How Often Is the Number of Divisors of $n$ a Divisor of $n$?

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Let $d(n)$ denote the number of positive integers dividing the positive integer $n$. We show that as $x$ approaches $\infty$, $\# \{ n \leq x : d(n) \text{ divides } n \} = (x/\sqrt{\log x})(\log \log x)^{-1+o(1)}$.

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1. INTRODUCTION

Let $d(n)$ denote the number of positive integers dividing the positive integer $n$. Here, we establish the following result:

**THEOREM 1.** $\# \{ n \leq x : d(n) \text{ divides } n \} = (x/\sqrt{\log x})(\log \log x)^{-1+o(1)}$.

In addition, we obtain an upper bound which is sharper than the upper estimate implied by Theorem 1:

**THEOREM 2.** Let $\{ \xi_k \}_{k=0}^{\infty}$ be defined recursively by

$$
\xi_0 = 0, \quad \xi_k = \xi_{k-1} + 2^\xi_{k-1} \quad \text{for } k \geq 1.
$$

Then there is a constant $C$ such that

$$
\# \{ n \leq x : d(n) \text{ divides } n \} \leq C x \sqrt{\log \log \log \log x \xi(x)} / \sqrt{\log x \log \log x}
$$

for $x \geq e^{16}$, where $\xi(x) = \# \{ k \geq 0 : \xi_k \leq x \}$.

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We remark that the first few terms of the sequence \{\xi_k\} are 0, 1, 3, 11, 2059, 2059 + 2^{2059}. The function \xi(x) grows extremely slowly—more slowly than any fixed number of iterated logarithms of x. In fact, when x is Skewes’ number \(10^{10^{10^{34}}}\), \(\xi(x)\) is merely 7.

In another paper [9] (see also, Theorem 4.4.1 of [6]), we establish that for every positive integer \(N\), there are computable constants \(R_1 > 0, R_2, \ldots, R_{N-1}\), for which

\[
\# \{n \leq x: d(n)|n + 1\} = \sum_{l=1}^{N-1} R_l x (\log x)^{(1/2)-l} + O_N(x)(\log x)^{(1/2) - N}). \tag{1}
\]

This result is more precise than Theorem 1, and shows that \(d(n)\) divides \(n + 1\) more often than \(d(n)\) divides \(n\). We remark that the arithmetic independence of \(n\) and \(n + 1\) enables us to obtain more precise estimates for \# \(\{n \leq x: d(n)|n + 1\}\) than we can get for \# \(\{n \leq x: d(n)|n\}\). Moreover, the fact that \(n\) and itself are not independent, whereas \(n\) and \(n + 1\) are arithmetically independent, explains why the right side of the estimate in Theorem 1 is smaller than the right side of (1) (for a fuller explanation, the reader is referred to the proof of Theorem 1).

Finally, we note that while nobody but the author appears to have studied how often \(d(n)\) divides \(n\), Bateman, Erdös, Pomerance, and Straus [1] have estimated \# \(\{n \leq x: d(n)|\sigma(n)\}\) and \# \(\{n \leq x: (d(n))^2|\sigma(n)\}\), where \(\sigma(n)\) denotes the sum of the positive integers dividing \(n\). Also, Scourfield [5] and the author [7] have obtained estimates for \# \(\{n \leq x: (d(n), n) = 1\}\).

2. NOTATION

Throughout this paper, \(p\) denotes a prime, and \(n\) and \(t\) are reserved for positive integers. The expression \(f(x) \sim g(x), f(x) = O(g(x))\), and \(f(x) = o(g(x))\) have their usual meanings. A sum or product of the form \(\sum_p\) or \(\prod_p\), respectively, denotes a sum or product over primes; thus, for example, \(\prod_{\substack{p|n \atop p \leq N}} (1 + 1/p)\) represents the product of \(1 + 1/p\), taken over all primes \(p\) dividing \(n\). Similarly, a sum of the form \(\sum_{n \leq x}\) is assumed to extend over all positive integers \(n\) not exceeding \(x\).

Other common notation frequently employed in this paper is summarized in the following table:
3. Preliminary Results

DEFINITION. We call a positive integer \( t \) squarefull if every prime \( p \mid t \) satisfies \( p^2 \mid t \).

**Lemma 1.** For \( y \geq 1 \), we have

\[
\sum_{t \leq y, \text{squarefull}} 1 \ll \sqrt{y},
\]
\[
\sum_{t \geq y, \text{squarefull}} t^{-1} \ll \frac{1}{\sqrt{y}}.
\]

**Proof.** The first inequality follows from a result of Erdös and Szekeres [2], and the second inequality can be obtained readily from the first one by applying partial summation. \( \square \)

**Lemma 2.** Let \( B > 0, C \in (0, 2) \) be fixed. If \( 0 \leq b \leq B, 0 \leq c \leq C, \) and \( k \in \mathbb{Z}^+ \), then

\[
\prod_{p \mid k} \left( 1 + \frac{b}{p} \right) \ll (L(3k))^b, \prod_{p \mid k} \left( 1 - \frac{c}{p} \right)^{-1} \ll (L(3k))^b.
\]

uniformly in \( b, c, \) and \( k \).
Proof. We will prove the first inequality; the second inequality can be deduced from the first inequality, or proved in a similar manner. The inequality \( \log(1 + x) < x \) for \( x > 0 \) implies that

\[
\prod_{p | k} \left(1 + \frac{b}{p}\right) = \exp \sum_{p | k} \log \left(1 + \frac{b}{p}\right) < \left(\exp \sum_{p | k} \frac{1}{p}\right)^b.
\]

(2)

It is enough to prove the lemma for \( k > 1 \). Let \( q \) denote the \( \omega(k) \)th prime. Since \( 1/p \) decreases with \( p \), we have

\[
\sum_{p | k} \frac{1}{p} \leq \sum_{p < q} \frac{1}{p} \leq L_2(3q) + c_1.
\]

(3)

Moreover, from the definition of \( q \), the fact that \( \log p \) increases with \( p \), and Chebyshev's inequality for \( \pi(x) \), we can conclude that

\[
q < c_2 \sum_{p < q} \log p \leq c_2 \sum_{p | k} \log p \leq c_2 \log k.
\]

(4)

The lemma now follows from (2), (3), and (4).

Notation. Define the entire function \( f(z) \) by

\[
f(z) = \frac{1}{\Gamma(1 + z)} \prod_p \left(1 - \frac{1}{p}\right)^z \left(1 + \frac{z}{p}\right).
\]

(5)

In addition, for every positive integer \( n \), put

\[
f_n(z) = \prod_{p | m} \left(1 + \frac{z}{p}\right)^{-1}.
\]

(6)

Also, for all positive integers \( k \), nonnegative integers \( j \), and real numbers \( x \), set

\[
\pi_j(x, k) = \# \{ n \leq x : \omega(n) = j, \mu(n) \neq 0, (n, k) = 1 \}.
\]

Lemma 3. Let \( B, u > 0 \) be fixed. Suppose that \( x \geq 3 \), and assume that \( j \) and \( k \) are positive integers with \( j \leq BL_2x \), \( k \leq \exp\{((\log x)^u\} \), and \( Q(k) \leq (\log x)^u \). Then

\[
\pi_j(x, k) = \frac{x}{\log x} \left(\frac{L_2(x)}{j-1}\right)^{j-1} \left\{ f \left(\frac{j-1}{L_2x}\right) f_k \left(\frac{j-1}{L_2x}\right) + O_{B,u}(j(L_2(x))^{-2} (L_3(16k))^2) \right\},
\]

uniformly in \( j \) and \( k \).
Proof. This lemma is an immediate consequence of Theorem 2 of [8]. If we apply the proof of Theorem 7 of [8] to principal characters only, we can show that the condition on $Q(k)$ is unnecessary.

Remark. Many related results are given or referenced in Section 4 of [4].

Lemma 4. For $y \geq 3$ and each integer $m \geq 0$, we have

$$\# \{k \in \mathbb{Z}^+: k \leq y \text{ and } k - v_2(k) = m\} \leq \frac{\xi((\log y)/\log 2)}{\log 2},$$

where the function $\xi$ is defined in the statement of Theorem 2.

Proof. Let $\{k_i\}_{i=1}^l$ be the elements in the set immediately above, arranged so that $v_2(k_1) < \cdots < v_2(k_l)$ (it is easy to see that no two values of $v_2(k_i)$ can be equal). For $1 \leq i \leq l$, let $m_i$ and $\lambda_i$ be nonnegative integers such that

$$k_i = 2^m \lambda_i, \quad 2 \nmid \lambda_i.$$

In view of our notation, if $1 \leq i \leq l-1$, we have

$$2^m \lambda_i - m_i = m = 2^{m_i+\lambda_i+1} - m_{i+1},$$

so that $m_i \equiv m_{i+1} \pmod{2^m}$. We can therefore conclude from the fact that $0 \leq m_1 < \cdots < m_l$ that $2^m + m_i \leq m_{i+1}$; hence $m_i \geq \xi_i-1$ for all $i$. Accordingly,

$$l \leq \xi(m_l) \leq \frac{\xi((\log k_l)/\log 2)}{\log 2} \leq \frac{\xi((\log y)/\log 2)}{\log 2}.$$

Lemma 5. As $x$ approaches $\infty$, we have

$$\sum_{|n - x| \gg x \log x} \frac{x^n}{n!} \ll \frac{e^x}{\sqrt{x \log x}}.$$

Remark. For much more precise estimates of this type, see [3].

Proof. We will show that

$$\sum_{n \ll x - \sqrt{x \log x}} \frac{x^n}{n!} \ll \frac{e^x}{\sqrt{x \log x}};$$

the proof that

$$\sum_{n \gg x + \sqrt{x \log x}} \frac{x^n}{n!} \ll \frac{e^x}{\sqrt{x \log x}}$$
is very similar, and we will omit it. It follows from Stirling's formula that
\[
\sum_{n \leq x - \sqrt{x \log x}} x^n/(n!) \ll \sum_{2 \leq n \leq x - \sqrt{x \log x}} e^{n \log x + n - (n + (1/2)) \log n} \ll \int_{1}^{x - \sqrt{x \log x}} e^{\log x + t - (t + (1/2)) \log t} \, dt \equiv I,
\]
since the exponent in the integrand is an increasing function of \( t \). The quantity
\[
(\log x - \log t - 1/(2t))/\sqrt{x \log x}
\]
is uniformly bounded away from zero throughout the range of integration, so that we have
\[
I \ll \sqrt{\log x} \int_{1}^{x - \sqrt{x \log x}} \left( \log x - \log t - \frac{1}{2t} \right) e^{\log x + t - (t + (1/2)) \log t} \, dt.
\]
When we integrate this expression, we find that it does not exceed
\[
\sqrt{x \log x} \exp\left\{ (x - \sqrt{x \log x})(1 + \log x) - (x - \sqrt{x \log x + \frac{1}{2}}) \log(x - \sqrt{x \log x}) \right\}.
\]
The asserted result now follows from the estimate
\[
\log(x - \sqrt{x \log x}) = \log x - \sqrt{\log x/x} - \frac{\log x}{2x} + O\left(\left(\frac{\log x}{x}\right)^{3/2}\right).
\]

\textbf{Lemma 6.} For \( y \geq 2 \), let \( A_{1,y} < A_{2,y} < \cdots \), be the sequence of positive integers with no prime divisor exceeding \( y \). Then
\[
\lim_{y \to \infty} \sup_{n \in \mathbb{Z}^+} (A_{n+1,y} - 1)/A_{n,y} = 1.
\]

\textit{Proof.} Since the supremum above is clearly at least 1, it suffices to show that for any fixed \( \delta > 1 \), the limit is at most \( \delta \). Now \( A_{n+1,y} \) is at least \( 1 + A_{n,y} \) for all \( n \), so that each subsequence \( \{A_{n,y}\} \) of \( \{A_{n,y}\} \) satisfies
\[
\sup_{n \in \mathbb{Z}^+} (B_{n+1,y} - 1)/B_{n,y} \geq \sup_{n \in \mathbb{Z}^+} (A_{n+1,y} - 1)/A_{n,y}.
\]
The idea is to exhibit a subsequence \( \{B_{n,y}\} \), for each sufficiently large value
of \( y \), such that the supremum on the left of (7) is at most \( \delta \). Choose a positive integer \( N \) so large that
\[
(2^N + 1)/2^N < \delta,
\]
and set \( M \) equal to the (positive) integer such that
\[
((2^N + 1)/2^N)^M < 2 \leq ((2^N + 1)/2^N)^{M + 1}.
\]
For \( y \geq 2^{MN} + 1 \), let \( \{B_{n,y}\}_{n=1}^\infty \) be the sequence of elements of
\[
\{1, 2, \ldots, 2^{MN} - 1\} \cup \sum_{\alpha=M}^{\infty} \{2^\alpha((2^N + 1)/2^N)^k : k = 0, 1, \ldots, M\},
\]
arranged in ascending order. Now \( \{B_{n,y}\} \) is a subsequence of \( \{A_{n,y}\} \), since each element \( B_{n,y} > y \) is the product of a power of 2 and a power of \( 2^N + 1 \). Furthermore, for \( 0 \leq k \leq m - 1 \), and \( x \geq MN \), we have
\[
2^x((2^N + 1)/2^N)^{k+1} - 1 < (2^N + 1)/2^N < \delta
\]
by (8). Similarly, (8) and (9) imply that
\[
2^{x+1} - 1 \leq 2^x((2^N + 1)/2^N)^M < 2^x((2^N + 1)/2^N)^{M+1} \leq 2^{x+1} - 1 < \delta.
\]
It follows from (10) and (11) that the supremum on the left of (7) does not exceed \( \delta \).

\section*{4. The Proof of Theorem 2}

Throughout this section and the next, every variable except for \( n \), \( x \), and \( y \) will denote a nonnegative integer, unless otherwise specified. For clarity of exposition, we put
\[
\mathcal{D} = \{n \in \mathbb{Z}^+ : d(n) \mid n\}, \mathcal{D}(x) = \# \{n \leq x : d(n) \mid n\}.
\]
If \( n \) is odd and \( d(n) \mid n \), then \( d(n) \) is odd, and therefore, \( n \) is a square. Hence,
\[
\mathcal{D}(x) = \# \{n \leq x : n \in \mathcal{D}, 2 \mid n\} + O(\sqrt{x}).
\]
Next, write \( n = 2^{k-1}m \), where \( m = \text{odd}(n) \). Since \( 2^{\lceil L_2x \rceil} \) divides \( n \) whenever \( k \) exceeds \( L_2x \), we find, upon partitioning the even elements of \( \mathcal{D} \) according to the value of \( k \), that
Now \( d(2^{k-1}m)|2^{k-1} \) if and only if

\[
d(kd(m))|2^{k-1}, \quad (14)
\]
since \( m \) is odd and the function \( d(n) \) is multiplicative. Since \( \text{odd}(k)|\text{odd}(kd(m)) \), we can deduce that

\[
\text{odd}(k)|m \quad \text{if} \quad 2^{k-1}m \in \mathcal{D}. \quad (15)
\]

Moreover, if we compare the exact power of 2 dividing each side of (14), we discover that

\[
v_2(d(m)) \leq k - 1 - v_2(k) \quad \text{if} \quad 2^{k-1}m \in \mathcal{D}. \quad (16)
\]

We can uniquely express \( m/\text{odd}(k) \) as \( tm' \), where \( t \) is squarefull, \( m' \) is squarefree, and \( (m', t) = 1 \). Then, in turn, we can write \( m' \) uniquely as \( hl \), where \( (h, k) = 1 \) and \( l|k \). Thus, we have

\[
m = thl \text{odd}(k), \quad (17)
\]

where \( (h, tk) = 1, l|Q(k), t \) is squarefull, and \( h \) is squarefree. Since \( (h, m/h) = 1 \), we conclude that \( d(h)|d(m) \), and therefore, the inequality of (16) implies that \( v_2(d(h)) \leq k - 1 - v_2(k) \). Furthermore, the fact that \( h \) is squarefree implies that \( d(h) = 2^{\omega(h)} \). Hence, we have

\[
\omega(h) \leq k - 1 - v_2(k) \quad \text{if} \quad 2^{k-1}m \in \mathcal{D}. \quad (18)
\]

Accordingly, (13), (17), and (18) yield

\[
\mathcal{D}(x) \leq O(x(\log x)^{-\log 2})
\]

\[
+ \sum_{2 \leq k \leq L_2(x)} \sum_{n \leq \frac{x}{2^k \text{odd}(k)}} \sum_{t \geq 1} \sum_{h < 2^{k-1}x/t|\text{odd}(k), (h,k) = 1} \sum_{\omega(h) \leq k - 1 - v_2(k), \mu(h) \neq 0} 1 \quad (19)
\]

The inner sum is trivially at most \( x/(tn2^{k-1}\text{odd}(k)) \). Consequently, if we separate the sum on \( t \) into the sum over those values of \( t \) not exceeding \( (\log x)^2 \) and the sum over those values of \( t \) exceeding \( (\log x)^2 \), and apply Lemma 1 to estimate the last of these sums, we find that the sum on \( t \) in (19) is

\[
\sum_{1 \leq t \leq (\log x)^2} \sum_{n \leq \frac{x}{2^k \text{odd}(k)}, (h,k) = 1} \sum_{\omega(h) \leq k - 1 - v_2(k), \mu(h) \neq 0} 1 + O\left(\frac{x/\log x}{2^k \text{odd}(k)}\right). \quad (20)
\]
If we partition according to the value of \(\omega(h)\), those integers \(h\) which contribute to the inner sum, we discover that the inner sum equals

\[
1 + \sum_{j=1}^{k-1 - \nu_2(k)} \pi_j(x/2^{k-1}l \text{ odd}(k), k).
\]

Thus, the sum on \(t\) in (19) equals

\[
\sum_{1 \leq t \leq (\log x)^2}^{k-1 - \nu_2(k)} \sum_{j=1}^{k-1 - \nu_2(k)} \pi_j\left(\frac{2^{1-k}x}{tl \text{ odd}(k)}, k\right)
+ O\left(\frac{x/\log x}{2^k l \text{ odd}(k)}\right) + O((\log x)^2).
\]

In the range of summation of the quadruple sum in (19), we have \(l \leq k \leq L_2 x\), from which we can deduce that

\[
2^{k-1}l \text{ odd}(k) \leq 2^k k^2 \leq (\log x)^{\log_2 (L_2(x))^2}.
\]

Consequently, the last error term in (21) can be absorbed into the first one. Since we have \(t \leq (\log x)^2\) in the range of summation of (21), we can conclude from (22) that

\[
x/(2^{k-1}l \text{ odd}(k)) \geq x(\log x)^{-2 - \log_2 (L_2(x))^{-2}}.
\]

Hence, we can utilize Lemma 3 with \(B = u = 2\), and with \(x\) replaced by \(x/(2^{k-1}l \text{ odd}(k))\). If we put \(w = x/(2^{k-1}l \text{ odd}(k))\), then that lemma implies that

\[
\pi_j\left(\frac{2^{1-k}x}{tl \text{ odd}(k)}, k\right)
- \frac{2^{1-k}}{tl \text{ odd}(k) \log w} \frac{1}{(L_2(w))/(j-1)!} \left\{ f_j\left(\frac{j-1}{L_2(w)}\right) f_k\left(\frac{j-1}{L_2(w)}\right) \right\}
+ O\left(\frac{j}{(L_2(w))^2 (L_3(16k))^2}\right).
\]

For \(j\) contributing to the inner sum in (21), we have \(1 \leq j \leq L_2(x)\). Consequently, (23) and our choice of \(w\) imply that

\[
0 \leq \frac{j-1}{L_2(x)} \leq \frac{j-1}{L_2(w)} < 2,
\]

\[
\log w = \log x + O(L_3(x)),
\]
where the implied constant in (26) is absolute. It follows from (26) and the inequalities \( j < k \leq L_2 x, w \leq x \) that the right side of (24) is

\[
O\left( \frac{2^{1-k}}{t \text{ odd}(k) \log x} \frac{1}{(j-1)!} \right) \frac{(L_2(x))^{j-1}}{(L_2(x))^2 (L_2(x))^{-1}} \left\{ f_k \left( \frac{j-1}{L_2(w)} \right) + (L_2(x)) \right\}.
\]

From (25) and the entirety of \( f(z) \), we can deduce that \( f((j-1)/L_2(w)) = O(1) \). Hence, the right side of (24) is

\[
O\left( \frac{2^{1-k}}{t \text{ odd}(k) \log x} \frac{1}{(j-1)!} \right) \frac{(L_2(x))^{j-1}}{(L_2(x))^2 (L_2(x))^{-1}} \left\{ f_k \left( \frac{j-1}{L_2(w)} \right) + (L_2(x)) \right\}.
\]

Since \( k \leq L_2 x \), we can conclude from (6), (25), and Lemma 2 with \( B = 2 \) that

\[
f_k \left( \frac{j-1}{L_2(w)} \right) \geq (L_4(x))^{-2}.
\]

Consequently, the summand \( (L_2(x))^{2} (L_2(x))^{-1} \) can be deleted from (27). Hence, by the sentence following (22), and the fact that the right side of (24) equals (27), the expression in (21) equals

\[
O\left( \sum_{1 \leq t \leq \sqrt{\log x}} \frac{2^{1-k}(\log x)^{-1} k - 1 - \sqrt{2}(k)}{t \text{ odd}(k)} \frac{(L_2(x))^{j-1}}{(j-1)!} f_k \left( \frac{j-1}{L_2(w)} \right) \right.
\]

\[
+ \frac{x/\log x}{2^k t \text{ odd}(k)}.
\]

According to (6) and (25), we can replace \( w \) by \( x \) in this expression. Thus, if we interchange the order of summation, we find that the quantity in (21) is

\[
O\left( \frac{x/\log x}{2^k t \text{ odd}(k)} \right) \left\{ 1 + \sum_{j=1}^{k-1} \frac{1}{j-1!} \right\}.
\]

Since the expression in (21) equals the sum on \( t \) in (19), we can conclude from Lemma 1 that
\[ \mathcal{D}(x) \leq x(\log x)^{-\log 2} \]

\[ + \frac{x}{\log x} \sum_{k=2}^{[L_2(x)]} \frac{2^{-k}}{\text{odd}(k)} \sum_{\substack{\ell, \mu(l) \neq 0 \quad \text{odd}(k) \equiv 1 \pmod{2} \quad \text{or} \quad \ell k \equiv 1 \pmod{2} \quad \text{or} \quad \ell k \equiv 1 \pmod{2}}} \times \left\{ 1 + \sum_{j=1}^{k-1} \frac{(L_2(x))^{j-1}}{(j-1)!} f_k \left( \frac{j-1}{L_2(x)} \right) \right\}. \]

From the inequalities \( \sum_{l \mid Q(k)} 1/l \leq Q(k) \) and \( Q(k) \leq 2 \text{ odd}(k) \), we deduce that

\[ \sum_{2 \leq k \leq L_2(x)} \frac{2^{-k}}{\text{odd}(k)} \sum_{\ell, \mu(l) \neq 0} \frac{1}{l} \leq \sum_{2 \leq k \leq L_2(x)} 2^{-k} \cdot 2 \leq 1. \]

Thus, we can delete the first 1 from the expression in curly brackets in (28). If we then interchange the order of summation in (28), we find that

\[ \mathcal{D}(x) \leq x(\log x)^{-\log 2} \]

\[ + \frac{x}{\log x} \sum_{2 \leq j+1 \leq L_2 x} \frac{(L_2(x))^{j-1}}{(j-1)!} \]

\[ \times \sum_{k \leq L_2(x)} \frac{2^{-k}}{\text{odd}(k)} \frac{1}{f_k \left( \frac{j-1}{L_2(x)} \right)} \sum_{\ell, \mu(l) \neq 0} \frac{1}{l} \]

\[ = x(\log x)^{-\log 2} + x(\log x)^{-1} \left( \sum_{f} + \sum_{ll} \right), \]

where

\[ \sum_{f} = \sum_{2 \leq j+1 \leq L_2(x) \text{, \ } |j-1-(1/2)L_2(x)| \ll \sqrt{L_2(x)} \text{, \ } \sqrt{L_2(x)}}, \quad \text{(30)} \]

\[ \sum_{ll} = \sum_{|j-1-(1/2)L_2(x)| \ll \sqrt{L_2(x)} \text{, \ } \sqrt{L_2(x)}}, \quad \text{(31)} \]

and the \( j \)th summand on either of the right sides of (30) and (31) is identical with the corresponding \( j \)th summand of the triple sum in (29).

Since the values of \( k \) which contribute to the sum on \( k \) in (29) do not exceed \( L_2 x \), we can conclude from (6) and Lemma 2 that

\[ \sum_{\substack{l \mid \ell, \mu(l) \neq 0 \quad \ell \leq 1 \pmod{2} \quad \text{or} \quad \ell k \equiv 1 \pmod{2} \quad \text{or} \quad \ell k \equiv 1 \pmod{2}}} \frac{1}{l} = \prod_{p \mid k} (1+1/p) \ll L_4(x). \quad \text{(32)} \]

Thus, if we use (6) to bound \( f_k(j-1)/L_2(x) \) by 1, we can deduce from (29), (30), and (32) that
\[
\sum \frac{(L_2(x))^{j-1}}{(j-1)!} \sum_{k \leq L_2(x) \text{ odd}} \frac{2^{-1}}{\text{odd}(k)}.
\]

In the range of summation of the sum on the right of (31), we have
\[
(j-1)/L_2x = \frac{1}{2} + O(\sqrt{L_3(x)}/\sqrt{L_2(x)}),
\]
so that
\[
\prod_{p | k} \left(1 + \frac{j-1}{pL_2(x)}\right)^{-1}
\]
\[
= \prod_{p | k} \left(1 + \frac{1}{2p} + O\left(\frac{1}{p} \sqrt{L_3(x)/L_2(x)}\right)\right)^{-1}
\]
\[
= \prod_{p | k} \left(1 + \frac{1}{2p}\right)^{-1} \left(1 + O\left(\frac{1}{p} \sqrt{L_3(x)/L_2(x)}\right)\right)^{-1}.
\]

Moreover, by Lemma 2 and the inequality \(k \leq L_2x\), we have
\[
\prod_{p | k} \left(1 + O\left(\frac{1}{p} \sqrt{L_3(x)/L_2(x)}\right)\right)^{-1}
\] 
\[\ll (L_2(3k))^{O(\sqrt{L_3(x)/L_2(x)})} \ll 1.\]

It follows at once from (6), (35), and (36) that
\[
f_k \left(\frac{j-1}{L_2x}\right) \ll \prod_{p | k} (1 + (2p)^{-1})^{-1},
\]
uniformly in \(j\) and \(k\) with \(|j - 1/2L_2x| < \sqrt{L_2(x)} \sqrt{L_3(x)}\), and \(k \leq L_2(x)\).
Combining (37) with the first statement of (32) yields
\[
f_k \left(\frac{j-1}{L_2(x)}\right) \sum_{\eta(k,\mu(f)) \neq 0} l^{-1} \ll \prod_{p | k} \left(1 + \frac{1}{2p}\right)^{-1} \left(1 + \frac{1}{p}\right).
\]

Now for \(x \geq 1\), we know that
\[
(1 + (2x)^{-1})^{-1} (1 + x^{-1}) \ll 1 + (2x)^{-1}.
\]
Hence, (38) and Lemma 2 imply that
\[
f_k \left(\frac{j-1}{L_2(x)}\right) \sum_{\eta(k,\mu(f)) \neq 0} l^{-1} \ll \prod_{p | k} \left(1 + \frac{1}{2p}\right) \ll \sqrt{L_2(3k)} \ll \sqrt{L_4(x)},
\]
uniformly in \( j \) and \( k \) with \( |j - 1 - \frac{1}{2} L_2(x)| < \sqrt{L_2(x)} \sqrt{L_3(x)} \), and \( k \leq L_2(x) \). Accordingly, (29), (31), and (39) yield

\[
\sum_{\mu} \ll \sqrt{L_4(x)} \sum_{|j - 1 - (1/2) L_2(x)| < \sqrt{L_2(x)} \sqrt{L_3(x)}} \frac{(L_2(x))^{j-1}}{(j-1)!} \times \sum_{k \leq L_2(x)} \frac{2^{-1}}{\text{odd}(k)}
\]

(40)

If \( 1 + j \leq k - v_2(k) \), then \( k \) exceeds \( j \), and we therefore have

\[
2^{-k/\text{odd}(k)} = 2^{-k + v_2(k)} k^{-1} \leq j^{-2} (k - v_2(k)).
\]

Thus, if we partition the positive integers \( k \) such that \( 1 + j \leq k - v_2(k) \) according to the value of \( k - v_2(k) \), we find that

\[
\sum_{k \leq L_2(x)} \frac{2^{-k}}{\text{odd}(k)} \leq \sum_{n \geq 1 + j} 2^{-n} \sum_{k \leq L_2(x)} 1.
\]

(41)

By Lemma 4, the inner sum is at most \( \xi(x) \); hence, the right side of (41) does not exceed \( j^{-1} 2^{-j} \xi(x) \). Therefore, (33), (40), and (41) yield

\[
\sum_{\mu} \ll (L_4(x)) \xi(x) \sum_{|j - 1 - (1/2) L_2(x)| \geq \sqrt{L_2(x)} \sqrt{L_3(x)}} \frac{((1/2) L_2(x))^{j-1} 1}{(j-1)!} j.
\]

(42)

\[
\sum_{\mu} \ll \sqrt{L_4(x)} \xi(x) \sum_{|j - 1 - (1/2) L_2(x)| < \sqrt{L_2(x)} \sqrt{L_3(x)}} \frac{((1/2) L_2(x))^{j-1} 1}{(j-1)!} j.
\]

(43)

To bound the right side of (42), we omit the factor \( 1/j \), and then apply Lemma 5. Thus,

\[
\sum_{\mu} \ll \sqrt{\log x} (L_4(x)) \xi(x)/(L_2(x)(L_3(x))).
\]

(44)

If we replace the factor \( 1/j \) by \( 1/L_2(x) \) on the right side of (43), and then let \( j \) run from \( 1 \) to \( \infty \), we find that

\[
\sum_{\mu} \ll \sqrt{L_4(x)} \sqrt{\log x} \xi(x)/L_2(x).
\]

(45)

The theorem now follows from (29), (44), and (45).
5. Derivation of the Lower Bound

To obtain a lower bound for \( \Omega(x) = \# \{ n \leq x : d(n) \mid n \} \), we first want to construct a large number of positive integers \( n \) such that \( d(n) \) divides \( n \). The next lemma will aid us in this construction.

**Lemma 7.** Let \( \varepsilon > 0 \) be given. Assume that \( k, y, \) and \( z \) are at least 2, with \( (k, \prod_{p \leq y + z} p) = 1 \), and that \( k \) has the prime factorization \( \prod_{i=1}^{t} p_i^{\beta_i} \). For \( i = 1, 2, \ldots, t \), let \( \beta_i \) be the smallest integer which is at least \( \alpha_i \), such that no prime divisor of \( \beta_i + 1 \) exceeds \( z \). For all odd primes \( p \leq z \), set

\[
\gamma_p = \min_{i \leq z, i \geq 0} \left\{ 2^i - 1 : \exists 2^i, 1 = \gamma_p \left( \prod_{i=1}^{t} (\beta_i + 1) \right) \right\}.
\]

If \( z \) is \( \varepsilon \)-sufficiently large, \( y \) is \( \lceil \varepsilon, z \rceil \)-sufficiently large, and \( k_z = (\prod_{i=1}^{t} p_i^{\beta_i}) \prod_{2 < p \leq z} p^{\gamma_p} \), then the following statements hold:

(i) \( k \mid k_z \), and if \( p \mid k_z \), then either \( p \mid k \) or \( p \leq z \);

(ii) if \( L \in \mathbb{Z}^+ \), \( \mu(L) \neq 0 \), \( (L, k \prod_{p \leq z} p) = 1 \), and \( \omega(L) \leq k - 1 - \frac{1}{2} \varepsilon \log k \), then \( d(2^{k-1}Lk_z) \mid 2^{k-1}k_z \);

(iii) \( k_z \mid k \cdot (1 + (1/2) \varepsilon) \).

Remark. As we saw in the proof of the last theorem, if \( k \) is an odd integer, \( 2^{k-1} \mid n \), and \( d(n) \mid n \), then we have \( 2^{k-1}k \mid n \). This lemma will be used to construct multiples \( n \) of \( 2^{k-1}k \) such that \( d(n) \mid n \). An example of a set of multiples \( n \) of this type is

\[
\{ 2^{48} \cdot (49 \cdot 3) : (m, 2 \cdot 3 \cdot 49) = 1, \mu(m) \neq 0, \omega(m) \leq 47 \}.
\]

(Here, \( k = 49 \).) The point of the lemma is to be able, for a positive proportion of all \( k \), to write \( n = 2^{k-1}k^*m \), where \( d(n) \mid n, \mu(m) \neq 0, \omega(m) \) can be almost as large as \( k \), \( (m, 2k^*) = 1 \), and \( k^* \) is at most \( k^{1+\varepsilon} \) (so that \( k^* \) is not much larger than \( k \)).

**Proof.** Part (i) is clear from the definition of \( k_z \). In the notation of Lemma 6, assume that \( z \) is so large that

\[
\sup_{n \in \mathbb{Z}^+} \left\{ (A_{n+1,z} - 1) / A_{n,z} \right\} < 1 + \varepsilon/3.
\]

Then we can conclude that \( \beta_i / x_i < 1 + \varepsilon/3 \) for all \( i \), so that

\[
\prod_{i=1}^{t} p_i^{\beta_i} < \left( \prod_{i=1}^{t} p_i^{x_i} \right)^{1 - \varepsilon/3} = k^{1 + \varepsilon/3}.
\]

(46)
Since \( z \geq 2 \), the lowest power of 2 exceeding \( \alpha_i \) is at least \( \beta_i + 1 \), and it follows that \( \beta_i + 1 < 2\alpha_i \). Accordingly,

\[
\frac{1}{\log 2} \log \prod_{i=1}^{t} (\beta_i + 1) \leq \sum_{i=1}^{t} (\log (2\alpha_i)) \leq \sum_{i=1}^{t} \alpha_i. \tag{47}
\]

Now each prime dividing \( k \) exceeds \( y \); hence (47) yields

\[
\log \prod_{i=1}^{t} (\beta_i + 1) < \frac{1}{\log y} \sum_{i=1}^{t} \alpha_i \log p_i = \frac{2}{\log y} \log k. \tag{48}
\]

If \( 2 < p \leq z \), then the definition of \( \gamma_p \) implies that

\[
\gamma_p \leq \frac{2\gamma_p \left( \prod_{i=1}^{t} (\beta_i + 1) \right)}{\log p \prod_{i=1}^{t} \log (\beta_i + 1)}. \tag{49}
\]

Hence, it follows from (48) that

\[
\gamma_p \leq 2(\log 2)(\log k)/\{(\log p) \log y\},
\]

and we immediately have

\[
\prod_{2 < p \leq z} p^{\gamma_p} \leq k^{2(\log 2)/\log y}. \tag{50}
\]

Part (iii) now follows from (46) and (50).

In view of our notation and the multiplicativity of the function \( d(n) \), we can conclude that for any squarefree, positive integer \( L \) coprime with \( k \prod_{p \leq z} p \), we have

\[
2^{k-1} k_z/d(2^{k-1} L k_z) = \frac{(\prod_{i=1}^{t} p^{\beta_i - \alpha_i})(\prod_{2 < p \leq z} p^{\gamma_p}) 2^{k-1 - \omega(L)}}{\prod_{i=1}^{t} (\beta_i + 1) \prod_{2 < p \leq z} (\gamma_p + 1)}. \tag{51}
\]

By construction, no prime divisor of the denominator on the right of (51) exceeds \( z \). Furthermore, we know that

\[
\gamma_p \geq \gamma_p \left( \prod_{i=1}^{t} (\beta_i + 1) \right)
\]

if \( 2 < p \leq z \), and that \( \gamma_p + 1 \) is a power of 2, so that if we put the fraction on the right of (51) into lowest terms, the denominator becomes a power of 2. If follows from (48) that

\[
v_2 \left( \prod_{i=1}^{t} (\beta_i + 1) \right) \leq \frac{1}{\log 2} \log \prod_{i=1}^{t} (\beta_i + 1) \leq (\log k)/\log y,
\]

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and from (49) that
\[
\nu_2 \left( \prod_{2 < p < z} (\gamma_p + 1) \right) = \sum_{2 < p < z} \frac{\log(\gamma_p + 1)}{\log 2} \leq \sum_{2 < p < z} \gamma_p \leq \frac{(2^z \log 2) \log k}{\log y}.
\]

Thus, if \( y \) is \([\epsilon, z]\)-sufficiently large, we have
\[
\nu_2 \left( \left( \prod_{2 < p < z} (\gamma_p + 1) \right) \prod_{i=1}^{t} (\beta_i + 1) \right) < \frac{1}{2} \log k.
\]

Hence, if \( y \) is \([\epsilon, z]\)-sufficiently large and \( \omega(L) \leq k - 1 - \frac{1}{2} \epsilon \log k \), then the power of 2 dividing the numerator on the right of (51) exceeds the power of 2 dividing the denominator of that fraction. Consequently, the quantity on the right of (51) is an integer, so that (ii) holds.

**Proof of Theorem 1.** In view of Theorem 2, it is enough to show that for any \( \epsilon > 0 \), we have
\[
\mathcal{D}(x) \geq x(I_{-2}(x))^{-1} e^{-\epsilon \sqrt{\log x}}. \tag{52}
\]

Without loss of generality, assume that \( \epsilon \leq \frac{1}{2} \). First let \( z \geq 2 \) be \( \epsilon \)-sufficiently large, and then let \( y \geq 2 \) be \([\epsilon, z]\)-sufficiently large, so that the conclusion of the last lemma holds for \( \epsilon, y, \) and \( z \). Set \( P = \prod_{p < y + z} p \). If we partition the positive integers \( n \) for which \( d(n) | n \) according to the value \( k - 1 \) of \( \nu_2(n) \), we find that
\[
\mathcal{D}(x) \geq \sum_{1 < k \leq \mathcal{L}(x)} \sum_{n \leq x, 2^{k-1} | d(n)| n} 1 \geq \sum_{1 < k \leq \mathcal{L}(x)} \sum_{n \leq x, 2^{k-1} | d(n)| n} 1. \tag{53}
\]

By Lemma 7, and in the notation therein, the set of \( n \) contributing to the inner sum on the right of (53) contains the set of positive integers of the form \( 2^{k-1} k_2 L \) which do not exceed \( x \), and which satisfy the conditions that
\( L \in \mathbb{Z}^+, \mu(L) \neq 0, (L, kP) = 1, \) and \( \omega(L) \leq k - 1 - \frac{1}{2} \epsilon \log k \). Hence,
\[
\mathcal{D}(x) \geq \sum_{1 < k \leq \mathcal{L}(x)} \sum_{\substack{L \leq 2^{1-k/2} k_2, (L, kP) = 1 \\omega(L) \leq k - 1 - (1/2) \epsilon \log k, \mu(L) \neq 0}} 1. \tag{54}
\]
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If we partition the numbers $L$ contributing to the inner sum according to the value of $\omega(L)$, we discover that this inner sum equals

$$D(x) \geq \sum_{1 < k \leq L_2(x), (k,P) = 1} \left( \sum_{1 < j \leq k - 1 - (1/2) \log k} \pi_j \left( \frac{2^{1-k}x}{k_z}, kP \right) + 1 \right). \quad (55)$$

Since $k \leq L_2(x)$, and Lemma 7 implies that $k_z \ll k^2$, we have

$$2^{1-k}x/k_z \gg x(\log x)^{-1} (L_2(x))^{-2} > \sqrt{x}. \quad (56)$$

Therefore, we can apply Lemma 3 with $B = u = 2$ to estimate each summand in (55). Thus, if we set $w = 2^{1-k}x/k_z$, we have

$$\pi_j \left( \frac{2^{1-k}x}{k_z}, kP \right) = \frac{2^{1-k}x}{k_z \log w} \left( \frac{j - 1}{L_2(w)} \right) f_{kp} \left( \frac{j - 1}{L_3(x)} \right) + O_{y,z}(j(L_2(w))^{-2} (L_3(k))^{-2}). \quad (57)$$

It follows from (5) that $f(z)$ is bounded away from 0 for $0 < z < 2$. Hence, (56) and our choice of $w$ imply that $f((j - 1)/L_2(w))$ is uniformly bounded away from zero for $j < k \leq L_2 x$. Furthermore, by the derivation of the displayed inequality following (27), $f_{kp}(j - 1/L_2(w)) \gg y,z (L_4(x))^{-2}$ uniformly in $j \leq k \leq L_2(x)$. In addition, (56) and our choice of $w$ imply that the error term in (57) is $O_{y,z}(L_5(x)^2/L_2(x))$, with the same range of uniformity in $j$ and $k$. Consequently, it follows from (57) that

$$\pi_j \left( \frac{2^{1-k}x}{k_z}, kP \right) \gg y,z \frac{2^{-k}x}{k_z \log w} \left( \frac{L_2(w))^{j - 1}}{(j - 1)!} \right) (L_4(x))^{-2}. \quad (58)$$

For $j \leq k \leq L_2 x$, we can conclude from (56) that

$$(L_2 w)^{j-1} \gg (L_2(x))^{j-1}, \quad (59)$$

uniformly in $j$ and $k$. Furthermore, Lemma 7 and our choices of $y$ and $z$ imply that

$$1/k_z \gg y,z k^{-1-\epsilon/2} \gg (L_2(x))^{-1-\epsilon/2}. \quad (60)$$

Combining (59) and (60) with (58), and the result with (55), yields

$$D(x) \gg y,z \sum_{1 < k \leq L_2(x), (k,P) = 1} \sum_{1 < j \leq k - 1 - (1/2) \log k} \frac{x(L_2(x))^{-1-\epsilon/2}}{(L_4(x))^2 \log x} \frac{2^{1-k}(L_2(x))^{j-1}}{(j-1)!}. \quad (61)$$
For $k \leq L_2(x)$, we have $k - 1 - \frac{1}{2} \varepsilon \log k \geq k - 1 - \frac{1}{2} \varepsilon L_2(x)$, so that
\[
\mathcal{D}(x) \geq \sum_{1 \leq k \leq L_2(x)} \sum_{1 \leq j \leq k - 1 - (1/2) \varepsilon L_2(x)} \frac{x(L_2(x))^{-1 - \varepsilon/2}}{(L_4(x))^2 \log x} \times 2^{1 - k} (L_2(x))^{j - 1} \frac{L_2(x)^j}{(j - 1)!}.
\]
If we interchange the order of summation, we find that
\[
\mathcal{D}(x) \geq \sum_{1 \leq j \leq 0.9 L_2(x)} \frac{x(L_2(x))^{-1 - \varepsilon/2}}{(L_4(x))^2 \log x} \sum_{1 \leq k \leq j + (1/2) \varepsilon L_2(x)} 2^{1 - k} \frac{(L_2(x))^j}{(j - 1)!}.
\]
If we bound the inner sum from below by the summand corresponding to the smallest value of $k$ which contributes to that inner sum, we can conclude that
\[
\mathcal{D}(x) \geq \sum_{1 \leq j \leq 0.9 L_2(x)} \frac{x(L_2(x))^{-1 - \varepsilon/2}}{(L_4(x))^2 \log x} 2^{- (1/2) \varepsilon L_2(x)} \sum_{1 \leq j \leq 0.9 L_2(x)} \frac{((1/2) \varepsilon L_2(x))^j}{(j - 1)!}.
\]
It follows from Lemma 5 that the sum above is asymptotic to $\sqrt{\log x}$. Since $y$ and $z$ depend only on $\varepsilon$, our theorem is now established.

6. Generalizations and Remarks

With a little more work, we can generalize Theorems 1 and 2 to give the following results:

**Theorem 3.** For every fixed positive integer $l$, we have
\[
\# \{ n \leq x: (d(n))' | n \} = x(\log x)^2 L_2(x)^{-1 - \varepsilon} + o(1).
\]

**Theorem 4.** Let $l$ be a positive integer, and let $\{ \xi_{k,l}, k = 0 \} \infty$ be recursively defined by
\[
\xi_{0,l} = v_2(l), \quad \xi_{k,l} = \xi_{k-1,l} + 2^{\xi_{k-1,l} - v_2(l)} \quad \text{for} \quad k \geq 1.
\]
Then
\[
\# \{ n \leq x: (d(n))' | n \} \leq \frac{x(\log x)}{(L_2(x))^l} \left( \frac{L_4(x)}{\log x} \right)^{1 - 2^{-l}}.
\]
where
\[ \zeta(x, l) = 1 + \# \{ k \geq 0 : \xi_{k,l} \leq x \}. \]

We remark that \( \xi(x, l) \leq k \cdot L_k(x) \) for any fixed positive integers \( k \) and \( l \), so that we can, in particular, replace \( \xi(x, l) \) by \( L_k(x) \) in the estimate of Theorem 4.

We can also extend our results to get an upper bound for \( \# \{ n \leq x : d_q(n) \mid n \} \) when \( q > 1 \) is a prime power, where the divisor functions \( d_q \) are defined by

\[ \sum_{n=1}^{\infty} d_q(n) n^{-s} = \left( \sum_{n=1}^{\infty} n^{-s} \right)^q, \quad \text{Re } s > 1. \]

Specifically, we can verify the following result:

**Theorem 5.** Set \( q = p^a \), where \( x \geq 1 \), and let \( \{ \zeta_{k,p} \}_{k=0}^{\infty} \) be recursively defined by

\[ \zeta_{0,p} = 0, \quad \zeta_{k,p} = \zeta_{k-1,p} + p^{\zeta_{k-1,p}} \quad \text{for } k \geq 1. \]

Then

\[ \# \{ n \leq x : d_q(n) \mid n \} \leq \frac{x^{\zeta_1(x, p)}}{q (L_2(x))^{q-1} \left( \log x \right)^{1/q}}, \]

where

\[ \zeta_1(x, p) = \# \{ k \geq 0 : \zeta_{k-1,p} \leq x \}. \]

The problem of obtaining good lower bounds for \( \# \{ n \leq x : d_q(n) \mid n \} \) for any \( q \geq 3 \) appears to be very difficult. We cannot even show the existence of a positive constant \( A \) such that

\[ \# \{ n \leq x : d_3(n) \mid n \} \geq x (\log x)^{-2/3} (L_2(x))^{-A}, \]

although we conjecture that

\[ \# \{ n \leq x : d_3(n) \mid n \} = x (\log x)^{-2/3} (L_2(x))^{-2 + o(1)}. \]

This conjecture states the best possible result for the exponent of \( L_2(x) \) on the right side, in view of Theorem 5.

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