More ZCPS-Wh(v) and several new infinite classes of Z-cyclic whist tournaments

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Abstract

A Z-cyclic whist tournament on v players having the property that the collection of initial round partner pairs form the patterned starter in \( Z_N \) is known as a Z-cyclic patterned starter whist tournament on v players, ZCPS-Wh(v). It is the case that a ZCPS-Wh(4) has been known for over 100 years. In this study we present for the first time examples of ZCPS-Wh(4n), \( n > 1 \).

Also presented is complete information regarding ZCPS-Wh(v) for all \( v \equiv 0, 1 \pmod{4} \), \( v \leq 48 \). A connection between ZCPS-Wh(4n+1) and (4n+1, 4, 1)-resolvable perfect Mendelsohn designs is established. A Z-cyclic Wh(148), a new result, is presented and used to establish several new infinite classes of Z-cyclic whist tournaments. © 1998 Elsevier Science B.V. All rights reserved.

1. Introduction

A whist tournament on v players, Wh(v) is a resolvable \((v, 4, 3)\)-BIBD having the interpretation that each block \((a, b, c, d)\) represents a game in which the partnership \(\{a, c\}\) opposes that of \(\{b, d\}\). The design is subject to the (whist) conditions that each player partners every other player exactly once and opposes every other player exactly twice. Wh(v) are known to exist for all \( v \equiv 0, 1 \pmod{4} \), \( v \geq 4 \) [1, 5, 14]. For a recent survey of the whist tournament problem see [2]. A Wh(v) is said to be Z-cyclic if (a) the players are elements in \( Z_N \cup A \) where if \( v \equiv 0 \pmod{4} \) then \( N = v - 1, A = \{\infty\} \) and if \( v \equiv 1 \pmod{4} \) then \( N = v, A = \emptyset \) and (b) the rounds are obtained by development modulo \( N \) from some "initial round". When \( \infty \) is present one has \( \infty + 1 = \infty \). If \( v \equiv 0 \pmod{4} \) it is conventional to designate the round in which \( \infty \) and 0 are partners as initial round and if \( v \equiv 1 \pmod{4} \) the initial round is conventionally that which omits 0. When \( N \) is an odd integer the patterned starter in \( Z_N \) is the set \( \{\{x, -x\}: x \in Z_N, x \neq 0\} \). A Z-cyclic Wh(v) for which the initial round partner pairs form the patterned starter is referred to as a ZCPS-Wh(v). Naturally, when \( v \equiv 0 \pmod{4} \) there

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is the additional partner pair \( \{ \infty, 0 \} \). ZCPS-\( Wh(v) \) were first studied in [9] and a large number of additional examples can be found in [15]. All of the results in [9, 15] relate to \( v \equiv 1 \pmod{4} \). In \( Z_4 \) the patterned starter consists of the single set \( \{1, 2\} \) and the initial round of a ZCPS-\( Wh(4) \) is \( (\infty, 1, 0, 2) \). Heretofore this was the only known example of a ZCPS-\( Wh(4n) \). In Examples 1.1–1.4 we present ZCPS-\( Wh(4n) \) for \( n = 7, 10, 19, 37 \). In each case the method of differences [1] can be employed to verify that the blocks (tables, games) listed constitute the initial round of a \( Wh(v) \). All of these results were obtained via the use of a computer. The cases \( n = 7 \) and 10 arose as part of an exhaustive search, the full results of which are discussed in Section 2. The cases \( n = 19 \) and 37 were found by employing hill-climbing methods.

**Example 1.1.** ZCPS-\( Wh(28) \).

\[
(\infty, 3, 0, 24), \ (4, 5, 23, 22), \ (6, 10, 21, 17), \ (11, 9, 16, 18),
\]

\[
(12, 7, 15, 20), \ (8, 2, 19, 25), \ (13, 1, 14, 26).
\]

**Example 1.2.** ZCPS-\( Wh(40) \).

\[
(\infty, 13, 0, 26), \ (1, 6, 38, 33), \ (2, 16, 37, 23), \ (3, 5, 36, 34),
\]

\[
(4, 19, 35, 20), \ (7, 10, 32, 29), \ (8, 12, 31, 27), \ (9, 18, 30, 21),
\]

\[
(11, 17, 28, 22), \ (14, 15, 25, 24).
\]

**Example 1.3.** ZCPS-\( Wh(76) \).

\[
(\infty, 25, 0, 50), \ (17, 20, 58, 55), \ (4, 15, 71, 60), \ (13, 27, 62, 48),
\]

\[
(2, 30, 73, 45), \ (3, 12, 72, 63), \ (8, 10, 67, 65), \ (24, 31, 51, 44),
\]

\[
(7, 19, 68, 56), \ (5, 26, 70, 49), \ (14, 22, 61, 53), \ (6, 36, 69, 39),
\]

\[
(18, 28, 57, 47), \ (34, 35, 41, 40), \ (9, 32, 66, 43), \ (1, 23, 74, 52),
\]

\[
(29, 33, 46, 42), \ (21, 37, 54, 38), \ (11, 16, 64, 59).
\]

**Example 1.4.** ZCPS-\( Wh(148) \).

\[
(\infty, 49, 0, 98), \ (34, 69, 113, 78), \ (22, 45, 125, 102), \ (28, 36, 119, 111),
\]

\[
(13, 33, 134, 114), \ (12, 66, 135, 81), \ (32, 54, 115, 93), \ (29, 61, 118, 86),
\]

\[
(48, 11, 99, 136), \ (6, 68, 141, 79), \ (72, 62, 75, 85), \ (20, 23, 127, 124),
\]

\[
(18, 27, 129, 120), \ (63, 56, 84, 91), \ (7, 43, 140, 104), \ (2, 50, 145, 97),
\]

\[
(26, 53, 121, 94), \ (30, 47, 117, 100), \ (35, 65, 112, 82), \ (10, 31, 137, 116),
\]

\[
(16, 17, 131, 130), \ (52, 57, 95, 90), \ (58, 73, 89, 74), \ (4, 15, 143, 132),
\]
In Section 2 we present, via orbits, the totality of ZCPS-Wh(28) and ZCPS-Wh(40). A summary of what is known about ZCPS-Wh(v) is also presented in Section 2. In Section 3 we develop a connection between ZCPS-Wh(4n + 1) and (4n + 1, 4, 1)-resolvable perfect Mendelsohn designs. There we also prove that no Z-cyclic (4n, 4, 1)-resolvable perfect Mendelsohn design can exist. In Section 4 we establish the existence of Z-cyclic Wh(3 · 7^2 · p^m + 1) for certain primes p \equiv 1 (mod 4) and for all m \geq 0.

2. Orbits

Consider the ring Z_N with N as described in Section 1. Each additive generator g in Z_N induces a mapping Φ_g: Z_N \rightarrow Z_N defined by Φ_g(x) = m whenever xg = m (mod N). It is well known that as g ranges over all additive generators in Z_N the set of all such maps Φ_g is a group, say G(N). It is also well known that |G(N)| = φ(N) where φ is the Euler phi function. If each Φ_g \in G(N) is extended to Z_N \cup \{∞\} by defining Φ_g(∞) = ∞ then it is straightforward to see that the image under Φ_g of any Wh(v) is again a whist tournament on v players. Thus for a given v and Wh(v) the orbit associated with Wh(v) is the set \{Φ_g(Wh(v)): Φ_g \in G(N)\}. If we denote by ZCPS-WHIST(v) the set of all ZCPS-Wh(v) then the action of G(N) on this set partitions it into orbits since if Φ_g(x) = m then Φ_g(−x) = −m. Furthermore, if g is coprime with N then so is −g. Consequently, if Wh(v) is a ZCPS-Wh(v) and if Φ_g(Wh(v)) = Wh*(v) then Φ_{−g}(Wh(v)) = Wh*(v). We conclude that the maximum order of any orbit is φ(N)/2. We list the orbits of the ZCPS-Wh(28) and ZCPS-Wh(40).

v = 28. Note that |G(27)| = φ(27) = 18. It is the case that |ZCPS-WHIST(28)| = 27 and there are three orbits each of order 9. In the listing below of the orbit representatives of the tables (a, b, −a, −b) is written in the abbreviated form (a, b). Similarly (∞, a, 0, −a) is condensed to (∞, a).

<table>
<thead>
<tr>
<th>Orbit</th>
<th>Representative</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(∞, 3), (1, 13), (2, 8), (4, 5), (6, 10), (7, 12), (9, 11)</td>
</tr>
<tr>
<td>2</td>
<td>(∞, 3), (1, 8), (2, 13), (4, 6), (5, 9), (7, 12), (10, 11)</td>
</tr>
<tr>
<td>3</td>
<td>(∞, 3), (1, 13), (2, 7), (4, 6), (5, 11), (8, 12), (9, 10)</td>
</tr>
</tbody>
</table>

v = 40. Here |G(39)| = 24 and |ZCPS-WHIST(40)| = 96. There are 8 orbits each of order 12. We use the same convention for the presentation of the orbit representatives as in the case v = 28.
We summarize what is presently known about ZCPS-\(Wh(v)\). In [9] all ZCPS-\(Wh(v)\) for \(v = 5, 13, 17, 25, 29, 37, 41\) are presented. Listed above are all ZCPS-\(Wh(v)\) for \(v = 4, 28, 40\). From [7] one can conclude that there are no ZCPS-\(Wh(v)\) for \(v = 8, 9, 12, 16, 20, 21\). Via exhaustive computer search we have determined that there are no ZCPS-\(Wh(v)\) for \(v = 24, 32, 33, 36, 44, 45, 48\). Thus everything is known about ZCPS-\(Wh(v)\) for all \(v \equiv 0, 1 \pmod{4}\), \(v \leq 48\). If \(q\) is a prime, \(q \equiv 3 \pmod{4}\), then ZCPS-\(Wh(q^2)\) for all \(7 \leq q \leq 31\) is established in [15]. Leonard has extended these results out to \(q \leq 499\) [3] but the associated data are as yet unpublished. For \(i = 1, \ldots, n\) let \(p_i\) denote a prime \(p_i \equiv 1 \pmod{4}\). The \(Wh(\prod_{i=1}^{n} p_i^{\alpha_i})\) with \(\alpha_i \geq 1, \alpha_i > 0, 2 \leq i \leq n\) presented in [8] are all ZCPS-\(Wh(v)\). Finally, there are the ZCPS-\(Wh(76), ZCPS-Wh(148)\) given in Examples 1.3 and 1.4 respectively.

3. A connection between ZCPS-\(Wh(4n + 1)\) and Mendelsohn designs

We begin with several definitions [6].

**Definition 3.1.** In the cyclic \(k\)-tuple \((a_1, \ldots, a_k)\), \(a_i, a_{i+t}\) are said to be \(t\)-apart, where \(i + t\) is taken modulo \(k\).

**Definition 3.2.** Let \(v\) and \(\lambda\) be positive integers and \(K\) a set of positive integers. A \((v, K, \lambda)\)-Mendelsohn design, \((v, K, \lambda)\)-MD, is a pair \((X, B)\) where \(X\) is a \(v\)-set (of points) and \(B\) is a collection of cyclic \(k\)-tuples of \(X\) (called blocks) with \(k \in K\) such that every ordered pair of points of \(X\) are consecutive in exactly \(\lambda\) of the blocks of \(B\). When \(K = \{k\}\), the design is denoted by \((v, k, \lambda)\)-MD. If for any \(t, 1 \leq t \leq k - 1\) each ordered pair of points appears \(t\)-apart in exactly \(\lambda\) blocks of \(B\) the design is called **perfect** and is denoted by \((v, k, \lambda)\)-PMD. If the design is resolvable then we write \((v, k, \lambda)\)-RMD (or \((v, k, \lambda)\)-RPMD).

If \((a, b, c, d)\) is a table of a \(Wh(v)\) then it is convenient to think of the players as sitting round a table with partner pairs sitting at the N–S and E–W positions of the compass. Obviously, \((a, b, c, d)\) can be considered as a cyclic set and one can speak
of such things as left-hand opponent and right-hand opponent. For instance, b is a's left-hand opponent and c's right-hand opponent.

**Definition 3.3.** A directed whist tournament on \( v \) players, \( DWh(v) \), is a \( Wh(v) \) that satisfies the additional conditions that every player opposes every other player exactly once as a left-hand opponent and exactly once as a right-hand opponent.

**Theorem 3.4** (Bennett and Zhu [6]). If \( v \equiv 0, 1 \pmod{4} \) then a \( DWh(v) \) is equivalent to a \((v, 4, 1)\)-RPMD.

Certainly if one considers \((a, b, c, d)\) as a cyclic set then the pairs of elements that are 1-apart are \(\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}\). Those that are 2-apart are \(\{a, c\}, \{b, d\}, \{c, a\}, \{d, b\}\) and those that are 3-apart are \(\{a, d\}, \{b, a\}, \{c, b\}, \{d, c\}\). Thus, if our design is Z-cyclic one can use differences to test the design. In this case, the 3-apart differences are the negatives of the 1-apart differences and depending upon orientation if one says that the 1-apart differences test for left-hand opponents then the 3-apart differences test for right-hand opponents. Clearly, the 2-apart differences test for partners.

**Theorem 3.5.** Every ZCPS-\( Wh(4n + 1) \) is a \((Z\text{-cyclic})\) \( DWh(4n + 1) \).

**Proof.** Let the initial round of the ZCPS-\( Wh(4n + 1) \) be given by \((a_i, b_i, -a_i, -b_i), i = 1, \ldots, n\). Because it is a \( Wh(v) \) we know that the partner differences cover \( Z_{4n+1}^* = Z_{4n+1} - \{0\} \) exactly once and the opponent differences cover \( Z_{4n+1}^* \) exactly twice. That is to say

\[
\bigcup_{i=1}^n \{\pm 2a_i\} \cup \bigcup_{i=1}^n \{\pm 2b_i\} = Z_{4n+1}^* \quad \text{and}
\]

\[
\bigcup_{i=1}^n \{\pm (a_i - b_i)\} \cup \bigcup_{i=1}^n \{\pm (a_i + b_i)\} = Z_{4n+1}^*,
\]

where (3.1) actually occurs twice. It suffices to show that the differences of the elements that are 1-apart cover \( Z_{4n+1}^* \). These differences are

\[
\bigcup_{i=1}^n \{b_i - a_i, -(a_i + b_i), -(b_i - a_i), a_i + b_i\},
\]

which, by (3.1) is precisely \( Z_{4n+1}^* \). □

**Corollary 3.6.** Every ZCPS-\( Wh(4n + 1) \) is a \((4n + 1, 4, 1)\)-RPMD.

Thus, the results of [15] extend the knowledge related to the existence of Z-cyclic \((4n + 1, 4, 1)\)-RPMD. The converse of Theorem 3.5 is not true as is illustrated by the \( DWh(17) \) of Example 3.7.

**Example 3.7.** The initial round of a Z-cyclic \( DWh(17) \) is given by \((7, 2, 15, 1), (3, 12, 6, 5), (11, 8, 13, 4), (9, 10, 14, 16)\).
That Theorem 3.5 is not true when \( v \equiv 0 \pmod{4} \) is established in Theorem 3.8. The proof given here is due to Lewis [16]. It is to be noted that the content of Theorem 3.8 appears, without proof, in [5].

**Theorem 3.8.** If \( v \equiv 0 \pmod{4} \) there is no Z-cyclic \( DWh(v) \).

**Proof.** Let \( v = 4n \) and suppose there is a \( Z \)-cyclic \( DWh(4n) \). Let \((\infty, a, 0, b), (a_i, b_i, c_i, d_i), i = 1, \ldots, n - 1 \) be the initial round tables of this \( DWh(4n) \). Since the differences of the elements 1-apart cover \( Z_{4n-1}^* \), the totality of these differences sum to zero. For each \( i = 1, \ldots, n - 1 \) the sum of the 1-apart differences for the table \((a_i, b_i, c_i, d_i)\) is zero whereas for the table \((\infty, a, 0, b)\) the 1-apart differences sum to \( b - a \), which is not zero, a contradiction. \( \square \)

Consequently, our \( ZCPS-Wh(4n) \) offer no new information regarding \( DWh(4n) \) or \((4n,4,1)\)-RPMD.

### 4. Several new infinite classes of \( Z \)-cyclic \( Wh(v) \)

As noted earlier, the tournament of Example 1.4 extends our knowledge of \( ZCPS-Wh(v) \). It does more, as it is the first known example of a \( Z \)-cyclic \( Wh(148) \). In this section we show how it leads to several new infinite classes of \( Z \)-cyclic \( Wh(v) \), \( v \equiv 0 \pmod{4} \). The construction uses the following structure for the rings \( Z_{3q^2p} \) and \( Z_{3q^2p} \) where \( q, p \) are primes, \( p \equiv 1 \pmod{4} \), \( q \equiv 3 \pmod{4} \), \( q \geq 7 \).

Partition \( Z_{3q^2p} \) as follows:

\[
Z_{3q^2p} = P \cup Q \cup T \cup E \cup \{0\}, \\
\tag{4.1}
\]

where \( P = \{x : p \mid x, \ x \neq 0\} \), \( Q = \{x : q \mid x, p \nmid x\} \), \( T = \{x : 3 \mid x, q \nmid x, p \mid x\} \) and \( E = \{x : 3 \nmid x, q \mid x, p \mid x\} \). Note that \( |P| = 3q^2 - 1 \), \( |Q| = 3q(p - 1) \), \( |T| = q(q - 1)(p - 1) \) and \( |E| = 2q(q - 1)(p - 1) \). Furthermore, \( Q = Q_1 \cup Q_2 \) where \( Q_2 = \{x \in Q : q^2 \mid x\} \) and \( Q_1 = Q - Q_2 \). Hence \( |Q_2| = 3(p - 1) \) and \( |Q_1| = 3(q - 1)(p - 1) \).

Partition \( Z_{3q^2p} \) as:

\[
Z_{3q^2p} = P' \cup Q' \cup T' \cup E' \cup \{0\}, \\
\tag{4.2}
\]

with \( P' = \{x : p \mid x, \ x \neq 0\} \), etc. Here \( |P'| = 3q - 1 \), \( |Q'| = 3(p - 1) \), \( |T'| = (q - 1)(p - 1) \) and \( |E'| = 2(q - 1)(p - 1) \).

Now, \( E \) in (4.1) is a group conventionally referred to as the reduced set of residues modulo \( 3q^2p \) and is denoted by \( RSR(3q^2p) \). Let \( W \) be a common primitive root of 3, \( q^2 \) and \( p \). Then \( \text{ord}_{3q^2p} W = q(q - 1)(p - 1)/\text{gcd}(q(q - 1), p - 1) \) and we define \( t \) by \( 4t = \text{ord}_{3q^2p} W \). The set

\[
H = \{1, W, W^2, \ldots, W^{4t-1}\} \cup \{-1, -W, \ldots, -W^{4t-1}\}
\]

198  
is a normal subgroup of $E$ and the factor group $E/H$ has a coset decomposition. Although $T'$ is not a group it is possible to use $H$ to obtain a “coset” decomposition of $T'$ also [12, 13]. Analogous comments apply to $E'$ and $T'$ in (4.2). Theorem 4.1 below is a paraphrase of Theorem 2.3 in [13].

**Theorem 4.1.** Let $x'_1$ be defined by the conditions $p_1^{x'_1 - 1} \parallel (p_2 - 1)$ and let $x'_2$ be an arbitrary integer $\geq 1$. Then for all $x_1 \geq x'_1$, $x_2 \geq x'_2$

(a) $|\text{RSR}(3p_1^{x_1} p_2^{x_2})/H| = |\text{RSR}(3p_1^{x'_1} p_2^{x'_2})/H'|$.

(b) If $C_1, C_2, \ldots, C_h$ and $C'_1, C'_2, \ldots, C'_h$ denote the cosets of $\text{RSR}(3p_1^{x_1} p_2^{x_2})/H$ and $\text{RSR}(3p_1^{x'_1} p_2^{x'_2})/H'$ respectively then $C'_i \subseteq C_i$ for all $i = 1, \ldots, h$.

**Definition 4.2.** A Z-cyclic $Wh(v)$ is said to be partitionable if there is a partition of $Z_N$, say $\bigcup_{i=1}^h A_i$, and an arrangement of the initial round tables into collections $S_1, \ldots, S_h$ such that for each cell of the partition all of the elements in that cell are found in exactly one of the collections $S_i$ and each collection $S_i$ is closed under the operation of subtraction modulo $N$.

**Theorem 4.3.** Let $p, q$ be primes such that $p \equiv 1 \pmod{4}$, $q \equiv 3 \pmod{4}$, $q \geq 7$ and $q \nmid (p - 1)$. Suppose there exists a Z-cyclic $Wh(3q^2 + 1)$ and a partitionable Z-cyclic $Wh(3q^2p)$ corresponding to the partition (4.2) then there exists a (partitionable) Z-cyclic $Wh(3q^2p + 1)$.

**Proof.** Although this theorem itself is original, the proof utilizes ideas and results that have previously been established in the literature. Consequently, we provide only an outline of the proof. Assume that $Z_{3q^2p}$, $Z_{3qp}$ are partitioned as in (4.1), (4.2) respectively. The initial round of the Z-cyclic $Wh(3q^2p + 1)$ is the union of the tables in (1)–(4) below.

1. For the set $P \cup \{0, \infty\}$ take the initial round tables of the Z-cyclic $Wh(3q^2 + 1)$ and multiply each non-\(\infty\) element by $p$.
2. For the set $Q_2$ apply the Moore construction as adapted by Anderson and Finizio [4].
3. The set $Q_1$ corresponds to $T' \cup E'$ via the bijection $x \rightarrow xq$, $x \in T' \cup E'$. Since the $Wh(3q^2p)$ is partitionable take all the initial round tables associated with $T' \cup E'$ and multiply each element by $q$.
4. Since $q \nmid (p - 1)$ the “coset structure” of $T' \cup E'$ is embedded in that of $T \cup E$ [13]. Consequently the initial round tables of the $Wh(3q^2p)$ associated with $T' \cup E'$ when adjusted as elements in $Z_{3q^2p}$ serve as initial round tables for $T \cup E$ (see [13] for details). □

Example 4.4 illustrates the construction contained in the proof of Theorem 4.3.

**Example 4.4.** $Wh(3 \cdot 7^2 \cdot 5 \mid 1) = Wh(736)$. 

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Construct the following tables:

1. Take the tables of the $W\ell(148)$ of Example 1.4 and multiply each non-$\infty$ element by 5.

2. For the set $Q_2$ use the Anderson–Finizio adaptation of the Moore construction [4]. Let $W$ be a common primitive root of $3$, $q^2$ and $p$. Define $\tau$ by $4\tau = p - 1$ and let $S \in Z_{3q^2p}$ be such that $S \equiv W^\tau (\text{mod } p)$, $S \equiv 2 (\text{mod } 3)$. Define $\alpha_{fg}$, $\alpha_{fgh}$ by:

$$
\alpha_{fg} = q^2 S_{f-1} W_{g-1} (\text{mod } 3q^2p), \quad f = 1, 2, 3, 4; \quad g = 1, \ldots, \tau
$$

$$
\alpha_{fgh} = (\alpha_{fg} + hq^2p) (\text{mod } 3q^2p), \quad h = 0, 1, 2.
$$

The tables

$$
(x_{1g0}, x_{3g0}, x_{2g0}, x_{4g0}), (x_{1g1}, x_{4g1}, x_{3g1}, x_{2g1}), (x_{1g2}, x_{2g2}, x_{4g2}, x_{3g2}), \quad g = 1, \ldots, \tau
$$

serve as initial round tables for $Q_2$. For the present example $\tau = 1$ and the choice $W = 17$ yields $S = 17$ and the tables corresponding to (4.3) are

$$
(49, 196, 98, 392), \quad (294, 637, 441, 343), \quad (539, 588, 147, 686).
$$

3. For $q = 7$, $p = 5$ it is shown in [11] that the tables

$$
(1, yW, 3, -3) \text{ times } 1, W, W^2, \ldots, W^{4r-1},
$$

$$
(-1, -W, -yW, -yW^2) \text{ times } 1, W^2, W^4, \ldots, W^{4r-2},
$$

with $W = 17$, $y = 2$ constitute a collection of initial round tables for $T' \cup E'$. Since 17 is a primitive root of $7^2$, the initial round tables for $Q_1$ are obtained by multiplying each entry of (4.4) and (4.5) by 7. Thus,

$$
(7, 238, 21, 714) \text{ times } 1, 17, 17^2, \ldots, 17^{11},
$$

and

$$
(728, 616, 497, 364) \text{ times } 1, 17^2, \ldots, 17^{10}.
$$

4. For the set $T' \cup E$ we adjust tables (4.4), (4.5) so that the additive inverses are taken modulo 735 and consider the appropriate value of "$r" to reflect the order of $W$ in $Z_{735}$.

$$
(1, 34, 3, 732) \text{ times } 1, 17, 17^2, \ldots, 17^{83},
$$

$$
(734, 718, 701, 157) \text{ times } 1, 17^2, \ldots, 17^{82}.
$$

**Theorem 4.5.** If in addition to the hypotheses of Theorem 4.3 we assume that $p \nmid (q - 1)$ then there exists a $Z$-cyclic $W\ell(3q^2p^n + 1)$ for all $n \geq 1$.

**Proof.** The proof is by induction on $n$. The case $n = 1$ is handled by Theorem 4.3. Assume the theorem true for $n - 1$ and consider the $n$ case, $n > 1$. Partition $Z_{3q^2p^n}$ exactly
as in (4.1) except now $|P| = 3q^2 p^{n-1} - 1$, $|Q| = 3q^p p^{n-1}(p - 1)$, $|T| = q(q - 1) p^{n-1}(p - 1)$, $|E| = 2q(q - 1) p^{n-1}(p - 1)$. Further we write $Q = Q_1 \cup Q_2$ with $|Q_2| = 3p^{n-1}(p - 1) + 1$ and $|Q_1| = 3(q-1)p^{n-1}(p-1)$. The construction of the initial round of the Z-cyclic $Wh(3q^2 p^n + 1)$ follows the format provided by (1)-(4) in the proof of Theorem 4.3 with the induction hypothesis supplying the tables for (1), (3), (4) and the Anderson–Finizio methodology [4] supplying the tables for (2)-(4) of the former is justified by Theorem 4.1 since $p + (q - 1)$. 

Let $A$ denote the following set of primes:


**Corollary 4.6.** Z-cyclic $Wh(3q^2 p^n + 1)$ exist for all $n \geq 1$ for $q = 7$ and $p \in A$.

**Proof.** For $q = 7$ and $p \in A$ the ingredients necessary for (3), (4) of the proofs of Theorems 4.3, 4.5 are presented in [10, 11]. 

**References**


[11] N.J. Finizio, Several special cases wherein the existence of $Z$ cyclic whist tournaments in $Z_{3q^n}$ guarantees $Z$-cyclic $Wh(3q^p n)$, $n \geq 1$, Bull. ICA 16 (1996) 49–64.
