Positive solutions for a class of boundary value problems on time scales

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Received 15 July 2006; received in revised form 10 January 2007; accepted 22 January 2007

Abstract

By obtaining intervals of the parameter $\lambda$, this paper is concerned with the existence and nonexistence of positive solution of the second-order nonlinear dynamic equation on time scales

$$
-\left[ p(t)x^{\Delta}(t) \right]^{\nabla} + q(t)x(t) = \lambda w(t)f(t,x),
$$

$$
\alpha x(\rho(a)) - \beta x^{\Delta}(\rho(a)) = 0,
$$

$$
\gamma x(b) + \delta x^{\Delta}(b) = 0
$$

for $t \in [a, b] \subset \mathbb{T}$, where $\mathbb{T}$ is a time scale, $\alpha \geq 0, \gamma \geq 0, \beta > 0, \delta > 0$ with $\alpha + \gamma > 0$. The arguments are based upon fixed point theorems in a cone.

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Keywords: Time scales; Delta and nabla derivatives; Positive solution; Fixed point theorem; Existence

1. Introduction

Consider the following second-order boundary value problem (BVP) on time scales

$$
Lx = \lambda w(t)f(t,x), \quad t \in [a, b] \subset \mathbb{T},
$$

$$
\alpha x(\rho(a)) - \beta x^{\Delta}(\rho(a)) = 0,
$$

$$
\gamma x(b) + \delta x^{\Delta}(b) = 0,
$$

where $\mathbb{T}$ is a time scale, $\alpha \geq 0, \gamma \geq 0, \beta > 0, \delta > 0$ with $\alpha + \gamma > 0$, $w : [a, b] \rightarrow [0, +\infty)$ is continuous and there exists $t_0 \in [a, b]$ such that $w(t_0) > 0$, $x^{\Delta}(t) = p(t)x^{\Delta}(t)$ is called the quasi $\Delta$-differentiable and

$$
Lx := -\left[ p(t)x^{\Delta}(t) \right]^{\nabla} + q(t)x(t),
$$

where $p : [a, b] \rightarrow (0, +\infty)$ is $\nabla$-differentiable on $\mathbb{T}_k$ and $q : [a, b] \rightarrow [0, +\infty)$ is continuous.

\textsuperscript{*} Supported by NNSF of China (10371006) and SRFDP of China (20050070111) and the Science Foundation of Beijing Information Technology Institute.

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doi:10.1016/j.camwa.2007.01.031
Recently, there have been many papers working on the existence of positive solutions of dynamic equations on time scales, see, [1–12]. These papers have relied on methods such as the Schauder fixed point theorem, the nonlinear alternative of Leray–Schauder or on disconjugacy to prove the existence of solutions to second-order BVPs on time scales subject to linear, separated boundary conditions. In particular, we would like to mention some results of Anderson [1] and Atici and Guseinov [4]. In [1], Anderson considered the following second-order BVP on a Measure Chain:

\[
\begin{align*}
-x^{\Delta\Delta} &= \lambda w(t)f(x^{\sigma}(t)), \quad t \in [a, b] \subset \mathbb{T}, \\
\alpha x(a) - \beta x^{\Delta}(a) &= 0, \\
\gamma x(\sigma(b)) + \delta x^{\Delta}(\sigma(b)) &= 0.
\end{align*}
\]

By using fixed point theory in a cone, the author established the existence of positive solutions and obtained the nonexistence of positive solutions of dynamic equations, even for the problem \(\lambda = 1, w(t) \equiv 1\), Atici et al. [4] established the existence of positive solutions for BVP (1.1) under the assumption that \(f\) satisfies \(f_0 = \lim_{x \to 0} \frac{f(t,x)}{x} = 0\) and \(f_\infty = \lim_{x \to \infty} \frac{f(t,x)}{x} = \infty\) or \(f_0 = \infty\) and \(f_\infty = 0\), uniformly on \(t \in [a, b]\).

Motivated by the results mentioned above, in this paper we show that appropriate combinations of superlinearity and sublinearity of \(f(t,x)\) with respect to \(x\) at zero and infinity guarantee the existence, multiplicity, and nonexistence of positive solutions for BVP (1.1) and describe the dependence of positive solutions of BVP (1.1) on the parameter \(\lambda\). By using new techniques to overcome difficulties arising from the appearances of \(Lx := \{-p(t)x^{\Delta}(t)\}^\sigma + q(t)x(t)\) and \(\mathbb{T}\) is a time scale, we will show that the number of positive solutions of BVP (1.1) is determined by the parameter \(\lambda\). On the other hand, to the best of our knowledge, there is no literature considering the existence, multiplicity, and nonexistence of positive solutions of dynamic equations, even for the problem (1.1). The arguments are based upon fixed point theorems in a cone.

The following lemmas are crucial to prove our main results.

**Lemma 1.1** ([13–17] Fixed Point Theorem of Cone Expansion and Compression of Norm Type). Let \(\Omega_1\) and \(\Omega_2\) be two bounded open sets in Banach space \(E\), such that \(\emptyset \in \Omega_1\) and \(\bar{\Omega}_1 \subset \Omega_2\). Let operator \(A : P \cap (\bar{\Omega}_2 \setminus \Omega_1) \to P\) be completely continuous, where \(\emptyset\) denotes the zero element of \(E\) and \(P\) is a cone in \(E\). Suppose that one of the two conditions

(i) \(\|Ax\| \leq \|x\|, \forall x \in P \cap \partial \Omega_1\) and \(\|Ax\| \geq \|x\|, \forall x \in P \cap \partial \Omega_2\),

or

(ii) \(\|Ax\| \geq \|x\|, \forall x \in P \cap \partial \Omega_1\), and \(\|Ax\| \leq \|x\|, \forall x \in P \cap \partial \Omega_2\),

is satisfied. Then \(A\) has at least one fixed point in \(P \cap (\bar{\Omega}_2 \setminus \Omega_1)\).

**Lemma 1.2** ([15]). Let \(\Omega_1, \Omega_2\) and \(\Omega_3\) be three bounded open sets in Banach space \(E\), such that \(\emptyset \in \Omega_1\) and \(\bar{\Omega}_1 \subset \Omega_2, \bar{\Omega}_2 \subset \Omega_3\). Let operator \(A : P \cap (\bar{\Omega}_3 \setminus \Omega_1) \to P\) be completely continuous, where \(\emptyset\) denotes the zero element of \(E\) and \(P\) is a cone in \(E\). Suppose that

(i) \(\|Ax\| \geq \|x\|, \forall x \in P \cap \partial \Omega_1\);

(ii) \(\|Ax\| \leq \|x\|, \forall x \in P \cap \partial \Omega_2\);

(iii) \(\|Ax\| \geq \|x\|, \forall x \in P \cap \partial \Omega_3\).

Then \(A\) has at least two fixed points \(x^*, x^{**}\) in \(P \cap (\bar{\Omega}_3 \setminus \Omega_1)\), and \(x^* \in P \cap (\bar{\Omega}_2 \setminus \Omega_1), x^{**} \in P \cap (\bar{\Omega}_3 \setminus \bar{\Omega}_2)\).

The paper is organized in the following fashion. In Section 2, we provide some necessary background and introduce several definitions on time scales. In particular, we state some properties of the Green’s function associated with BVP (1.1). In Section 3, the main result will be stated and proved.

### 2. Preliminaries

Let \(J = [a, b]\), where \(a, b \in \mathbb{T}\) and \(a \leq b\). The basic space used in this paper is \(E = C[a, b]\) of real-valued continuous (in the topology of \(\mathbb{T}\)) functions \(x(t)\) defined on \([a, b]\). It is well known that \(E\) is a Banach space with the norm \(\|\cdot\|\) defined by \(\|x\| = \max_{t \in J} |x(t)|\). Let \(K\) be a cone of \(E\), \(K_r = \{x \in K : \|x\| < r\}, \partial K_r = \{x \in K : \|x\| = r\}\), where \(r > 0\).

The following assumptions will stand throughout this paper:
(H₁) \( w : [a, b] \to [0, +\infty) \) is continuous and there exists \( t_0 \in [a, b] \) such that \( w(t_0) > 0 \);
(H₂) \( p : [a, b] \to (0, +\infty) \) is \( \nabla \)-differentiable on \( T_k \);
(H₃) \( q : [a, b] \to [0, +\infty) \) is continuous, and if \( q \equiv 0 \), then \( \alpha + \gamma > 0 \);
(H₄) \( f : [a, b] \times \mathbb{R} \to \mathbb{R} \) is continuous in \( (t, \xi) \) with respect to the topology of \( \mathbb{R} \times \mathbb{R} \) and \( f(t, \xi) \geq 0 \) for \( \xi \in \mathbb{R}^+ \),
where \( \mathbb{R}^+ \) denotes the set of nonnegative real numbers.

For convenience, we introduce several definitions on time scales we refer to [18–22], and give some properties of the Green’s function associated with BVP (1.1) and some lemmas which are useful in proving our main results in the rest of this section.

**Definition 2.1.** A time scale \( T \) is a nonempty closed subset of \( \mathbb{R} \).

**Definition 2.2.** Define the forward (backward) jump operator \( \sigma(t) \) for \( t < \sup \mathbb{R} \) \( (\rho(t) \text{ at } t \to \inf \mathbb{R}) \) by

\[
\sigma(t) = \inf \{ t > t : t \in T \} \quad (\rho(t) = \sup \{ t < t : t \in T \})
\]

for all \( t \in T \). We assume that \( T \) has the topology that it inherits from the standard topology on \( \mathbb{R} \) and say \( t \) is right-scattered, left-scattered, right-dense and left-dense if \( \sigma(t) > t, \rho(t) < t, \sigma(t) = t \) and \( \rho(t) = t \), respectively. Finally, we introduce the sets \( T^k \) and \( T_k \) which are derived from the time scale \( T \) as follows. If \( T \) has a left-scattered maximum \( t_1 \), then \( T^k = T - t_1 \), otherwise \( T^k = T \). If \( T \) has a right-scattered minimum \( t_2 \), then \( T_k = T - t_2 \), otherwise \( T_k = T \).

**Definition 2.3.** Fix \( t \in T \) and let \( y : T \to \mathbb{R} \). Define \( y^\Delta(t) \) to be the number (if it exists) with the property that given \( \varepsilon > 0 \) there is a neighbourhood \( U \) of \( t \) with

\[
| [y(\sigma(t)) - y(s)] - y^\Delta(t)[\sigma(t) - s] | < \varepsilon |\sigma(t) - s|
\]

for all \( s \in U \). Call \( y^\Delta(t) \) the (delta) derivative of \( y(t) \) at the point \( t \).

**Definition 2.4.** Fix \( t \in T \) and let \( y : T \to \mathbb{R} \). Define \( y^\nabla(t) \) to be the number (if it exists) with the property that given \( \varepsilon > 0 \) there is a neighbourhood \( U \) of \( t \) with

\[
| [y(\rho(t)) - y(s)] - y^\nabla(t)[\rho(t) - s] | < \varepsilon |\rho(t) - s|
\]

for all \( s \in U \). Call \( y^\nabla(t) \) the (nabla) derivative of \( y(t) \) at the point \( t \).

If \( T = \mathbb{R} \) then \( f^\Delta(t) = f^\nabla(t) = f'(t) \). If \( T = \mathbb{Z} \) then \( f^\Delta(t) = f(t + 1) - f(t) \) is the forward difference operator while \( f^\nabla(t) = f(t) - f(t - 1) \) is the backward difference operator.

**Definition 2.5.** A function \( f : T \to \mathbb{R} \) is called rd-continuous provided it is continuous at all right dense points of \( T \) and its left sided limit exists (finite) at left dense points of \( T \).

**Definition 2.6.** A function \( f : T \to \mathbb{R} \) is called ld-continuous provided it is continuous at all left dense points of \( T \) and its right sided limit exists (finite) at right dense points of \( T \).

**Definition 2.7.** If \( f : T \to \mathbb{R} \) is \( \Delta \)-differentiable at \( t_0 \in T^k \) and \( f^\Delta(t_0) > 0 \) \( (f^\Delta(t_0) < 0) \) then \( f \) is right-increasing (right-decreasing), at \( t_0 \).

**Remark 2.1.** All right-dense continuous bounded functions on \([a, b]\) are delta integrable from \( a \) to \( b \), and all left-dense continuous bounded functions on \([a, b]\) are nabla integrable from \( a \) to \( b \).

Throughout this paper, we assume \( T \) is a closed subset of \( \mathbb{R} \) with \( a \in T_k, b \in T^k \).

In this paper, the Green’s function of the corresponding homogeneous BVP defined by

\[
G(t, s) = \frac{1}{A} \begin{cases} 
\phi(s)\psi(t), & \text{if } \rho(a) \leq s \leq t \leq \sigma(b), \\
\phi(t)\psi(s), & \text{if } \rho(a) \leq t \leq s \leq \sigma(b)
\end{cases}
\]

where \( \phi \) and \( \psi \) satisfy

\[
L\phi = 0, \quad \phi(\rho(a)) = \beta, \quad \phi^\Delta(\rho(a)) = \alpha, \\
L\psi = 0, \quad \psi(b) = \delta, \quad \psi^\Delta(b) = -\gamma.
\]

(2.1)
It is not difficult from [4] to show that \( A := -[\phi(t)\psi^{\Delta}(t) - \phi^{\Delta}(t)\psi(t)] > 0 \) and (i) \( \phi \) is nondecreasing on \( J \) and \( \phi \geq 0 \) on \( J \); (ii) \( \psi \) is nonincreasing on \( J \) and \( \psi \geq 0 \) on \( J \).

It is easy to prove that \( G(t, s) \) has the following properties.

**Proposition 2.1.** For \( t, s \in (\rho(a), b) \), we have

\[
G(t, s) > 0.
\]  

**Proposition 2.2.** For \( t, s \in J \), we have

\[
0 \leq G(t, s) \leq G(s, s).
\]  

**Proposition 2.3.** Let \( \bar{\theta} \in \mathbb{T} \) and \( \bar{\theta} \in (a, \frac{b + a}{2}) \). We define \( J_{\bar{\theta}} = [\bar{\theta}, b + a - \bar{\theta}] \). Then for all \( t \in J_{\bar{\theta}}, s \in (a, b) \) we have

\[
G(t, s) \geq \Gamma G(s, s),
\]

where

\[
\Gamma (= \Gamma_{\bar{\theta}}, \text{ i.e., } \Gamma \text{ is dependent on } \bar{\theta}) = \min \left\{ \frac{\psi(b + a - \bar{\theta})}{\psi(a)}, \frac{\phi(\bar{\theta})}{\phi(b)} \right\}.
\]

In fact, for \( t \in [\bar{\theta}, b + a - \bar{\theta}] \), we have

\[
\frac{G(t, s)}{G(s, s)} \geq \min \left\{ \frac{\psi(b + a - \bar{\theta})}{\psi(s)}, \frac{\phi(\bar{\theta})}{\phi(s)} \right\} \geq \min \left\{ \frac{\psi(b + a - \bar{\theta})}{\psi(a)}, \frac{\phi(\bar{\theta})}{\phi(b)} \right\} =: \Gamma^*.
\]

It is easy to see that \( 0 < \Gamma < 1 \).

For the sake of applying Lemmas 1.1 and 1.2, we construct a cone in \( E = C[a, b] \) by

\[
K = \{ x \in C[a, b] : x \geq 0, \min_{t \in J_{\bar{\theta}}} x(t) \geq \Gamma \| x \| \}.
\]

It is easy to see \( K \) is a closed convex cone of \( E \).

Define \( T_{\lambda} : K \to K \) by

\[
(T_{\lambda}x)(t) = \lambda \int_{\rho(a)}^{b} G(t, s)w(s)f(s, x(s))\nabla s, \quad t \in [a, b].
\]

By (2.9), it is well known that BVP (1.1) has a positive solution \( x \) if and only if \( x \in K \) is a fixed point of \( T_{\lambda} \).

**Lemma 2.1.** Let (H1)–(H4) hold. Then \( T_{\lambda}K \subset K \) and \( T_{\lambda} : K \to K \) is completely continuous.

**Proof.** For \( x \in K \), by (2.9), we have \( T_{\lambda}x(t) \geq 0 \) and

\[
\| T_{\lambda}x \| \leq \lambda \int_{\rho(a)}^{b} G(s, s)w(s)f(s, x(s))\nabla s.
\]

On the other hand, by (2.6), (2.9) and (2.10), we obtain

\[
\min_{t \in J_{\bar{\theta}}} (T_{\lambda}x)(t) = \min_{t \in J_{\bar{\theta}}} \lambda \int_{\rho(a)}^{b} G(t, s)w(s)f(s, x(s))\nabla s
\]

\[
\geq \lambda \Gamma \int_{\rho(a)}^{b} G(s, s)w(s)f(s, x(s))\nabla s
\]

\[
\geq \Gamma \| T_{\lambda}x \|.
\]

Therefore \( T_{\lambda}(x) \in K \), i.e. \( T_{\lambda}K \subset K \).

Next by standard methods and Ascoli–Arzela theorem one can prove \( T_{\lambda} : K \to K \) is completely continuous. So it is omitted. \( \square \)
3. Main results

Write

\[ f_\beta = \limsup_{x \to \beta} \max_{t \in J} \frac{f(t, x)}{x}, \quad f_\beta = \liminf_{x \to \beta} \min_{t \in J} \frac{f(t, x)}{x}, \]

where \( \beta \) denotes 0 or \( \infty \).

In this section, we apply the Lemmas 1.1 and 1.2 to establish the existence of positive solutions for BVP (1.1).

**Theorem 3.1.** Let \((H_1)–(H_4)\) hold. In addition, letting one of the following two conditions

(i) \( f_0 = 0 \) and \( f_\infty = \infty \); 
(ii) \( f_0 = \infty \) and \( f_\infty = 0 \)

be satisfied, then, for all \( \lambda > 0 \), BVP (1.1) has at least one positive solution \( x^*(t) \).

**Proof.** Let \( T_\lambda \) be cone preserving, completely continuous operator that was defined by (2.9).

(i) Considering \( f_0 = 0 \), there exists \( r_1 > 0 \) such that \( f(t, x) \leq \varepsilon_1 x \), for \( 0 \leq x \leq r_1, t \in J \), where \( \varepsilon_1 > 0 \) satisfies

\[ \varepsilon_1 \frac{1}{A} \phi(b) \psi(\rho(a)) \int_{\rho(a)}^{b} w(s)\,\nabla s \leq 1. \]

So, for \( x \in \partial K_{r_1} \), we have from (2.5)

\[ (T_\lambda)(x)(t) \leq \lambda \int_{\rho(a)}^{b} G(s, s)w(s)f(s, x(s))\nabla s \]

\[ \leq \lambda \int_{\rho(a)}^{b} G(s, s)w(s)\varepsilon_1 x(s)\nabla s \]

\[ \leq \lambda \frac{1}{A} \phi(b) \psi(\rho(a)) \| x \| \varepsilon_1 \int_{\rho(a)}^{b} w(s)\nabla s \]

\[ \leq \| x \|. \]

Consequently, for \( x \in \partial K_{r_1}, t \in J \), we have

\[ \| T_\lambda x \| \leq \| x \|. \quad (3.1) \]

Next, turning to \( f_\infty = \infty \), there exists \( r_2 > 0 \) such that \( f(t, x) \geq \varepsilon_2 x \) for \( x \geq r_2, t \in J \), where \( \varepsilon_2 \) satisfies

\[ \varepsilon_2 \frac{1}{A} \Gamma^2 \phi(\tilde{\theta}) \psi(b + a - \tilde{\theta}) \| x \| \int_{\tilde{\theta}}^{b+a-\tilde{\theta}} w(s)\nabla s \geq 1. \]

Choose \( r_2 = \max\{ \frac{r_1}{2}, r_1 + 1 \} \), then \( r_2 > r_1 \). If \( x \in \partial K_{r_2}, t \in J \), then

\[ \min_{t \in J} x(t) \geq \Gamma \| x \| = \Gamma r_2 \geq r_2, \]

and

\[ (T_\lambda)(x)(t) \geq \lambda \int_{\tilde{\theta}}^{b+a-\tilde{\theta}} G(t, s)w(s)f(s, x(s))\nabla s \]

\[ \geq \lambda \int_{\tilde{\theta}}^{b+a-\tilde{\theta}} G(t, s)w(s)\varepsilon_2 x(s)\nabla s \]

\[ \geq \min_{t \in J} \lambda \int_{\tilde{\theta}}^{b+a-\tilde{\theta}} G(t, s)w(s)\varepsilon_2 x(s)\nabla s \]

\[ \geq \lambda \Gamma^2 \int_{\tilde{\theta}}^{b+a-\tilde{\theta}} G(s, s)w(s)\nabla s \varepsilon_2 \| x \| \]

\[ \geq \lambda \frac{1}{A} \Gamma^2 \phi(\tilde{\theta}) \psi(b + a - \tilde{\theta}) \varepsilon_2 \| x \| \int_{\tilde{\theta}}^{b+a-\tilde{\theta}} w(s)\nabla s \]

\[ \geq \| x \|. \]
Thus, \( \| T_\lambda x \| \geq \| x \| \). Hence, for \( x \in \partial K_{r_2} \) we have
\[
\| T_\lambda x \| \geq \| x \|. \tag{3.2}
\]

Applying (i) of Lemma 1.1 to (3.1) and (3.2) yields that \( T_\lambda \) has a fixed point \( x^* \in \bar{K}_{r_1, r_2} \), \( r_1 \leq \| x^* \| \leq r_2 \) and \( x^*(t) \geq \Gamma \| x^* \| > 0, t \in J_\bar{\theta} \). Thus it follows that BVP (1.1) has a positive solution \( x^* \) for all \( \lambda > 0 \).

(ii) Considering \( f_0 = \infty \), there exists \( r_3 > 0 \) such that \( f(t, x) \geq \varepsilon_3 x \), for \( 0 \leq x \leq r_3, t \in J_\bar{\theta} \), where \( \varepsilon_3 > 0 \) satisfies \( \frac{1}{4} \lambda \int \phi(\bar{\theta}) \psi(b + a - \bar{\theta}) \varepsilon_3 \int_{\bar{\theta}}^{b + a - \bar{\theta}} w(s) \| s \| \geq 1 \).

So, for \( x \in \partial K_{r_3}, t \in J_\bar{\theta} \), we have from (2.6)
\[
(T_\lambda x)(t) \geq \lambda \int_{\bar{\theta}}^{b + a - \bar{\theta}} G(t, s) w(s) f(s, x(s)) \| s \| \geq \min_{t \in J_\bar{\theta}} \lambda \int_{\bar{\theta}}^{b + a - \bar{\theta}} \lambda \int_{\bar{\theta}}^{b + a - \bar{\theta}} G(t, s) w(s) \| s \| \geq \lambda \int_{\bar{\theta}}^{b + a - \bar{\theta}} G(s, s) w(s) \| s \| \geq \frac{1}{\lambda} \lambda \int \phi(\bar{\theta}) \psi(b + a - \bar{\theta}) \varepsilon_3 \int_{\bar{\theta}}^{b + a - \bar{\theta}} w(s) \| s \| \geq \| x \|. \tag{3.3}
\]

Consequently, for \( x \in \partial K_{r_3}, t \in J_\bar{\theta} \), we have
\[
\| T_\lambda x \| \geq \| x \|. \tag{3.4}
\]

Next, turning to \( f^\infty = 0 \), there exists \( \bar{r}_4 > 0 \) such that \( f(t, x) \leq \varepsilon_4 x \), for \( x \geq \bar{r}_4, t \in J \), where \( \varepsilon_4 > 0 \) satisfies \( \varepsilon_4 \lambda \int \phi(b) \psi(\rho(a)) \int_{\rho(a)}^{b} w(s) \| s \| \leq \frac{1}{2} \).

Let
\[
M = \lambda \sup_{x \in \partial K_{\bar{r}_4}, t \in [\rho(a), b]} f(t, x(t)) \int_{\rho(a)}^{b} G(t, t) w(t) dt.
\]

It is not difficult to see that \( M < +\infty \).

Choosing \( r_4 > \max\{r_3, \bar{r}_4, 2M\} \), then we get \( M < \frac{1}{2} r_4 \).

Now, we choose \( x \in \partial K_{r_4} \) arbitrarily. Letting \( \bar{x}(t) = \min\{x(t), \bar{r}_4\} \), then \( \bar{x} \in \partial K_{\bar{r}_4} \). In addition, writing \( e(x) = \{t \in [\rho(a), b] : x(t) > \bar{r}_4\} \). Therefore, for \( t \in e(x) \), we get \( \bar{r}_4 < x(t) \leq \| x \| = r_4 \). By the choosing of \( \bar{r}_4 \), for \( t \in e(x) \), we have \( f(t, x(t)) \leq \varepsilon_4 r_4 \).

Thus for \( x \in \partial K_{r_4} \), we have from (2.5)
\[
(T_\lambda x)(t) \leq \lambda \int_{\rho(a)}^{b} G(s, s) w(s) f(s, x(s)) \| s \| = \lambda \int_{e(x)} G(s, s) w(s) f(s, x(s)) \| s \| + \lambda \int_{[\rho(a), b] \setminus e(x)} G(s, s) w(s) f(s, x(s)) \| s \| \leq \lambda \varepsilon_4 r_4 \int_{\rho(a)}^{b} G(s, s) w(s) \| s \| + \lambda \int_{\rho(a)}^{b} G(s, s) w(s) f(s, \bar{x}(s)) \| s \| \leq \lambda \varepsilon_4 r_4 \frac{1}{\lambda} \phi(b) \psi(\rho(a)) \int_{\rho(a)}^{b} w(s) \| s \| + M \leq \frac{1}{2} r_4 + \frac{1}{2} r_4 = r_4 = \| x \|. \tag{3.5}
\]
Consequently, from (3.5), for \( x \in \partial K_{r_4}, t \in J \), we have
\[
\| T_\lambda x \| \leq \| x \|. \tag{3.6}
\]

Applying (ii) of Lemma 1.1 to (3.4) and (3.6) yields that \( T_\lambda \) has a fixed point \( x^* \in \tilde{K}_{r_3, r_4}, r_3 \leq \| x^* \| \leq r_4 \) and \( x^*(t) \geq \Gamma \| x^* \| > 0, t \in J_c \). Thus it follows that BVP (1.1) has a positive solution \( x^* \) for all \( \lambda > 0 \). The proof is complete. \( \square \)

**Theorem 3.2.** Let \((H_1)-(H_4)\) hold. In addition, letting the following two conditions
(i) \( f^0 = 0 \) or \( f^\infty = 0 \);
(ii) there exists \( \rho_1 > 0 \), for \( 0 \leq \lambda \leq \rho_1 \), \( t \in J_0 \) such that \( f(t, x) \geq \epsilon_1 t \rho_1 \), where \( \epsilon_1 = [\lambda \Gamma \frac{1}{A} \phi(b) \psi(\rho(a))]^{\frac{1}{2}} \int_{\tilde{\theta}}^{b-a-\tilde{\theta}} w(s) \nabla s \geq 1 \)

be satisfied, then there exists \( \lambda_0 > 0 \) such that for all \( \lambda > \lambda_0 \), BVP (1.1) has at least one positive solution \( x^*(t) \).

**Proof.** Considering \( f^0 = 0 \), there exists \( 0 < \epsilon_5 < \rho_1 \) such that \( f(t, x) \leq \epsilon_5 x \), for \( 0 \leq x \leq \epsilon_5, t \in J \), where \( \epsilon_5 > 0 \) satisfies \( \epsilon_5 \lambda \frac{1}{A} \phi(b) \psi(\rho(a)) \| x \| \int_{\rho(a)}^{b} w(s) \nabla s \leq 1 \).

So, for \( x \in \partial K_{r_5} \), we have from (2.5)
\[
(T_\lambda x)(t) \leq \lambda \int_{\rho(a)}^{b} G(s, s)w(s) f(s, x(s)) \nabla s
\leq \lambda \int_{\rho(a)}^{b} G(s, s)w(s) \epsilon_5 x(s) \nabla s
\leq \lambda \frac{1}{A} \phi(b) \psi(\rho(a)) \| x \| \epsilon_5 \int_{\rho(a)}^{b} w(s) \nabla s
\leq \| x \|. \tag{3.7}
\]

Consequently, for \( x \in \partial K_{r_5} \), we have
\[
\| T_\lambda x \| \leq \| x \| . \tag{3.8}
\]

If \( f^\infty = 0 \), similar to the proof of (3.6), there exists \( \rho_6 > \rho_1 \) such that \( f(t, x) \leq \epsilon_6 x \), for \( x \geq \epsilon_6, t \in J \), where \( \epsilon_6 > 0 \) satisfies \( \epsilon_6 \lambda \frac{1}{A} \phi(b) \psi(\rho(a)) \int_{\rho(a)}^{b} w(s) \nabla s \leq 1 \), and, for \( x \in \partial K_{r_6}, t \in J \), we have
\[
\| T_\lambda x \| \leq \| x \|. \tag{3.9}
\]

By (ii), for \( x \in \partial K_{\rho_1}, t \in J_0 \), we have
\[
(T_\lambda x)(t) \geq \lambda \int_{\tilde{\theta}}^{b-a-\tilde{\theta}} G(t, s)w(s) f(s, x(s)) \nabla s
\geq \lambda \tau_1 \rho_1 \int_{\tilde{\theta}}^{b-a-\tilde{\theta}} G(t, s)w(s) \nabla s
\geq \min_{t \in J_0} \lambda \tau_1 \rho_1 \int_{\tilde{\theta}}^{b-a-\tilde{\theta}} G(t, s)w(s) \nabla s
\geq \lambda \Gamma \tau_1 \rho_1 \frac{1}{A} \phi(\tilde{\theta}) \psi(b + a - \tilde{\theta}) \int_{\tilde{\theta}}^{b-a-\tilde{\theta}} w(s) \nabla s
\geq \rho_1 \geq \| x \| .
\]

Consequently, for \( x \in \partial K_{\rho_1}, t \in J_0 \), we have \( \| T_\lambda x \| \geq \| x \| \), which implies that there exists \( \lambda_0 > 0 \) such that for \( x \in \partial K_{\rho_1}, \lambda > \lambda_0 \), we have
\[
\| T_\lambda x \| > \| x \|. \tag{3.9}
\]
By Lemma 1.1, for all $\lambda > \lambda_0$, (3.7) and (3.9), respectively, yield that $T_\lambda$ has a fixed point $x^* \in \bar{K}_{r_5, \rho_1}$, $r_5 \leq \|x^*\| < \rho_1$ and $x^*(t) \geq \Gamma \|x^*\| > 0$, $t \in J_{\bar{\theta}}$ or $x^* \in \bar{K}_{\rho_1, r_6}$, $\rho_1 < \|x^*\| \leq r_6$ and $x^*(t) \geq \Gamma \|x^*\| > 0$, $t \in J_{\bar{\theta}}$. Thus it follows that BVP (1.1) has a positive solution $x^*$ for all $\lambda > \lambda_0$. $\square$

**Theorem 3.3.** Let (H1)–(H4) hold. In addition, letting the following two conditions

(i) $f_0 = \infty$ or $f_\infty = \infty$;

(ii) there exists $\rho_2 > 0$, for $0 \leq x \leq \rho_2$, $t \in J_{\bar{\theta}}$ such that $f(t, x) \leq \tau_2 \rho_2$, where $\tau_2 = \lfloor \lambda \Gamma \frac{1}{\lambda} \phi(b) \psi(\rho(a)) f^{b+a-\bar{\theta}} w(s) \nabla s \rfloor^{-1}$

be satisfied, then there exists $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$, BVP (1.1) has at least one positive solution $x^*(t)$.

The remaining of the proof is similar to that of Theorem 3.2.

**Theorem 3.4.** Let (H1)–(H4) hold. In addition, letting the following two conditions

(i) $\int_0^1 = \infty$ and $\int_{\infty} = \infty$;

(ii) there exists $\rho_1 > 0$, for $0 \leq x \leq \rho_1$, $t \in J_{\bar{\theta}}$ such that $f(t, x) \geq \tau_1 \rho_1$, where $\tau_1 = \lfloor \lambda \Gamma \frac{1}{\lambda} \phi(b) \psi(1 - \bar{\theta}) f^{1-\bar{\theta}} w(s) \nabla s \rfloor^{-1}$

be satisfied, then there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$, BVP (1.1) has at least two positive solutions $x^*(t), x^{**}(t)$.

**Proof.** The proof similar to Theorem 3.2. By Lemma 1.2, (3.7)–(3.9) yield that $T_\lambda$ has at least two fixed points $x^*$, $x^{**}$, where $x^* \in \bar{K}_{r_5, \rho_1}$, $r_5 \leq \|x^*\| < \rho_1$ and $x^*(t) \geq \Gamma \|x^*\| > 0$, $t \in J_{\bar{\theta}}$, $x^{**} \in \bar{K}_{\rho_1, r_6}$, $\rho_1 < \|x^{**}\| \leq r_6$ and $x^{**}(t) \geq \Gamma \|x^{**}\| > 0$, $t \in J_{\bar{\theta}}$. Thus it follows that BVP (1.1) has at least two positive solutions $x^*, x^{**}$ for all $\lambda > \lambda_0$. $\square$

**Theorem 3.5.** Let (H1)–(H4) hold. In addition, letting the following two conditions

(i) $f_0 = \infty$ and $f_\infty = \infty$;

(ii) there exists $\rho_2 > 0$, for $0 \leq x \leq \rho_2$, $t \in J_{\bar{\theta}}$ such that $f(t, x) \leq \tau_2 \rho_2$, where $\tau_2 = \lfloor \lambda \Gamma \frac{1}{\lambda} \phi(b) \psi(\rho(a)) f^{b+a-\bar{\theta}} w(s) \nabla s \rfloor^{-1}$

be satisfied, then there exists $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$, BVP (1.1) has at least two positive solutions $x^*(t), x^{**}(t)$.

The remaining of the proof is similar to that of Theorem 3.4.

**Theorem 3.6.** Let (H1)–(H4) hold. In addition, letting $f_0, f_\infty < \infty$ be satisfied, then there exists $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$, BVP (1.1) has no positive solution.

**Proof.** Since $f_0 < \infty$ and $f_\infty < \infty$, then there exist $\eta_3 > 0$, $\eta_4 > 0$, $h_3 > 0$ and $h_4 > 0$ such that $h_3 < h_4$ and for $t \in J$, $0 \leq x \leq h_3$, we have

$$f(t, x) \leq \eta_3 x,$$  \hspace{1cm} (3.10)

and for $t \in J$, $x \geq h_4$, we have

$$f(t, x) \leq \eta_4 x.$$  \hspace{1cm} (3.11)

Let

$$\eta^* = \max \left\{ \eta_3, \eta_4, \max \left\{ \frac{f(t, x)}{x} : t \in J, h_3 \leq x \leq h_4 \right\} \right\} > 0.$$  \hspace{1cm} (3.12)

Thus, for $t \in J$, $x \in \mathbb{R}^+$, we have

$$f(t, x) \leq \eta^* x.$$  \hspace{1cm} (3.12)

Assume $y$ is a positive solution of BVP (1.1). We will show that this leads to a contradiction for $0 < \lambda < \lambda_0 = \left\lfloor \frac{1}{\lambda} \eta^* \phi(b) \psi(\rho(a)) f^{b} w(s) \nabla s \right\rfloor^{-1}$. 

In fact, for $0 < \lambda < \lambda_0$, $t \in J$, since $(Ty)(t) = y(t)$, we have
\[
\|y\| = \|(T_{\lambda}y)\|
\leq \max_{t \in J} \lambda \int_{\rho(a)}^{b} G(t, s) w(s) f(s, y(s)) \nabla s
\leq \lambda \int_{\rho(a)}^{b} G(s, s) w(s) f(s, y(s)) \nabla s
\leq \lambda \int_{\rho(a)}^{b} G(s, s) w(s) \eta^* y(s) \nabla s
\leq \lambda \|y\| \eta^* \frac{1}{\Lambda} \phi(b) \psi(\rho(a)) \int_{\rho(a)}^{b} w(s) \nabla s
< \|y\|,
\]
which is a contradiction. The proof is complete. \[\square\]

Acknowledgement

The authors thank the referee for his/her valuable comments and suggestions.

References