_h(f(b_j)),
\]
\[
(h \circ f)(m_i) := \text{Ad}_h(f(m_i)).
\]
Note that the condition in (2.2) is preserved by this action of \( G \). Thus writing explicitly,
\[
\prod_{i=1}^{g} \exp((h \circ f)(a_i)) \exp((h \circ f)(b_i)) \exp(-(h \circ f)(a_i)) \exp(-(h \circ f)(b_i)) \prod_{j=1}^{n} \exp(h \circ f)(m_j) = e.
\]
This produces an action of \( G \) on \( A \) where any \( h \in G \) acts as \( f \mapsto h \circ f \). Define the quotient space for this action of \( G \) on \( A \)
\[
W := \frac{A}{G}
\]
and the quotient map
\[
\psi : A \rightarrow W. \quad (2.3)
\]
For any given \( f \in A \), by sending each \( a_j, b_j, m_i \in \Gamma \) to \( \exp(f(a_j)), \exp(f(b_j)), \exp(f(m_i)) \) respectively, we obtain a homomorphism from \( \Gamma \) to \( G \). Define
\[
\mathcal{R} := \left\{ \rho \in \text{Hom}(\Gamma, G) \mid \rho(m_i) \in C_{s_i} \right\} \subset \frac{\text{Hom}(\Gamma, G)}{G},
\]
where \( C_{s_i} = \exp(\tilde{C}_{s_i}) \). Let
\[
H : W \rightarrow \mathcal{R} \quad (2.4)
\]
be the map defined by
\[ H(f)(a_j) := \exp(f(a_j)), \quad H(f)(b_j) := \exp(f(b_j)), \]
\[ H(f)(m_i) := \exp(f(m_i)) \in C_{S_j}, \]
where \( f \in W. \)

Let \( B \) be the \( G \)-invariant symmetric bilinear 2-form on \( g \) defined by the Killing form. Let \( E \rightarrow P \) be a flat principal \( G \)-bundle. Let \( \iota : P \hookrightarrow M \) be the inclusion map. The \( G \)-invariant bilinear form \( B \) induces a bilinear form on the adjoint vector bundle \( \text{ad}(E) \) of \( E \). The connection on \( E \) induces a connection on \( \text{ad}(E) \). Define the direct image \( t_* \Sigma(\text{ad}(E)) \rightarrow M \).

Let \( \omega \) be the space of all flat connections on \( E \). The space \( \mathcal{R} \) is the space of all isomorphism classes of \( (E, \nabla) \), where \( \nabla \in \omega \) satisfying the condition that for any \( s_i \in S \) and any loop \( c_i \) around \( s_i \),

\[ \exp \int_{c_i} \nabla \in C_{S_i} \]

(\( C_{S_i} \) is the fixed conjugacy class associated to \( s_i \)). The tangent space at any \( \rho \in \mathcal{R} \) is given by

\[ T_{\rho} \mathcal{R} = H^1(M, t_* \Sigma(\text{ad}(E))) \]

(see [3]). Hence

\[ T_{\rho} \mathcal{R} \otimes T_{\rho} \mathcal{R} = H^1(M, t_* \Sigma(\text{ad}(E))) \otimes H^1(M, t_* \Sigma(\text{ad}(E))) \rightarrow H^2(M, t_* \Sigma(\text{ad}(E))) \]

\[ \otimes t_* \Sigma(\text{ad}(E)). \]

As \( B \) is \( G \)-invariant, we have a bilinear pairing

\[ \widetilde{B} : t_* \Sigma(\text{ad}(E)) \otimes t_* \Sigma(\text{ad}(E)) \rightarrow \mathbb{R}, \]

and hence

\[ H^2(M, t_* \Sigma(\text{ad}(E))) \otimes t_* \Sigma(\text{ad}(E)) \rightarrow H^2(M, \mathbb{R}) = \mathbb{R}. \]

The natural symplectic form \( \Omega_0 \) on \( \mathcal{R} \) is given by the homomorphism

\[ T_{\rho} \mathcal{R} \otimes T_{\rho} \mathcal{R} \rightarrow \mathbb{R} \]

obtained by combining the above two homomorphisms [3,1]. We define the form

\[ \Omega := H^* \Omega_0 \]

on \( W \), where \( H \) is constructed in (2.4).

Next we describe the tangent space of \( A \). Let us define for any \( a_j, b_j \) or \( m_i \), and \( f \in A, \)

\[ g(f(a_j)) := [f(a_j), g] \subset g, \quad g(f(b_j)) := [f(b_j), g] \subset g, \]
\[ g(f(m_i)) := [f(m_i), g] \subset g, \]

where \( 1 \leq j \leq g \) and \( 1 \leq i \leq n \). Let \( V_f \) be the space of all functions \( v : \Gamma \rightarrow g \) satisfying the following conditions:
\( v(a_j) \in \mathfrak{g}(f(a_j)) \),
\( v(b_j) \in \mathfrak{g}(f(b_j)) \),
\( v(m_i) \in \mathfrak{g}(f(m_i)) \),
\( \sum_{\mu=1}^{4g+n} \text{Ad}_{f_\mu}(v(\alpha_\mu)) = 0 \),

where \( f_\mu \) is defined as follows:
\[
\begin{align*}
f_1 &= \exp f(a_1), \\
f_2 &= \exp f(a_1) \exp f(b_1), \\
f_3 &= \exp f(a_1) \exp f(b_1) \exp(-f(a_1)), \\
f_4 &= \exp f(a_1) \exp f(b_1) \exp(-f(a_1)) \exp(-f(b_1)), \\
\vdots \\
f_{4g} &= \prod_{j=1}^{g} \exp f(a_j) \exp f(b_j) \exp(-f(a_j)) \exp(-f(b_j)), \\
f_{4g+k} &= f_{4g} \prod_{i=1}^{n} \exp f(m_i), \quad 1 \leq k \leq n,
\end{align*}
\]

and \( \alpha_1 = a_1, \alpha_2 = b_1, \alpha_3 = a_1^{-1}, \alpha_4 = b_1^{-1}, \alpha_5 = a_2, \ldots, \alpha_{4g} = b_1^{-1}, \alpha_{4g+i} = m_i, \ 1 \leq i \leq n \).

**Proposition 2.1.** The tangent space \( T_f A \) is identified with \( V_f \) defined above.

**Proof.** As \( \mathfrak{g} \) is a Lie algebra, the conditions (1) and (2) are obtained. Also from (2.1) we know that \( f(m_i) \) is in the conjugacy class \( C_{s_i} \), hence under the Lie bracket it remains in \( C_{s_i} \); so (2.1) leads to condition (3). On the other hand, (2.2) is preserved under the adjoint action, hence (2.2) leads to the condition (4). Consequently, \( V_f \) gets identified with \( T_f A \). \( \Box \)

In view of Proposition 2.1, we have the following identity (see [1, (3.5)] and [1, Theorem 1]):

**Proposition 2.2.**
\[
\sum_{\mu=1}^{n+4g} B\big(f(\alpha_\mu), [v(\alpha_\mu), w(\alpha_\mu)]\big) = (\psi^* H^* \Omega_0)(v, w),
\]

where \( \alpha_1 = a_1, \alpha_2 = b_1, \alpha_3 = a_1^{-1}, \alpha_4 = b_1^{-1}, \alpha_5 = a_2, \ldots, \alpha_{4g+i} = m_i, \ 1 \leq i \leq n \), \( \Omega_0 \) is the canonical symplectic form on \( \mathcal{R} \) and \( v, w \in V_f \).

For \( g = 0 \), this was proved in [2, Proposition 2.2].

Let us construct a \((4g + n)\)-sided geodesic polygon from a given \( f \in W \). Vertices of the polygon are:
\[
\begin{align*}
f_1 &= \exp(f(a_1)), \\
f_2 &= \exp(f(a_1)) \exp(f(b_1)), \\
f_3 &= \exp(f(a_1)) \exp(f(b_1)) \exp(-f(a_1)), \\
\vdots \\
f_{4g} &= \prod_{j=1}^{g} \exp(f(a_j)) \exp(f(b_j)) \exp(-f(a_j)) \exp(-f(b_j)), \\
f_{4g+k} &= f_{4g} \prod_{i=1}^{n} \exp(f(m_i)), \quad 1 \leq k \leq n.
\end{align*}
\]
\[ f_4 = \exp(f(a_1)) \exp(f(b_1)) \exp(-f(a_1)) \exp(-f(b_1)), \]
\[
\vdots \]
\[ f_{4g} = \prod_{j=1}^{g} \exp(f(a_j)) \exp(f(b_j)) \exp(-f(a_j)) \exp(-f(b_j)), \]
\[ f_{4g+k} = f_{4g} \prod_{i=1}^{n} \exp(f(m_i)), \quad 1 \leq k \leq n. \quad (2.7) \]

So from (2.2) we have
\[ f_{4g+n} = e. \]

Following the construction of [2], we set \( f_0 = e \). For any \( 1 \leq \mu \leq 4g + n \), the edge that connects the vertices \( f_{\mu - 1} \) and \( f_{\mu} \) will be denoted by \( l^\mu_{\mu - 1} \). Each edge is the geodesic segment defined as follows:
\[
\begin{align*}
    l^0_0 &: t \mapsto \exp(tf(a_1)), \\
    l^1_0 &: t \mapsto f_1 \exp(tf(b_1)), \\
    l^2_0 &: t \mapsto f_2 \exp(-tf(a_1)), \\
    l^3_0 &: t \mapsto f_3 \exp(-tf(b_1)), \\
    l^4_0 &: t \mapsto f_4 \exp(tf(a_2)), \\
    \vdots \\
    l^{4g}_{4g-1} &: t \mapsto f_{4g-1} \exp(-tf(b_g)), \\
    l^{4g+k}_{4g+k-1} &: t \mapsto f_{4g+k-1} \exp(tf(m_k)), \quad 1 \leq k \leq n, \quad (2.8)
\end{align*}
\]

where \( t \in [0, 1] \).

Let us denote the space of all such geodesic \( N \)-gons by \( \mathcal{D}(N) \). The polygon described above in (2.7) and (2.8) is a geodesic \((4g + n)\)-gon in \( G \). So we have a map
\[
\phi : W \longrightarrow \mathcal{D}(4g + n) \quad (2.9)
\]
that sends any \( f \in W \) to the polygon described in (2.8).

We recall the construction of the 2-form \( \theta \) on \( \mathcal{D}(N) \) (see [2]). Let \( \theta_0 \) be the trilinear form on \( g \) given by
\[
\theta_0 : g \otimes g \otimes g \longrightarrow \mathbb{R}; \quad (a, b, c) \mapsto B(a, [b, c]). \quad (2.10)
\]
As \( B \) is \( G \)-invariant, we get a 3-form \( \tilde{\theta} \) on \( G \) using translations
\[
(g', \theta_0) = \tilde{\theta}(g') \quad (2.11)
\]
where \( g' \in G \). Let \( \tau \) be a geodesic \( N \)-gon on \( G \). A tangent vector \( v \in T_\tau \mathcal{D}(N) \) is given by Jacobi fields along each \( l^i_{i-1} \), where \( l^i_{i-1} \) is the edge (geodesic line segment) joining vertices \( i-1 \) and \( i \). Take \( v, w \in T_\tau \mathcal{D}(N) \); let \( \tilde{v} \) and \( \tilde{w} \) be the corresponding sections of \( \tau^* T \mathcal{D}(N) \).

Now \( \tau^* \tilde{\theta}(v, w) \) gives a piece-wise smooth 1-form on \([1, N]\). Denoting integral of this 1-form on \([1, N]\) by \( \theta(v, w) \in \mathbb{R} \), we have a 2-form \( \theta \) on \( \mathcal{D}(N) \), which is given by
\[
\theta : (v, w) \mapsto \theta(v, w) \tag{2.12}
\]
for any \(v, w \in T_\tau \mathcal{D}(N)\).

Let \(g' \in G\) and \(\alpha \in g\). Consider the geodesic segment
\[
l : [0, 1] \longrightarrow G
\]
defined by \(t \mapsto g' \exp(t\alpha)\). For any \(\beta, \gamma \in g\), let \(\bar{\beta}\) and \(\bar{\gamma}\) be the corresponding Jacobi field along \(l\). Contracting \(\bar{\theta}\) (constructed in (2.11)) by \(\bar{\beta}\) and \(\bar{\gamma}\) we obtain a 1-form on \([0, 1]\). Denote this 1-form by \(\sigma(\beta, \gamma)\). We have
\[
\int_0^1 \sigma(\beta, \gamma) = \theta_0(\alpha, \beta, \gamma) \tag{2.13}
\]
(see [2, Proposition 2.3]); so, from the definition of \(\theta_0\) in (2.10),
\[
\int_0^1 \sigma(\beta, \gamma) = B(\alpha, [\beta, \gamma]).
\]

Let \(\tau\) be a geodesic \((4g + n)\)-gon as described in (2.8) and (2.7). For each edge \(l_\mu^{\mu-1}\), \(1 \leq \mu \leq 4g + n\), we apply (2.13); from Proposition 2.2,
\[
\phi^* \theta \big|_{l_\mu^{\mu-1}} = \psi^* H^* \Omega_0 \big|_{l_\mu^{\mu-1}},
\]
where \(\phi\) is defined in (2.9). Thus we have proven:

**Theorem 2.1.** For a surface of genus \(g\) with \(n\) punctures, if the symplectic form on \(\mathcal{R}\) is \(\Omega_0\) and \(\phi, \psi, H\) and \(\theta\) are defined as in (2.9), (2.3), (2.4) and (2.12) respectively, then
\[
\psi^* H^* \Omega_0 = \phi^* \theta.
\]

**References**


