Invariant Hamming graphs in infinite quasi-median graphs

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Abstract

It is shown that a quasi-median graph $G$ without isometric infinite paths contains a Hamming graph (i.e., a cartesian product of complete graphs) which is invariant under any automorphism of $G$, and moreover if $G$ has no infinite path, then any contraction of $G$ into itself stabilizes a finite Hamming graph.

0. Introduction

For several classes of graphs, it has been shown that each member of these classes contains a regular subgraph of the same class which is invariant under any automorphism, or that any contraction of that graph into itself stabilizes a regular subgraph of the same class. One can find various examples of such classes, particularly with finite graphs. See for example: Nowakowski and Rival [8] for trees, Poston [12] for finite contractible graphs, Quillot [13] for finite ball-Helly graphs, Polat [10, 11] for infinite dismantlable graphs and infinite ball-Helly graphs, Bandelt and Mulder [1] for finite pseudo-median graphs, Bandelt and van de Vel [3] for finite median graphs, and Tardif [16] for infinite median graphs.

The graphs that we consider in this paper are the quasi-median graphs. These graphs have been defined independently by several authors and with various approaches. The finite quasi-median graphs were introduced as a generalization of median graphs (see [7]), as connected subgraphs of Hamming graphs (i.e., cartesian products of complete graphs) that are closed under the quasi-median operation (see [7, 5]), as retracts of Hamming graphs (see [17, 5]), as graphs in which there exists an optimal strategy for a particular dynamic location problem (see [5]). Note that median graphs are the bipartite quasi-median graphs and that the regular quasi-median graphs are precisely the Hamming graphs. Bandelt et al. [2] gave several characterizations of (finite or infinite) quasi-median graphs, by bringing together

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different approaches, and in particular by linking those graphs with some ternary algebras called quasi-median algebras.

Some special sets of vertices, called prefibers, related to the structure of metric space which is naturally associated with a graph, are very important for the study of quasi-median graphs. In any graph, the family of all prefibers has the Helly property (i.e., every finite family of pairwise non-disjoint prefibers has a nonempty intersection). Moreover this property also holds for every infinite family of nondisjoint prefibers if the graph has no isometric rays (i.e., no distance-preserving one-way infinite paths), and this enables to prove that:

A quasi-median graph without isometric rays contains a Hamming graph which is invariant under any automorphism.

This result holds a fortiori if the graph is rayless (i.e., without infinite paths), but in this case we also have the following:

Any contraction of a rayless quasi-median graph stabilizes a finite Hamming graph.

These results generalize those recently obtained by the first author for finite quasi-median graphs [4] as well as some results of Tardif on median graphs [16].

1. Notation and definitions

The graphs we consider are undirected, without loops and multiple edges. We denote by \( V(G) \) the vertex set of a graph \( G \), and by \( E(G) \) its edge set. If \( x \) and \( y \) are two vertices of a graph \( G \) we write \( x \equiv a y \) if \( x = y \) or \( \{x, y\} \in E(G) \). If \( x \in V(G) \), the set \( V(x; G) := \{y \in V(G): \{x, y\} \in E(G)\} \) is the neighborhood of \( x \). The subgraph of \( G \) induced by a subset \( A \) of \( V(G) \) is denoted by \( G[A] \), or simply by \( A \) whenever no confusion is likely; and we set \( G - A := G[V(G) - A] \). A path \( W := <x_0, \ldots, x_n> \) is a graph with \( V(W) = \{x_0, \ldots, x_n\} \), \( x_i \neq x_j \) if \( i \neq j \), and \( E(W) = \{\{x_i, x_{i+1}\}: 0 \leq i < n\} \); \( x_0 \) and \( x_n \) are its endpoints, and \( W \) is also called an \( x_0x_n \)-path. A ray or one-way infinite path \( R := <x_0, x_1, \ldots> \) is defined similarly.

The (geodesic) distance in \( G \) between two vertices \( x \) and \( y \), that is the length of an \( xy \)-geodesic (i.e. a shortest \( xy \)-path) in \( G \), is denoted by \( d_G(x, y) \); and every graph \( G \) is endowed with the structure of metric space associated with this distance. A subgraph \( H \) of \( G \) is isometric if \( d_H(x, y) = d_G(x, y) \) for all vertices \( x \) and \( y \) of \( H \). If \( x \) is a vertex of \( G \) and \( r \) a nonnegative integer, the set \( B_G(x, r) := \{y \in V(G): d_G(x, y) \leq r\} \) is the ball of center \( x \) and radius \( r \) in \( G \). If \( x \) and \( y \) are two vertices of \( G \), then the interval \( I_G(x, y) \) is the set of vertices of all \( xy \)-geodesics. Clearly \( I_G(x, y) := \{z \in V(G): d_G(x, z) + d_G(z, y) = d_G(x, y)\} \). A subset \( C \) of \( V(G) \) is geodesically convex, for short convex, if it contains the interval \( I_G(x, y) \) for all \( x, y \in C \). The convex hull \( co_G(C) \) of \( C \) in \( G \) is the smallest convex set of \( G \) containing \( C \). Thus \( co_G(C) = \bigcup_{n \geq 0} C_n \) where \( C_0 = C \) and \( C_{n+1} = \bigcup_{x, y \in C_n} I_G(x, y) \).
Let \((u_1, u_2, u_3)\) be a triple of vertices of a graph \(G\). A quasi-median of \((u_1, u_2, u_3)\) is a triple of vertices \((x_1, x_2, x_3)\) such that

- \(x_i, x_j\) lie on a \(u_iu_j\)-geodesic, \(i, j \in \{1, 2, 3\}\);
- \(d_G(x_1, x_2) = d_G(x_2, x_3) = d_G(x_3, x_1) = k\);
- \(k\) is minimal with respect to these conditions.

If \(k = 0\), then the quasi-median is reduced to a single vertex \(x\), which is called a median of the triple \((u_1, u_2, u_3)\). A median graph is a graph in which every triple of vertices has a unique median.

If \(G\) and \(H\) are two graphs, a map \(f : V(G) \rightarrow V(H)\) is a contraction if \(f\) preserves the relation \(\equiv\), i.e., \(x \equiv_G y\) implies \(f(x) \equiv_H f(y)\). Notice that a contraction \(f : G \rightarrow H\) is a non-expansive map between the metric spaces \((V(G), \text{dist}_G)\) and \((V(H), \text{dist}_H)\), i.e., \(\text{dist}_H(f(x), f(y)) \leq \text{dist}_G(x, y)\) for all \(x, y \in V(G)\). A contraction \(f\) from \(G\) onto an induced subgraph \(H\) of \(G\) is a retraction, and \(H\) is a retract of \(G\), if its restriction \(f|_H\) to \(H\) is the identity. The cartesian product \(G \times H\) of two graphs \(G\) and \(H\) is defined by \(V(G \times H) = V(G) \times V(H)\), and \((x, y) \equiv_{G \times H} (x', y')\) if and only if \(x = x'\) and \(y \equiv_H y'\), or \(x \equiv_G x'\) and \(y = y'\). Clearly \(d_{G \times H} = d_G + d_H\). A contraction \(f\) of \(G\) (into itself) is said to stabilize a set \(A\) of vertices (resp. a subgraph \(H\) of \(G\)) if \(f(A) = A\) (resp. \(f(H) = H\)). A subgraph \(H\) of \(G\) is said to be invariant if it is stabilized by any automorphism of \(G\).

A complete graph is simply called a simplex, and a clique is a simplex which is maximal with respect to inclusion. A Hamming graph (resp. hypercube) is a cartesian product of simplices (resp. \(K_2\)). As usual \(K_{2,3}\) (resp. \(K_{1,1,2}\)) denotes the complete bipartite (resp. tripartite) graph whose subsets of vertices have 2 and 3 (resp. 1, 1 and 2) elements, respectively. Roughly \(K_{1,1,2}\) is \(K_4\) minus an edge.

2. Prefibers

The concept of prefiber generalizes that of fiber of a cartesian product of metric spaces; it has been studied in particular by Dress and Scharlau [6] and by Tardif [14, 15].

2.1. Definition. Let \((\mathcal{X}, d)\) be a metric space. A prefiber (or gated set) of \(\mathcal{X}\) is a subset \(A\) of \(\mathcal{X}\) such that, for all \(x \in \mathcal{X}\), there is \(y \in A\) with \(d(x, z) = d(x, y) + d(y, z)\) for every \(z \in A\). The element \(y\) is unique, and the map \(\text{proj}_A : \mathcal{X} \rightarrow A\) defined by \(y = \text{proj}_A(x)\) is the projection onto \(A\).

In this paper we will use the following properties:

2.2. Properties (Tardif [14]). (i) If \(A\) and \(B\) are two prefibers of a metric space \((\mathcal{X}, d)\), then \(\text{proj}_A(B)\) is a prefiber of \(\mathcal{X}\). Moreover, if \(A \cap B \neq \emptyset\), then \(A \cap B\) is a prefiber of \((\mathcal{X}, d)\) and

\[
\text{proj}_{A \cap B} = \text{proj}_A \circ \text{proj}_B = \text{proj}_B \circ \text{proj}_A.
\]
(ii) The family of prefibers of a metric space has the Helly property (i.e., every finite family of pairwise nondisjoint prefibers has a nonempty intersection).

(iii) If \((X, d)\) is a complete metric space, and \(F\) a family of prefibers of \(X\) such that \(\bigcap F \neq \emptyset\), then \(\bigcap F\) is a prefiber of \(X\).

When the metric space of a graph is concerned, the projection associated with a prefiber is clearly an idempotent contraction, and a prefiber is a retract of this graph. We will give two properties that will be useful in the following. The first, which is a property on nested prefibers, enables to show the existence of some particular geodesic.

2.3. Proposition. Let \((W_n)_{n \geq 1}\) be a nested sequence of prefibers of connected graph \(G\) such that \(W_{n+1} \subseteq W_n\) for all \(n \geq 0\). If, for \(x_0 \in W_0\), the sequence \((x_n)_{n \geq 0}\) is defined by \(x_{n+1} = \text{proj}_{W_{n+1}}(x_n)\), then, for every \(n \geq 0\), there is an \(x_0x_n\)-geodesic \(P_n\) such that \(P_n \supseteq P_k\) for all \(k\) with \(0 \leq k \leq n\).

Proof. Construct \(P_0, P_1, \ldots\) such that \(P_n\) is an \(x_0x_n\)-geodesic containing \(P_k\) for \(0 \leq k \leq n\), as follows. Set \(P_0 := \langle x_0 \rangle\). Suppose that \(P_0, \ldots, P_n\) have already been constructed. If \(x_0 \in W_n\) then \(x_n = x_0\); let \(P_{n+1} := P_n\). If \(x_0 \notin W_n\) then \(x_n = \text{proj}_{W_n}(x_0)\) and \(x_n \in I_G(x_0, z)\) for every \(z \in W_n\), thus in particular for \(x_{n+1}\), therefore there exists an \(x_0x_{n+1}\)-geodesic \(P_{n+1}\) containing \(P_n\).

By 2.2(ii) the family of prefibers of a graph has the Helly property, but this can be strengthened if the graph has no isometric rays.

2.4. Proposition (Strong Helly property). If \(G\) is a graph without isometric rays, then any (finite or infinite) family of pairwise nondisjoint prefibers has a nonempty intersection.

Proof. Let \(F\) be a family of pairwise nondisjoint prefibers of \(G\). Assume that \(\bigcap F = \emptyset\). Construct prefibers \(W_0, W_1, \ldots\) of \(F\) and vertices \(x_0, x_1, \ldots\) such that \(x_n \in \bigcap_{0 \leq i \leq n} W_i\), as follows. Let \(W_0 \in F\) and \(x_0 \in W_0\). Suppose that \(W_0, \ldots, W_n\) and \(x_0, \ldots, x_n\) have already been constructed. There is \(W_{n+1} \in F\) such that \(x_n \notin W_{n+1}\), otherwise \(\bigcap F \neq \emptyset\); moreover \(\bigcap_{0 \leq i \leq n} W_i \neq \emptyset\) by the Helly property, thus \(\bigcap_{0 \leq i \leq n} W_i\) is a prefiber of \(G\) that is strictly included in the prefiber \(\bigcap_{0 \leq i \leq n} W_i\) and which does not contain \(x_n\). Let \(x_{n+1} := \text{proj}_{W_{n+1}}(x_n)\); clearly \(d(x_n, x_{n+1}) \geq 1\).

By Proposition 2.3 there exists a sequence \((P_n)_{n \geq 0}\) such that \(P_n\) is an \(x_0x_n\)-geodesic with \(P_n \subseteq P_{n+1}\) (where \(\subseteq\) denotes the strict inclusion) since \(d(x_n, x_{n+1}) \geq 1\). Thus \(\bigcup_{n \geq 0} P_n\) is an isometric ray, which is a contradiction with the hypothesis. Therefore \(\bigcap F \neq \emptyset\). Note that, by 2.2(iii), this intersection is a prefiber.
3. Quasi-median graphs

3.1. Definition. A graph $G$ is quasi-median if

(i) each triple of vertices of $G$ has a unique quasi-median,
(ii) $K_{1,1,2}$ is not an induced subgraph of $G$,
(iii) the convex hull of any isometric 6-cycle of $G$ is a 3-cube.

3.2. In order to recall some characterizations of quasi-median graphs given by Bandelt et al. [2] we introduce the following notation, definition and properties.

3.2.1. A graph $G$ has the triangle property if, for any vertices $u, v, w$ with $d_G(u, v) = d_G(u, w) = k > 1$ and $d_G(v, w) = 1$, there exists a common neighbor $x$ of $v$ and $w$ with $d_G(u, x) = k - 1$. $G$ has the quadrangle property if, for any vertices $u, v, w, z$ with $d_G(u, v) = d_G(u, w) = d_G(u, z) = k > 1$ and $d_G(v, w) = 2$ with $z$ a common neighbor of $v$ and $w$, there exists a common neighbor $x$ of $v$ and $w$ with $d_G(u, x) = k - 1$. A connected graph is said to be weakly modular if it has the triangle property as well as the quadrangle property.

3.2.2. The prefibers of a quasi-median graph $G$ has the following properties:

- A subset $S$ of $V(G)$ is prefiber if and only if it is convex and $A$-closed (i.e., if $S$ contains two vertices of a $K_3$ then it contains the third as well).
- As a convex set, any prefiber is closed under the quasi-median operation.

3.2.3. If $\{a, b\}$ is an edge of a graph $G$ we denote

$$W_{ab} := \{w \in V(G): d_G(a, w) < d_G(b, w)\}.$$ 

$$U_{ab} := \{u \in W_{ab}: u \text{ has a neighbor in } W_{ba}\}.$$ 

If $G$ is quasi-median, then these sets are such that:

- $W_{ab}$ and $U_{ab}$ are prefibers of $G$.
- The map $f: U_{ab} \to U_{ba}$, defined by $f(u) = v$ if and only if $\{u, v\}$ is an edge, is an isomorphism.

Theorem 3.3 (Bandelt et al. [2]). For a connected graph $G$ the following are equivalent:

(i) $G$ is a quasi-median graph;
(ii) Every interval of $G$ induces a median graph, and, for any three vertices $u, v, w$ of $G$,

$$I_G(u, v) \cap I_G(u, w) = \{u\} \implies d_G(v, w) \geq \max\{d_G(u, v), d_G(u, w)\};$$

(iii) $G$ is weakly modular and contains neither $K_{1,1,2}$ nor $K_{2,3}$ as induced subgraphs;
(iv) Every clique of $G$ is a prefiber, and the set $U_{ab}$ is convex for every edge $\{a, b\}$ of $G$. 

Since Hamming graphs are the quasi-median graphs that contain no convex path of length 2, one deduces the following result:

**Corollary 3.4** (Bandelt et al. [2]). For a connected graph $G$ the following are equivalent:

(i) $G$ is a Hamming graph;

(ii) Every interval of $G$ induces a hypercube, and, for any three vertices $u$, $v$, $w$ of $G$,

$$I_G(u, v) \cap I_G(u, w) = \{u\} \Rightarrow d_G(v, w) \geq \max\{d_G(u, v), d_G(u, w)\};$$

(iii) $G$ is weakly modular and contains neither $K_{1,1,2}$ nor $K_{2,3}$ as induced subgraphs, nor path of length 2 as a convex subgraph;

(iv) Every clique of $G$ is a prefiber, and the set $U_{ab}$ and $W_{ab}$ are equal and convex for every edge $\{a, b\}$ of $G$.

4. Invariant Hamming graph

In order to find an invariant Hamming graph in a quasi-median graph without isometric rays we introduce a particular family of prefibers satisfying the strong Helly property.

4.1. **Definition.** A prefiber $W$ of a graph $G$ is said of maximal type if

(i) $d_G(x, W) = 1$ for all $x \in V(G - W)$;

(ii) $G - \bigcup_{x \in V(G - W)} B_G(x, 1) \neq \emptyset$ (i.e., $W - \text{proj}_w(G - W) \neq \emptyset$).

We denote by $G'$ the intersection of all prefibers of maximal type of $G$.

4.2. **Lemma.** Let $G$ be a graph. We have the following:

(i) if $W$ and $W'$ are two prefibers of maximal type, then $W \cap W'$ is nonempty;

(ii) $G'$ is empty or is a prefiber;

(iii) if $G$ is nonempty and without isometric rays, then $G'$ is nonempty;

(iv) if $f$ is an automorphism of $G$, then, for every prefiber $W$ of maximal type, $f(W)$ is also a prefiber of maximal type;

(v) if $G'$ is nonempty, then $G'$ is a subgraph of $G$ that is invariant under any automorphism of $G$.

**Proof.**

(i) If $W \cap W' = \emptyset$, then $W \subseteq G - W'$ and $W' \subseteq G - W$. Hence $d_G(y, W') = 1$ for all $y \in W$. This implies that $W \subseteq \bigcup_{z \in W'} B_G(z, 1)$. Thus $W \subseteq \bigcup_{z \in G - W} B_G(z, 1)$. Therefore $G - \bigcup_{z \in V(G - W)} B_G(z, 1) = \emptyset$, a contradiction to Definition 4.1(ii).

(ii) This is a consequence of Property 2.2(iii).

(iii) The prefibers of maximal type, being pairwise nondisjoint by (i), have a nonempty intersection if $G$ has no isometric rays by Proposition 2.4.
(iv) $f(W)$ is a prefiber of $G$ since an automorphism is distance preserving. Let $y \in G - f(W)$. Then $d_{G}(y, f(W)) \geq 1$; and also $y = f(x)$ for some $x \in G - W$, which implies that $d_{G}(y, f(W)) \leq 1$. Consequently $d_{G}(y, f(W)) = 1$.

(v) Consequence of (iv).

4.3. Definition. Let $G$ be a graph. For any ordinal $\alpha$, we define $G^{(\alpha)}$ inductively as follows:

- $G^{(0)} := G$
- $G^{(\alpha + 1)} := (G^{(\alpha)})'$
- $G^{(\alpha)} := \bigcap_{\beta < \alpha} G^{(\beta)}$ if $\alpha$ is a limit ordinal.

We will denote $d(G) := \min \{\alpha : G^{(\alpha)} = G^{(\alpha + 1)}\}$ and $G^{(\infty)} := G^{d(G)}$.

4.4. Lemma. Let $G$ be a graph. We have the following:

(i) For any ordinal $\alpha$, the graph $G^{(\alpha)}$ (or more precisely its vertex set) is a prefiber of $G$ if it is nonempty.

(ii) If $G^{(\alpha)}$ is nonempty, then, for all ordinals $\alpha$ and $\beta$ with $\alpha \leq \beta$, $G^{(\beta)} \subseteq G^{(\alpha)}$, and the map $f_{\alpha\beta} : G^{(\alpha)} \to G^{(\beta)}$ defined by $f_{\alpha\beta} = \text{proj}_{\alpha\beta}$ is a retraction of $G^{(\alpha)}$.

(iii) If $G$ is nonempty and without isometric rays, then $G^{(\alpha)}$ is nonempty.

(iv) $G^{(\alpha)}$ is a subgraph of $G$ which is invariant under any automorphism of $G$.

(v) If $G$ is quasi-median, then so is $G^{(\alpha)}$ for every ordinal $\alpha$.

Proof. (i) is a consequence of 2.2(iii), and (ii) is obvious.

(iii) We will prove by induction on $\alpha$ that $G^{(\alpha)}$ is nonempty if $G$ has no isometric rays. This is clear if $\alpha = 0$. Let $\alpha \geq 0$. Suppose that this holds for any ordinal less than $\alpha$. If $\alpha = \beta + 1$, then $G^{(\alpha)} = (G^{(\beta)})'$ and the result is then a consequence of Lemma 4.2(iii). If $\alpha$ is a limit ordinal, then $G^{(\alpha)}$ is the intersection of a sequence of prefibers that are nonempty by (i) and the induction hypothesis, and pairwise nondisjoint since nested. Thus $G^{(\alpha)}$ is nonempty by Proposition 2.4.

(iv) We will also prove by induction on $\alpha$ that $G^{(\alpha)}$ is invariant. This is clear if $\alpha = 0$. Let $\alpha \geq 0$. Suppose that this holds for any ordinal less than $\alpha$. If $\alpha = \beta + 1$, then $G^{(\alpha)} = (G^{(\beta)})'$ and the result is then a consequence of Lemma 4.2(v). Assume that $\alpha$ is a limit ordinal, and let $f$ be an automorphism of $G$, and $x \in V(G^{(\alpha)})$. For every $\beta < \alpha$, $x \in V(G^{(\beta)})$ and $G^{(\beta)}$ is invariant. Thus $f(x) \in V(G^{(\beta)})$. Hence $f(x) \in \bigcap_{\beta < \alpha} V(G^{(\beta)}) = V(G^{(\alpha)})$. Therefore $G^{(\alpha)}$ is invariant under $f$.

(v) is a consequence of the convexity of every prefiber of a quasi-median graph.

4.5. We will now show that $G^{(\infty)}$ is a Hamming graph whenever $G$ has no isometric rays. If $\{a, b\}$ is an edge of a subgraph $H$ of $G$, we denote by $K_{H}(a, b)$ the maximal simplex included in $H$ and containing $a$ and $b$. When no confusion is likely we write $K(a, b)$ for $K_{G}(a, b)$.

If $G$ as well as $H$ are quasi-median, the simplex $K_{H}(a, b)$ is necessarily unique because of the lack of $K_{1,1,2}$ as an induced subgraph.
If $H$ is a prefiber of $G$, then $K_H(a, b) = K(a, b)$, since a simplex is included in any prefiber which contains one of its edges, and a prefiber is a $A$-closed convex set. 

The subgraph induced by $\bigcup \{ U_{xy} : x, y \in K(a, b) \}$ and $x \neq y$ is isomorphic to $U_{ab} \times K(a, b)$ since, if $x$ and $y$ are neighbors of $a$ and $b$ then $x \equiv_G y$ by the lack of $K_{1,1,2}$, and since the prefibers $U_{xy}$ and $U_{ab}$ are isomorphic (see (3.2.3)); furthermore $d_G(z, U_{ba}) = d_G(z, W_{ba}) = 1$ for every $z \in \bigcup \{ U_{xy} : x, y \in K(a, b) \}$. 

4.6. Lemma. Let $G$ be a quasi-median graph without isometric rays. If $G$ is not a Hamming graph, then there exists an edge $\{a, b\}$ of $G$ such that 

(i) $U_{ab} \neq W_{ba}$, 
(ii) $U_{xb} = W_{xb}$ for all $x \in V(K(a, b)) \setminus \{b\}$. 

Proof. Note that the finite case of this lemma was already proved by Wilkeit in [17, Proposition 7.2]. Let $\{x, y\}$ be an edge of $G$. We distinguish two cases. 

Case 1: $U_{zx} = W_{zx}$ for every vertex $z$ of $K(x, y)$. We construct inductively two sequences $(x_n)_{n \geq 0}$ and $(G_n)_{n \geq 0}$ as follows. Let $x_0 := x$, $x_1 := y$, $G_0 := G$ and $G_1 := U_{xx} = U_{x_1x_0}$ (i.e., $G_0 = G_1 \times K(x_1, x_0)$). Suppose that $x_n$ and $G_n$ have already been constructed for some $n \geq 1$. If $U_{xx_n} \cap G_n = W_{xx_n} \cap G_n$ for every vertex $z$ of $K(x_n, x_{n-1})$, and if $|U_{xx_n} \cap G_n| > 1$, take $x_{n+1} \in V(x_n; G) \cap U_{xx_n} \cap G_n$ and $G_{n+1} := U_{xx_{n+1}} \cap G_n$ ($G_{n+1}$ is a relative prefiber of $G_n$, thus a prefiber of $G$). Then one has $G_n = G_{n+1} \times K(x_{n+1}, x_n)$. 

The path $\langle x_0, \ldots, x_n \rangle$ of length $n$ is isometric by Proposition 2.3, thus, since $G$ has no isometric rays, the sequence $(x_n)_{n \geq 0}$ must be finite and there are two possibilities: 

- $U_{xx} \cap G_n = \langle x_n \rangle$ for some $n \geq 0$. Then $G = K(x_0, x_1) \times \cdots \times K(x_{n-1}, x_n)$. Thus $G$ is a Hamming graph; 
- $U_{xx_n} \cap G_n \neq W_{xx_n} \cap G_n$ for some vertex $z$ of $K(x_n, x_{n-1})$. We are then in the second case with $x = z$ and $y = x_{n-1}$. 

Case 2: $U_{zx} \neq W_{zx}$ for some vertex $z$ of $K(x, y)$. If $z$ is unique we can take $a := x$ and $b := z$. Otherwise there are at least two different vertices $z$ and $z'$ of $K(x, y)$ such that $U_{zx} \neq W_{zx}$ and $U_{z'x} \neq W_{z'x}$. Note that $U_{zz} \neq W_{zz}$ and $U_{z'z} \neq W_{z'z}$ since $K(x, y) = K(z, z')$. Construct by induction three sequences $(x_n)_{n \geq 0}$, $(z_n)_{n \geq 0}$ and $(t_n)_{n \geq 0}$ of vertices and a sequence $(P_n)_{n \geq 0}$ of $x_0x_n$-geodesics, as follows (cf. Fig. 1). 

Let $x_0 := z_0 := z$, $t_0 := z'$, and $P_0 := \langle x_0 \rangle$. Clearly $U_{tz_0} \neq W_{tz_0}$. Let $n \geq 0$. Suppose that $x_n, z_n, t_n$ and $P_n$ have already been constructed so that $U_{tz_n} \neq W_{tz_n}$ and $P_{n-1} \subseteq P_n$. Define $x_{n+1} := \text{proj}_{t_n}(x_n)$, and $z_{n+1}$ as any vertex of $U_{tz_n}$ having a neighbor $t$ in $W_{tz_n} - U_{tz_n}$. One has once again two possibilities: 

- if $U_{zx_{n+1}} = W_{zx_{n+1}}$ for all vertices $z$ of $K(t, z_{n+1}) \setminus \{z_{n+1}\}$, then take $a := t$ and $b := z_{n+1}$; 
- otherwise $U_{tz_{n+1}} \neq W_{tz_{n+1}}$ for some vertex $t_{n+1}$ of $K(t, z_{n+1}) \setminus \{z_{n+1}\}$. Then, by Proposition 2.3, there is an $x_0x_{n+1}$-geodesic $P_{n+1}$ of length greater than $n$ such that $P_{n+1} \supseteq P_n$; and this implies that the sequence $(x_n)_{n \geq 0}$ must be finite since $G$ has no isometric rays. \[\square\]
The prefiber $W_{ba}$ we got in the preceding lemma is of maximal type and is strictly included in $G$, hence:

**4.7. Corollary.** Let $G$ be a quasi-median graph without isometric rays. If $G$ is not a Hamming graph, then $G' \neq G$.

We can then prove the first of our main results:

**4.8. Theorem.** Every quasi-median graph without isometric rays contains a Hamming graph which is invariant under any automorphism.

**Proof.** Since $G^{(x)}$ is quasi-median for every ordinal $x$, $G^{(\infty)}$ is then a nonempty Hamming graph by Corollary 4.7, which is invariant by Lemma 4.4(iv).

**4.9. Remarks.** This invariant Hamming graph must be the cartesian product of finitely many simplices since it has no isometric rays.

Note that a quasi-median graph may contain rays but no isometric rays. Tardif in [16] exhibited a median graph, thus a quasi-median graph, with this property. For rayless quasi-median graphs Theorem 4.8 enables to get the following fixpoint-like result.

**4.10. Theorem.** Every contraction of a rayless quasi-median graph stabilizes a finite Hamming graph.

To prove it we need several lemmas and the following concept.
4.11. **Definition.** A set $A$ of vertices of a graph $G$ is *fragmented* if there is a finite subset $S$ of $V(G)$ such that the elements of $A$ are pairwise separated by $S$ (i.e., every path joining two distinct elements of $A$ contains a vertex in $S$).

4.12. **Lemma** (Polat [9, Theorem 3.12]). A graph $G$ is rayless if and only if every infinite set of vertices of $G$ contains an infinite fragmented subset.

4.13. **Lemma.** Let $G$ be a rayless graph. If all intervals of $G$ are finite, then the convex hull of any finite set of vertices of $G$ is finite.

**Proof.** Let $F$ be a finite subset of $V(G)$. Its convex hull is $\hat{F} = \bigcup_{n \geq 0} F_n$ where $F_0 := F$ and $F_{n+1} := \bigcup_{x, y \in F_n} I_G(x, y)$. Clearly every $F_n$ is finite since so are all intervals of $G$. Suppose that $\hat{F}$ is infinite. By Lemma 4.12 $\hat{F}$ contains an infinite fragmented subset $H$. Then there is a finite set $S$ that pairwise separates the elements of $H$, and that is minimal with respect to inclusion. W.l.o.g. we can suppose that $S \subseteq F$, otherwise we would consider the set $F' := F \cup S$ and take its convex closure.

Since $H$ is infinite and $S \subseteq F$, there exist infinitely many components of $G - F$ that contain an element of $H$. Moreover, among those, there are infinitely many of them that are disjoint from $F_1$, since $F_1$ is finite. Let $C$ be one of them, and let $n$ be the least integer greater than 1 with $C \cap F_n \neq \emptyset$. Such an integer exists since $C \cap H \neq \emptyset$ and $H \subseteq \hat{F} = \bigcup_{n \geq 0} F_n$. Let $x \in C \cap F_n$. Then, by the definition of $F_n$, $x$ belongs to an $a$-$b$-geodesic $P$ for some $a, b \in F_{n-1}$. As $C \cap F_{n-1} = \emptyset$, the endpoints $a$ and $b$ of $P$ do not belong to $C$. Hence, since $C$ is a component of $G - F$, the $ax$-subpath (resp. $bx$-subpath) of $P$ contains at least a vertex of $F$, say $a'$ (resp. $b'$). Then the $a'b'$-subpath $P'$ of $P$ is an $a'b'$-geodesic that contains $x$ and whose endpoints $a'$ and $b'$ belong to $F = F_0$. Therefore $x \in V(P') \subseteq V(F_1)$, contrary to the hypothesis $C \cap F_1 = \emptyset$. \[\square\]

4.14. **Lemma** (Polat [10, Corollary 2.4]). Every contraction of a rayless connected graph stabilizes a finite nonempty subgraph.

4.15. **Proof of Theorem 4.10.** Let $f$ be a contraction of a rayless quasi-median graph $G$. By the preceding lemma there is a finite nonempty subset $F$ of $V(G)$ such that $f(F) = F$. Moreover any interval of a quasi-median graph is convex [7] and induces a median graph (Theorem 3.3); but the convex hull of a finite set of vertices of a median graph is finite [16]. Consequently every interval of a quasi-median graph is finite. Therefore, by Lemma 4.13, the convex hull $co(F)$ of $F$ is finite, and also quasi-median by convexity.

Let $F_0 := co(F)$ and, for every nonnegative integer $n$, let $F_{n+1} := f(F_n) \cap F_0$. Obviously $F_{n+1} \subseteq F_n$ for all $n$. Finally let $F := \bigcap_{n \geq 0} F_n$. This is clearly a finite subgraph of $G$ containing $F$ such that $f(F) = F$. We will prove that it is quasi-median using the characterization 3.3(iii). First $F$, being an induced subgraph of $F_0$ which is quasi-median, cannot contain $K_{1,1,2}$ or $K_{2,3}$ as induced subgraphs. Now let $u, v, w$ be three...
vertices of \( \tilde{F} \) with \( d_G(u, v) = d_G(u, w) = k > 1 \) and \( d_G(v, w) = 1 \); since \( F_0 \) is quasi-median and finite, the set \( X \) of common neighbors \( x \in F_0 \) of \( v \) and \( w \) with \( d_G(u, x) = k - 1 \) is nonempty and finite. We will show that \( X \cap \tilde{F} \neq \emptyset \).

As \( \tilde{F} \) is finite, there is \( p \geq 0 \) with \( f^p(z) = z \) for every \( z \) in \( \tilde{F} \). Thus \( d_G(f^{mp}(u), f^{mp}(v)) = d_G(u, v) = k \) for all \( m \geq 0 \), and more generally \( d_G(f^n(u), f^n(v)) = k \) for all \( n \geq 0 \). Let \( x \in X \). We claim that \( f^n(x) \in F_n \) for all \( n \geq 0 \). This is trivial if \( n = 0 \). Suppose that \( f^n(x) \in F_n \) for some \( n \geq 0 \). Note that, since \( \langle u, x, v \rangle \) is a geodesic and since \( f \) is a nonexpansive map with \( d_G(f^{n+1}(u), f^{n+1}(v)) = k \), \( \langle f^{n+1}(u), f^{n+1}(x), f^{n+1}(v) \rangle \) is also a geodesic. Hence \( f^{n+1}(x) \in F_0 \) by the convexity of this set. Thus \( f^{n+1}(x) \in f(F_n) \cap F_0 = F_{n+1} \). Therefore \( f^{mp}(x) \in X \cap F_{mp} \) for all \( m \geq 0 \). This implies that \( X \cap F_n \neq \emptyset \) for all \( n \geq 0 \). Thus \( X \cap \tilde{F} \neq \emptyset \) by the finiteness of \( X \).

Consequently \( F \) has the triangle property. One can prove analogously that \( \tilde{F} \) has the quadrangle property too.

\( \tilde{F} \) is then a finite and quasi-median induced subgraph of \( G \) for which \( \tilde{f} \), restriction of \( f \) to \( \tilde{F} \), is an automorphism. Therefore, by Theorem 4.8, there exists a finite Hamming graph in \( \tilde{F} \) that is invariant under \( \tilde{f} \), thus under \( f \).

Tardif proved in [16] that every contraction of a median graph without isometric rays stabilizes a finite hypercube. Thus in that result the absence of isometric ray is a sufficient restriction. We conjecture that this weaker constraint is also sufficient for quasi-median graphs, i.e., that the statement of Theorem 4.10 holds true if one replaces ‘rayless graph’ by ‘graph without isometric rays and infinite simplices’.

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References