Available online at www.sciencedirect.com

ScienceDirect

J. Math. Anal. Appl. 340 (2008) 252–281

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

# Distributed order calculus and equations of ultraslow diffusion

Anatoly N. Kochubei [1]

*Institute of Mathematics, National Academy of Sciences of Ukraine, Tereshchenkivska 3, Kiev 01601, Ukraine*

Received 21 March 2007

Available online 19 August 2007

Submitted by Goong Chen

**Abstract**

We consider equations of the form

$$\left(\mathbb{D}^{(\mu)}u\right)(t,x)-\Delta u(t,x)=f(t,x),\quad t>0,\ x\in\mathbb{R}^n,$$

where $\mathbb{D}^{(\mu)}$ is a distributed order derivative, that is

$$\mathbb{D}^{(\mu)}\varphi(t)=\int_0^1\left(\mathbb{D}^{(\alpha)}\varphi\right)(t)\mu(\alpha)\,d\alpha,$$

$\mathbb{D}^{(\alpha)}$ is the Caputo–Dzhrbashyan fractional derivative of order $\alpha$, $\mu$ is a positive weight function.

The above equation is used in physical literature for modeling diffusion with a logarithmic growth of the mean square displacement. In this work we develop a mathematical theory of such equations, study the derivatives and integrals of distributed order.


*Keywords:* Distributed order derivative; Distributed order integral; Ultraslow diffusion; Fundamental solution of the Cauchy problem

## 1. Introduction

Fractional diffusion equations with the Caputo–Dzhrbashyan fractional time derivatives

$$\left(\mathbb{D}_t^{(\alpha)}u\right)(t,x)-Bu(t,x)=f(t,x),\quad t>0,\ x\in\mathbb{R}^n,\tag{1.1}$$

where $0<\alpha<1$, $B$ is an elliptic differential operator in the spatial variables, are widely used in physics to model anomalous diffusion in fractal media.

*E-mail addresses:* kochubei@i.com.ua, kochubei@imath.kiev.ua.

[1] Partially supported by the Ukrainian Foundation for Fundamental Research, Grant 14.1/003.

0022-247X/$ – see front matter 
doi:10.1016/j.jmaa.2007.08.024

Physically, the most important characteristic of diffusion is the mean square displacement

$$\overline{(\Delta x)^2} = \int\limits_{\mathbb{R}^n} |x-\xi|^2 Z(t, x-\xi)\, d\xi$$

of a diffusive particle, where $Z$ is a fundamental solution of the Cauchy problem for the diffusion equation. In normal diffusion (described by the heat equation or more general parabolic equations) the mean square displacement of a diffusive particle behaves like const $\cdot\, t$ for $t \to \infty$. A typical behavior for anomalous diffusion on some amorphous semiconductors, strongly porous materials etc. is const $\cdot\, t^\alpha$, and this was the reason to invoke Eq. (1.1), usually with $B = \Delta$, where this anomalous behavior is an easy mathematical fact. There are hundreds of physical papers involving equations (1.1); see the surveys [23,24]. The mathematical theory was initiated independently by Schneider and Wyss [37] and the author [18,19]; for more recent developments see [9,10,15] and references therein.

A number of recent publications by physicists (see [3–5,25,38] and references there) is devoted to the case where the mean square displacement has a logarithmic growth. This ultraslow diffusion (also called "a strong anomaly") is encountered in polymer physics (a polyampholyte hooked around an obstacle), as well as in models of a particle's motion in a quenched random force field, iterated map models etc. In order to describe ultraslow diffusion, it is proposed to use evolution equations

$$\left(\mathbb{D}_t^{(\mu)} u\right)(t,x) - Bu(t,x) = f(t,x), \quad t > 0,\ x \in \mathbb{R}^n, \tag{1.2}$$

where $\mathbb{D}^{(\mu)}$ is the distributed order derivative (introduced by Caputo [2]) of the form

$$\left(\mathbb{D}^{(\mu)}\varphi\right)(t) = \int\limits_0^1 \left(\mathbb{D}^{(\alpha)}\varphi\right)(t)\mu(\alpha)\, d\alpha, \tag{1.3}$$

$\mu$ is a positive weight function.

The above physical papers contain some model calculations for such evolution equations. There are only two mathematical papers on this subject. Meerschaert and Scheffler [22] developed a stochastic model based on random walks with a random waiting time between jumps. Scaling limits of these random walks are subordinated random processes whose density functions solve the ultraslow diffusion equation. The solutions in [22] are understood as solutions of "algebraic" equations obtained if the Laplace transform in $t$ and the Fourier transform in $x$ are applied.

Umarov and Gorenflo [39] applied to Eqs. (1.2) Dubinskij's theory [8] of analytic pseudo-differential operators. This leads to solvability results for (1.2) in the spaces of analytic functions and dual spaces of analytic functionals. Such a theory is very different from the theory of parabolic equations; results obtained this way "do not feel" the difference between Eqs. (1.2) with $B = \Delta$ and $B = -\Delta$.

The aim of this paper is to develop a theory of the model equation (1.2) with $B = \Delta$ comparable with the classical theory of the Cauchy problem for the heat equation. In particular, we construct and investigate in detail a fundamental solution of the Cauchy problem for the homogeneous equation ($f = 0$) and the corresponding kernel appearing in the volume potential solving the inhomogeneous equation, prove their positivity and subordination properties. This leads to a rigorous understanding of a solution of the Cauchy problem—it is important to know, in which sense a solution satisfies the equation. In its turn, this requires a deeper understanding of the distributed order derivative (1.3), the availability of its various forms resembling the classical fractional calculus [36]. We also introduce and study a kind of a distributed order fractional integral corresponding to the derivative (1.3). A Marchaud-type representation of the distributed order derivative (based on a recent result by Samko and Cardoso [35]) is the main tool for obtaining, in the spirit of [9,19], uniqueness theorems for the Cauchy problem for general equations (1.2) in the class of bounded functions and, for $n = 1$ and $B = d^2/dx^2$, in the class of functions of sub-exponential growth.

Comparing with the theory of fractional diffusion equation (1.1) we see that the distributed order equations (under reasonable assumptions regarding $\mu$) constitute the limiting case equations, as $\alpha \to 0$. That is readily observed from estimates of fundamental solutions having, as $|x| \to \infty$, the estimate $\exp(-a|x|^{\frac{2}{2-\alpha}})$, $a > 0$, for the fractional diffusion equations, and $\exp(-a|x|)$ in the case of ultraslow diffusion.

In fact, we begin with the "ordinary" equation $\mathbb{D}^{(\mu)}u = \lambda u$, $\lambda \in \mathbb{R}$. If $\lambda < 0$, already this equation demonstrates a logarithmic decay of solution at infinity; see Theorem 2.3 below.

In general, the theory presented here is an interesting example of subtle analysis (with kernels from $L_1$ belonging to no $L_p$, $p > 1$, etc.) appearing in problems of a direct physical significance.

## 2. Distributed order derivative

### 2.1. Definitions

Recall that the regularized fractional derivative of a function $\varphi \in C[0,T]$ (also called the Caputo or Caputo–Dzhrbashyan derivative) of an order $\alpha \in (0,1)$ is defined as

$$\left(\mathbb{D}^{(\alpha)}\varphi\right)(t) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{d}{dt}\int_0^t (t-\tau)^{-\alpha}\varphi(\tau)\,d\tau - t^{-\alpha}\varphi(0)\right], \quad 0 < t \leqslant T, \tag{2.1}$$

if the derivative in (2.1) exists. If $\varphi$ is absolutely continuous on $[0,T]$, then

$$\left(\mathbb{D}^{(\alpha)}\varphi\right)(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\varphi'(\tau)\,d\tau \tag{2.2}$$

(see [9]).

Let $\mu(t)$, $0 \leqslant t \leqslant 1$, be a continuous non-negative function, different from zero on a set of a positive measure. If a function $\varphi$ is absolutely continuous on $[0,T]$, then by (1.3) and (2.2)

$$\left(\mathbb{D}^{(\mu)}\varphi\right)(t) = \int_0^t k(t-\tau)\varphi'(\tau)\,d\tau \tag{2.3}$$

where

$$k(s) = \int_0^1 \frac{s^{-\alpha}}{\Gamma(1-\alpha)}\mu(\alpha)\,d\alpha, \quad s > 0. \tag{2.4}$$

It is obvious that $k$ is a positive decreasing function.

Note that for an absolutely continuous $\varphi$,

$$\frac{d}{dt}\int_0^t k(t-\tau)\varphi(\tau)\,d\tau = \frac{d}{dt}\int_0^t k(s)\varphi(t-s)\,ds = \int_0^t k(s)\varphi'(t-s)\,ds + k(t)\varphi(0),$$

so that

$$\left(\mathbb{D}^{(\mu)}\varphi\right)(t) = \frac{d}{dt}\int_0^t k(t-\tau)\varphi(\tau)\,d\tau - k(t)\varphi(0). \tag{2.5}$$

The right-hand side of (2.5) makes sense for a continuous function $\varphi$, for which the derivative $\frac{d}{dt}\int_0^t k(t-\tau)\varphi(\tau)\,d\tau$ exists. Below we use (2.5) as a general definition of the distributed order derivative $\mathbb{D}^{(\mu)}\varphi$.

The necessity to use the regularized fractional derivatives, not the Riemann–Liouville ones (defined as in (2.1), but without subtracting $t^{-\alpha}\varphi(0)$), in the relaxation and diffusion problems, is caused by the fact that a solution of an equation with a Riemann–Liouville derivative typically has a singularity at the origin $t = 0$ (see, for example, [9]), so that the initial state of a system to be described by the equation is not defined and requires a regularization. However, mathematically such problems are legitimate. A distributed order derivative with a constant weight, based on the Riemann–Liouville fractional derivative, was introduced by Nakhushev (see [26]). The diffusion equation with such a time derivative and a single spatial variable was investigated by Pskhu [30]. For other definitions of variable order and distributed order derivatives see also [17,21] and references therein. In this paper we use only the derivatives (2.1) and (2.5).

### 2.2. Asymptotic properties

Since the kernel (2.4) is among the main objects of the distributed order calculus, it is important to investigate its properties.

**Proposition 2.1.** *If* $\mu \in C^3[0,1]$, $\mu(1) \neq 0$, *then*

$$k(s) \sim s^{-1}(\log s)^{-2}\mu(1), \quad s \to 0, \tag{2.6}$$

$$k'(s) \sim -s^{-2}(\log s)^{-2}\mu(1), \quad s \to 0. \tag{2.7}$$

**Proof.** Denote $r = -\log s$ ($\to \infty$), $\psi(\alpha) = \frac{\mu(\alpha)}{\Gamma(1-\alpha)}$. Then

$$k(s) = \int_0^1 \psi(\alpha) e^{r\alpha}\, d\alpha.$$

Integrating twice by parts, we get

$$k(s) = r^{-2}\int_0^1 \psi''(\alpha) e^{r\alpha}\, d\alpha - r^{-1}\mu(0) - r^{-2}\bigl[\psi'(1)e^r - \psi'(0)\bigr].$$

We have

$$\psi'(\alpha) = \frac{\mu'(\alpha)\Gamma(1-\alpha) + \mu(\alpha)\Gamma'(1-\alpha)}{[\Gamma(1-\alpha)]^2},$$

so that $\psi'(1) = -\mu(1)$, and another integration by parts yields the relation

$$k(s) = \mu(1)r^{-2}e^r + O\bigl(r^{-3}e^r\bigr), \quad r \to \infty,$$

which implies (2.6). The proof of (2.7) is similar. □

It follows from (2.6) that $k \in L_1(0,T)$; however $k \notin L_\beta$ for any $\beta > 1$. Note also that one cannot integrate by parts in (2.3) because, by (2.7), $k'$ has a non-integrable singularity.

Throughout this paper we use the Laplace transform

$$\mathcal{K}(p) = \int_0^\infty k(s) e^{-ps}\, ds, \quad \operatorname{Re} p > 0.$$

Using (2.4) and the relation

$$\int_0^\infty s^{-\alpha} e^{-ps}\, ds = \frac{\Gamma(1-\alpha)}{p^{1-\alpha}}$$

(see 2.3.3.1 in [28]), we find that

$$\mathcal{K}(p) = \int_0^1 p^{\alpha-1}\mu(\alpha)\, d\alpha. \tag{2.8}$$

It will often be useful to write (2.8) as

$$\mathcal{K}(p) = \int_0^1 e^{(\alpha-1)\log p}\mu(\alpha)\, d\alpha. \tag{2.9}$$

Taking the principal value of the logarithm we extend $\mathcal{K}(p)$ to an analytic function on the whole complex plane cut along the half-axis $\mathbb{R}_- = \{\operatorname{Im} p = 0, \operatorname{Re} p \leqslant 0\}$.

**Proposition 2.2.**

(i) *Let* $\mu \in C^2[0,1]$. *If* $p \in \mathbb{C} \setminus \mathbb{R}_-$, $|p| \to \infty$, *then*

$$\mathcal{K}(p) = \frac{\mu(1)}{\log p} + O\big((\log|p|)^{-2}\big). \tag{2.10}$$

*More precisely, if* $\mu \in C^3[0,1]$, *then*

$$\mathcal{K}(p) = \frac{\mu(1)}{\log p} - \frac{\mu'(1)}{(\log p)^2} + O\big((\log|p|)^{-3}\big). \tag{2.10$'$}$$

(ii) *Let* $\mu \in C[0,1]$, $\mu(0) \neq 0$. *If* $p \in \mathbb{C} \setminus \mathbb{R}_-$, $p \to 0$, *then*

$$\mathcal{K}(p) \sim p^{-1}\left(\log \frac{1}{p}\right)^{-1} \mu(0). \tag{2.11}$$

(iii) *Let* $\mu \in C[0,1]$, $\mu(\alpha) \sim a\alpha^\lambda$, $a > 0$, $\lambda > 0$. *If* $p \in \mathbb{C} \setminus \mathbb{R}_-$, $p \to 0$, *then*

$$\mathcal{K}(p) \sim a\Gamma(1+\lambda) p^{-1}\left(\log \frac{1}{p}\right)^{-1-\lambda}. \tag{2.11$'$}$$

**Proof.** (i) Integrating by parts, as in the proof of Proposition 2.1, we find that

$$\int_0^1 e^{\alpha r} \mu(\alpha)\, d\alpha = \frac{\mu(1)e^r}{r} + O\big(r^{-2}e^r\big), \quad r \to \infty,$$

which implies (2.10). The relation (2.10′) is proved similarly.

(ii), (iii) The relations (2.11) and (2.11′) follow from the complex version of Watson's lemma [27, Chapter 4]. □

In some cases it is convenient to use a rough estimate

$$\big|\mathcal{K}(p)\big| \leqslant C|p|^{-1}\left(\log \frac{1}{|p|}\right)^{-1}, \quad |p| \leqslant p_0,$$

valid for any $\mu \in C[0,1]$. This estimate follows from general results about the behavior of the Laplace transform near the origin (see Chapter II, §1 of [7]).

*2.3. "Ordinary" equations*

Let us consider the simplest equation with a distributed order derivative, that is

$$\big(\mathbb{D}^{(\mu)} u_\lambda\big)(t) = \lambda u_\lambda(t), \quad t > 0, \tag{2.12}$$

where $\lambda \in \mathbb{R}$, and it is assumed that a solution satisfies the initial condition $u(0) = 1$. A solution of (2.12) should be seen as an analog of the exponential function $t \mapsto e^{\lambda t}$ of the classical analysis and the function $t \mapsto E_\alpha(\lambda t^\alpha)$, where $E_\alpha$ is the Mittag–Leffler function, appearing for the equation with the regularized fractional derivative of order $\alpha \in (0,1)$ (see [9]). Equation (2.12) with $\lambda < 0$ is discussed in [16] as the one describing distributed order relaxation. The uniqueness of a solution will follow from the uniqueness theorem for Eq. (1.2); see Theorem 6.1. Of course the method of proof of the latter theorem can be used to prove separately the uniqueness for a much simpler equation (2.12). Below we assume that $\mu \in C^2[0,1]$, $\mu(1) \neq 0$, $\lambda \neq 0$; evidently, $u_0(t) \equiv 1$.

Applying formally the Laplace transform to Eq. (2.12) and taking into account the initial condition $u(0) = 1$, for the transformed solution $\widetilde{u_\lambda}(p)$ we get

$$\widetilde{u_\lambda}(p) = \frac{\mathcal{K}(p)}{p\mathcal{K}(p) - \lambda}. \tag{2.13}$$

The function (2.13) is analytic on the half-plane $\operatorname{Re} p > \gamma$, if $\gamma > 0$ is large enough. We have $\widetilde{u_\lambda}(p) \sim p^{-1}$, $p = \sigma + i\tau$, $\sigma, \tau \in \mathbb{R}$, $|\tau| \to \infty$. Therefore [7] $\widetilde{u_\lambda}$ is indeed the Laplace transform of some function $u_\lambda(t)$, and for almost all values of $t$,

$$u_\lambda(t) = \frac{d}{dt}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{pt}}{p}\frac{\mathcal{K}(p)}{p\mathcal{K}(p)-\lambda}\,dp. \tag{2.14}$$

Let $\frac{1}{2} < \omega < 1$. We will often use the contour $S_{\gamma,\omega}$ in $\mathbb{C}$ consisting of the arc

$$T_{\gamma,\omega} = \{p \in \mathbb{C}\colon |p| = \gamma,\ |\arg p| \leqslant \omega\pi\},$$

and two rays

$$\Gamma^+_{\gamma,\omega} = \{p \in \mathbb{C}\colon |\arg p| = \omega\pi,\ |p| \geqslant \gamma\},$$

$$\Gamma^-_{\gamma,\omega} = \{p \in \mathbb{C}\colon |\arg p| = -\omega\pi,\ |p| \geqslant \gamma\}.$$

The contour $S_{\gamma,\omega}$ is oriented in the direction of growth of $\arg p$.

By Jordan's lemma,

$$u_\lambda(t) = \frac{d}{dt}\frac{1}{2\pi i}\int_{S_{\gamma,\omega}} \frac{e^{pt}}{p}\frac{\mathcal{K}(p)}{p\mathcal{K}(p)-\lambda}\,dp.$$

In contrast to (2.14), here we may differentiate under the integral, so that

$$u_\lambda(t) = \frac{1}{2\pi i}\int_{S_{\gamma,\omega}} e^{pt}\frac{\mathcal{K}(p)}{p\mathcal{K}(p)-\lambda}\,dp. \tag{2.15}$$

Note that $\gamma$ is chosen in such a way that $p\mathcal{K}(p) \neq \lambda$ for all $p \in S_{\gamma,\omega}$ (this is possible, since $p\mathcal{K}(p) \to \infty$, as $|p| \to \infty$).

The next result establishes qualitative properties of the function $u_\lambda$ resembling those of the exponential function and the Mittag–Leffler function. Recall that a function $u \in C^\infty(0,\infty)$ is called completely monotone [13], if $(-1)^n u^{(n)}(t) \geqslant 0$ for all $t > 0$, $n = 0, 1, 2, \ldots.$

**Theorem 2.3.**

(i) *The function $u_\lambda(t)$ is continuous at the origin $t$ and belongs to $C^\infty(0,\infty)$.*
(ii) *If $\lambda > 0$, then $u_\lambda(t)$ is non-decreasing, and $u_\lambda(t) \geqslant 1$ for all $t \in (0,\infty)$.*
(iii) *If $\lambda < 0$, then $u_\lambda(t)$ is completely monotone.*
(iv) *Let $\lambda < 0$. If $\mu(0) \neq 0$, then*

$$u_\lambda(t) \sim C(\log t)^{-1}, \quad t \to \infty. \tag{2.16}$$

*If $\mu(\alpha) \sim a\alpha^\nu$, $\alpha \to 0$ ($a > 0$, $\nu > 0$), then*

$$u_\lambda(t) \sim C(\log t)^{-1-\nu}, \quad t \to \infty. \tag{2.17}$$

*Here and below $C$ denotes various positive constants.*

**Proof.** The smoothness of $u_\lambda$ for $t > 0$ is evident from (2.15). The integral in (2.15) is the sum of the integral over $T_{\gamma,\omega}$ (obviously continuous at $t = 0$) and the functions

$$u^\pm_\lambda(t) = \frac{1}{2\pi i}\int_{\Gamma^\pm_{\gamma,\omega}} e^{pt}\frac{\mathcal{K}(p)}{p\mathcal{K}(p)-\lambda}\,dp.$$

We have

$$u_\lambda^+(t)+u_\lambda^-(t)=\frac{1}{\pi}\operatorname{Im}\left\{e^{i\omega\pi}\int\limits_\gamma^\infty e^{\operatorname{tr}e^{i\omega\pi}}\frac{\mathcal{K}(re^{i\omega\pi})}{re^{i\omega\pi}\mathcal{K}(re^{i\omega\pi})-\lambda}\,dr\right\}$$

$$=\frac{1}{\pi}\operatorname{Im}\int\limits_\gamma^\infty r^{-1}e^{\operatorname{tr}e^{i\omega\pi}}\,dr+\frac{\lambda}{\pi}\operatorname{Im}\int\limits_\gamma^\infty \frac{e^{\operatorname{tr}e^{i\omega\pi}}}{r(re^{i\omega\pi}\mathcal{K}(re^{i\omega\pi})-\lambda)}\,dr.$$

The second summand is obviously continuous at $t=0$. The first summand equals

$$\frac{1}{\pi}\int\limits_\gamma^\infty r^{-1}e^{\operatorname{tr}\cos\omega\pi}\sin(\operatorname{tr}\sin\omega\pi)\,dr=\frac{1}{\pi}\int\limits_{-\gamma t\cos\omega\pi}^\infty s^{-1}e^{-s}\sin(-s\tan\omega\pi)\,ds$$

(recall that $\cos\omega\pi<0$), and this expression is continuous at $t=0$.

Let $\lambda>0$. The function $p\mapsto\frac{1}{p-\lambda}$ is the Laplace transform of the function $t\mapsto e^{\lambda t}$. Therefore [13] it is completely monotone. The Laplace transform of the function $u_\lambda'(t)$ equals

$$p\widetilde{u_\lambda}(p)-u_\lambda(0)=\frac{p\mathcal{K}(p)}{p\mathcal{K}(p)-\lambda}-1=\frac{\lambda}{p\mathcal{K}(p)-\lambda}$$

(strictly speaking, we have to use this formula, together with the asymptotics of $\mathcal{K}(p)$, to prove the existence of the Laplace transform of $u_\lambda'$).

On the other hand, the function

$$p\mathcal{K}(p)=\int\limits_0^1 p^\alpha\mu(\alpha)\,d\alpha$$

is positive, while its derivative is completely monotone. By the criterion 2 of the complete monotonicity (see Chapter XIII of [13]), the Laplace transform of the function $u_\lambda'(t)$ is completely monotone. It follows from the Bernstein theorem about completely monotone functions and the uniqueness property for the Laplace transform that $u_\lambda'(t)\geqslant 0$ for all $t>0$, whence $u_\lambda$ is non-decreasing and $u_\lambda(t)\geqslant 1$.

Let $\lambda<0$. Up to now, $\gamma$ was chosen so big that $p\mathcal{K}(p)\neq\lambda$ for all $p\in S_{\gamma,\omega}$. In fact,

$$\operatorname{Im}p\mathcal{K}(p)=\int\limits_0^1 |p|^\alpha\mu(\alpha)\sin(\alpha\arg p)\,d\alpha,$$

so that $\operatorname{Im}p\mathcal{K}(p)=0$ only for $\arg p=0$. Meanwhile, if $\arg p=0$ and $\lambda<0$, then $p\mathcal{K}(p)-\lambda=\int_0^1|p|^\alpha\mu(\alpha)\,d\alpha-\lambda>0$. Therefore, the above integral representation of $u_\lambda$ holds for any $\gamma>0$.

Since $|\mathcal{K}(p)|\leqslant C|p|^{-1}(\log\frac{1}{|p|})^{-1}$ for small $|p|$, we find that

$$\left|\frac{\mathcal{K}(p)}{p\mathcal{K}(p)-\lambda}\right|\leqslant C|p|^{-1}\left(\log\frac{1}{|p|}\right)^{-1}$$

whence

$$\left|\int\limits_{T_{\gamma,\omega}} e^{pt}\frac{\mathcal{K}(p)}{p\mathcal{K}(p)-\lambda}\,dp\right|\leqslant\frac{Ce^{\gamma t}}{\log\frac{1}{\gamma}}\to 0,$$

as $\gamma\to 0$.

Considering other summands in the integral representation of $u_\lambda$, we see that

$$\frac{1}{\pi}\int\limits_\gamma^\infty r^{-1}e^{\operatorname{tr}e^{i\omega\pi}}\,dr=-\frac{1}{\pi}\int\limits_{-\gamma t\cos\omega\pi}^\infty s^{-1}e^{-s}\sin(s\tan\omega\pi)\,ds\longrightarrow-\frac{1}{\pi}\int\limits_0^\infty s^{-1}e^{-s}\sin(s\tan\omega\pi)\,ds,$$

as $\gamma\to 0$. Next, we have to consider the expression

$$\frac{\lambda}{\pi}\int_{\gamma}^{\infty}\operatorname{Im}\left(\frac{e^{\mathrm{tr}\,e^{i\omega\pi}}}{r}\right)\operatorname{Re}\left(\frac{1}{re^{i\omega\pi}\mathcal{K}(re^{i\omega\pi})-\lambda}\right)dr+\frac{\lambda}{\pi}\int_{\gamma}^{\infty}\operatorname{Re}\left(\frac{e^{\mathrm{tr}\,e^{i\omega\pi}}}{r}\right)\operatorname{Im}\left(\frac{1}{re^{i\omega\pi}\mathcal{K}(re^{i\omega\pi})-\lambda}\right)dr\stackrel{\mathrm{def}}{=}I_1+I_2.$$

We have

$$\operatorname{Im}\left(\frac{e^{\mathrm{tr}\,e^{i\omega\pi}}}{r}\right)=r^{-1}e^{\mathrm{tr}\cos\omega\pi}\sin(\mathrm{tr}\sin\omega\pi),$$

and this expression has a finite limit, as $r\to 0$. Since also $p\mathcal{K}(p)\to 0$, as $p\to 0$, we see that we may pass to the limit in $I_1$, as $\gamma\to 0$.

In order to consider $I_2$, we have to study the function

$$\Phi(r,\omega)=\operatorname{Im}\frac{1}{re^{i\omega\pi}\mathcal{K}(re^{i\omega\pi})-\lambda}.$$

We have

$$re^{i\omega\pi}\mathcal{K}\big(re^{i\omega\pi}\big)=\int_0^1\big(re^{i\omega\pi}\big)^{\alpha}\mu(\alpha)\,d\alpha=\int_0^1 e^{-\alpha(s-i\omega\pi)}\mu(\alpha)\,d\alpha,$$

$s=-\log r\to\infty$, as $r\to 0$, so that

$$\Phi(r,\omega)=-\frac{\int_0^1 e^{-\alpha s}\sin(\alpha\omega\pi)\mu(\alpha)\,d\alpha}{[\int_0^1 e^{-\alpha s}\cos(\alpha\omega\pi)\mu(\alpha)\,d\alpha-\lambda]^2+[\int_0^1 e^{-\alpha s}\sin(\alpha\omega\pi)\mu(\alpha)\,d\alpha]^2}.$$

As $s\to\infty$, the denominator tends to $\lambda^2$, while in the numerator

$$\int_0^1 e^{-\alpha s}\sin(\alpha\omega\pi)\mu(\alpha)\,d\alpha\leqslant C\int_0^1\alpha e^{-\alpha s}\leqslant\frac{C}{s^2}=\frac{C}{(\log r)^2}.$$

This makes it possible to pass to the limit in $I_2$, as $\gamma\to 0$, so that

$$u_\lambda(t)=-\frac{1}{\pi}\int_0^\infty s^{-1}e^{-s}\sin(s\tan\omega\pi)\,ds+\frac{\lambda}{\pi}\int_0^\infty r^{-1}e^{\mathrm{tr}\cos\omega\pi}\sin(\mathrm{tr}\sin\omega\pi)\Psi(r,\omega)\,dr$$
$$+\frac{\lambda}{\pi}\int_0^\infty r^{-1}e^{\mathrm{tr}\cos\omega\pi}\cos(\mathrm{tr}\sin\omega\pi)\Phi(r,\omega)\,dr,\qquad(2.18)$$

where

$$\Psi(r,\omega)=\frac{\int_0^1 e^{-\alpha s}\cos(\alpha\omega\pi)\mu(\alpha)\,d\alpha-\lambda}{[\int_0^1 e^{-\alpha s}\cos(\alpha\omega\pi)\mu(\alpha)\,d\alpha-\lambda]^2+[\int_0^1 e^{-\alpha s}\sin(\alpha\omega\pi)\mu(\alpha)\,d\alpha]^2},$$

$s=-\log r$.

In (2.18), we may pass to the limit, as $\omega\to 1$. It is easy to see that the first two terms in (2.18) tend to zero, so that

$$u_\lambda(t)=\frac{\lambda}{\pi}\int_0^\infty r^{-1}e^{-\mathrm{tr}}\Phi(r,1)\,dr,\qquad(2.19)$$

$$\Phi(r,1)=-\frac{\int_0^1 r^{\alpha}\sin(\alpha\pi)\mu(\alpha)\,d\alpha}{[\int_0^1 r^{\alpha}\cos(\alpha\pi)\mu(\alpha)\,d\alpha-\lambda]^2+[\int_0^1 r^{\alpha}\sin(\alpha\pi)\mu(\alpha)\,d\alpha]^2}.$$

Since $\lambda<0$, it is seen from (2.19) that $u_\lambda$ is the Laplace transform of a positive function. Therefore $u_\lambda$ is completely monotone.

Let $\lambda < 0$ and $\mu(0) \neq 0$. As we have proved, $u_\lambda$ is monotone decreasing. It follows from (2.13) and (2.11) that $\widetilde{u_\lambda}(p) \sim \frac{C}{p \log \frac{1}{p}}$, $p \to +0$. Applying the Karamata–Feller Tauberian theorem (see Chapter XIII in [13]) we get (2.16). Similarly, if $\mu(\alpha) \sim a\alpha^\nu$, $\alpha \to 0$, we use the asymptotic relation (2.11′), and the same Tauberian theorem yields (2.17). □

A non-rigorous "physicist-style" proof of the statement (iii) was given in [16], where the asymptotics (2.16) was also found for the case $\mu(\alpha) \equiv 1$.

## 3. Distributed order integral

### 3.1. Definition and properties

Suppose that $\mathbb{D}^{(\mu)}u = f$, $u(0) = 0$. Applying formally the Laplace transform $u \mapsto \tilde{u}$, we find that $\tilde{u}(p) = \frac{1}{p\mathcal{K}(p)}\tilde{f}(p)$. The asymptotic properties of $\mathcal{K}(p)$ show (see [7]) that the function $p \mapsto \frac{1}{p\mathcal{K}(p)}$ is the Laplace transform of some function $\varkappa(t)$, and

$$\varkappa(t) = \frac{d}{dt}\frac{1}{2\pi i}\int\limits_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{pt}}{p} \cdot \frac{1}{p\mathcal{K}(p)}\,dp, \quad \gamma > 0. \tag{3.1}$$

It is natural to define *the distributed order integral* $\mathbb{I}^{(\mu)}$, as the convolution operator

$$\big(\mathbb{I}^{(\mu)} f\big)(t) = \int\limits_0^t \varkappa(t-s) f(s)\,ds.$$

**Proposition 3.1.** *Suppose that $\mu \in C^3[0,1]$, $\mu(1) \neq 0$, and either $\mu(0) \neq 0$ or $\mu(\alpha) \sim a\alpha^\nu$, $a > 0$, $\nu > 0$. Then*

(i) $\varkappa \in C^\infty(0,\infty)$*, and* $\varkappa$ *is completely monotone*;
(ii) *for small values of t,*

$$\varkappa(t) \leqslant C \log\frac{1}{t}, \tag{3.2}$$

$$\big|\varkappa'(t)\big| \leqslant Ct^{-1}\log\frac{1}{t}. \tag{3.3}$$

**Proof.** As in Section 2, we deform the contour of integration in (3.1) and differentiate

$$\varkappa(t) = \frac{1}{2\pi i}\int\limits_{S_{\gamma,\omega}} \frac{e^{pt}}{p\mathcal{K}(p)}\,dp. \tag{3.4}$$

We will need information about the asymptotic behavior of $\frac{1}{\mathcal{K}(p)}$. By (2.10′),

$$\mathcal{K}(p) = \frac{\mu(1)}{\log p} - \frac{\mu'(1)}{(\log p)^2} + c(p), \quad c(p) = O\left(\frac{1}{(\log|p|)^3}\right), \quad |p| \to \infty.$$

Then we can write

$$\frac{1}{\mathcal{K}(p)} - \frac{\log p}{\mu(1)} - \frac{\mu'(1)}{[\mu(1)]^2} = \frac{-\mu(1)c(p)(\log p)^3 + [\mu'(1)]^2 - \mu'(1)c(p)(\log p)^2}{[\mu(1)]^2[\mu(1)\log p - \mu'(1) + c(p)(\log p)^2]},$$

whence

$$\frac{1}{\mathcal{K}(p)} = \frac{\log p}{\mu(1)} + \frac{\mu'(1)}{[\mu(1)]^2} + O\left(\frac{1}{(\log|p|)}\right), \quad p \to \infty. \tag{3.5}$$

The integral in (3.4) consists of the integral over $T_{\gamma,\omega}$ (a function from $C^\infty[0,\infty)$) and integrals over $\Gamma^{\pm}_{\gamma,\omega}$. Each of the latter ones is estimated, due to (3.5), by an expression

$$C\int_\gamma^\infty e^{-art}r^{-1}\log r\,dr\sim C\log\frac{1}{t},\quad t\to 0$$

($a, C>0$; see the asymptotic formula (13.49) in [32]). This implies (3.2). The proof of (3.3) is similarly based on the same asymptotic relation from [32].

In order to prove that $\varkappa$ is completely monotone, we proceed as in the proof of Theorem 2.3, to transform (3.4) into a representation by a Laplace integral. First we pass to the limit, as $\gamma\to 0$. This is possible because, by Proposition 2.2, either

$$\frac{1}{p\mathcal{K}(p)}\sim\mu(0)\log\frac{1}{p},\quad p\to 0, \tag{3.6}$$

if $\mu(0)\neq 0$, or

$$\frac{1}{p\mathcal{K}(p)}\sim C\left(\log\frac{1}{p}\right)^{1+\nu},\quad p\to 0, \tag{3.7}$$

if $\mu(\alpha)\sim a\alpha^\nu$, $\alpha\to 0$. Both the relations (3.6) and (3.7) are sufficient to prove that the integral over $T_{\gamma,\omega}$ tends to 0, as $\gamma\to 0$, while the $\gamma\to 0$ limits of both the integrals over $\Gamma^{\pm}_{\gamma,\omega}$ exist. We come to the representation

$$\varkappa(t)=\frac{1}{\pi}\operatorname{Im}\left\{e^{i\omega\pi}\int_0^\infty e^{\mathrm{tr}\,e^{i\omega\pi}}\frac{dr}{re^{i\omega\pi}\mathcal{K}(re^{i\omega\pi})}\right\}. \tag{3.8}$$

We find, introducing the parameter $s=-\log r\to\infty$, as $r\to 0$, that

$$\begin{aligned}re^{i\omega\pi}\mathcal{K}\big(re^{i\omega\pi}\big)&=\int_0^1\big(re^{i\omega\pi}\big)^\alpha\mu(\alpha)\,d\alpha=\int_0^1 e^{-\alpha(s-i\omega\pi)}\mu(\alpha)\,d\alpha\\&=\int_0^1 e^{-\alpha s}\big(\cos(\alpha\omega\pi)+i\sin(\alpha\omega\pi)\big)\mu(\alpha)\,d\alpha.\end{aligned}$$

Taking into account the logarithmic behavior of the integrand of (3.8) near the origin, we may pass to the limit in (3.8), as $\omega\to 1$, and we get that

$$\varkappa(t)=\frac{1}{\pi}\int_0^\infty e^{-\mathrm{tr}}\frac{\int_0^1 r^\alpha\sin(\alpha\pi)\mu(\alpha)\,d\alpha}{[\int_0^1 r^\alpha\cos(\alpha\pi)\mu(\alpha)\,d\alpha]^2+[\int_0^1 r^\alpha\sin(\alpha\pi)\mu(\alpha)\,d\alpha]^2}\,dr,$$

as desired. □

Note that, by (3.2), $\varkappa\in L_1^{\mathrm{loc}}(0,\infty)$.

*3.2. The Marchaud form of the distributed order derivative*

If $f\in L_1(0,T)$, $u=\mathbb{I}^{(\mu)}f$, then $u=\varkappa*f$,

$$\big(\mathbb{D}^{(\mu)}u\big)(t)=\frac{d}{dt}(k*\varkappa*f)(t)=\frac{d}{dt}(1*f)=\frac{d}{dt}\int_0^t f(\tau)\,d\tau=f(t)$$

almost everywhere. Thus $\mathbb{D}^{(\mu)}\mathbb{I}^{(\mu)}=I$ on $L_1(0,T)$.

The identity $(k * \varkappa)(t) \equiv 1$ (almost everywhere), which follows from the fact that the product of the Laplace transforms $\mathcal{K}(p)$ and $\frac{1}{p\mathcal{K}(p)}$ equals $\frac{1}{p}$, means that $\varkappa$ is *a Sonine kernel* (see [35]). Since both the functions $k$ and $\varkappa$ are monotone decreasing (obviously, $k$ is completely monotone), we are within the conditions [35], under which the operator $\mathbb{D}^{(\mu)}$, on functions $u = \mathbb{I}^{(\mu)} f$, $f \in L_p(0,T)$, $1 < p < \infty$, can be represented in the form

$$\bigl(\mathbb{D}^{(\mu)}u\bigr)(t) = k(t)u(t) + \int_0^t k'(\tau)\bigl[u(t-\tau) - u(t)\bigr]\,d\tau, \quad 0 < t \leqslant T, \tag{3.9}$$

where the representation (3.9) is understood as follows. Let

$$(\Psi_\varepsilon u)(t) = \begin{cases} \int_\varepsilon^t k'(\tau)[u(t-\tau) - u(t)]\,d\tau, & \text{if } t \geqslant \varepsilon, \\ 0, & \text{if } 0 < t < \varepsilon. \end{cases}$$

Then

$$\lim_{\varepsilon \to 0} \bigl\| \bigl(\mathbb{D}^{(\mu)}u\bigr)(t) - k(t)u(t) - (\Psi_\varepsilon u)(t) \bigr\|_{L_p(0,T)} = 0. \tag{3.10}$$

The representation (3.9), similar to the Marchaud form of a fractional derivative [36], will be useful for our proofs of uniqueness theorems, because the integral operator in (3.9) has the form enabling the maximum principle approach. On the other hand, the precaution we made understanding (3.9) in terms of (3.10) cannot be easily avoided due to a strong singularity of $k'$ in accordance with the asymptotics (2.7).

## 4. The equation of ultraslow diffusion

### 4.1. A fundamental solution of the Cauchy problem

Let us consider Eq. (1.2) with $B = \Delta$, that is

$$\bigl(\mathbb{D}_t^{(\mu)}u\bigr)(t,x) = \Delta u(t,x), \quad x \in \mathbb{R}^n,\ t > 0. \tag{4.1}$$

In this section we construct fundamental solution $Z(t,x)$ of the Cauchy problem, a solution of (4.1) with $Z(0,x) = \delta(x)$, and obtain its estimates.

Below we use the following normalization of the Fourier transform:

$$\widehat{u}(\xi) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} u(x)\,dx,$$

so that

$$u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \widehat{u}(\xi)\,d\xi.$$

For a radial function $u(r)$, $r = |x|$,

$$\widehat{u}(r) = 2\pi^{n/2} \left(\frac{r}{2}\right)^{1-\frac{n}{2}} \int_0^\infty \rho^{n/2} u(\rho) J_{\frac{n}{2}-1}(r\rho)\,d\rho, \tag{4.2}$$

where $J_\nu$ is the Bessel function.

Applying formally the Laplace transform in $t$ and the Fourier transform in $x$, we find that

$$\widehat{\widetilde{Z}}(p,\xi) = \frac{\mathcal{K}(p)}{p\mathcal{K}(p) + |\xi|^2}.$$

By (4.2),

$$\widetilde{Z}(p,x) = (2\pi)^{-\frac{n}{2}} |x|^{1-\frac{n}{2}} \mathcal{K}(p) \int_0^\infty \frac{s^{n/2}}{p\mathcal{K}(p) + s^2} J_{\frac{n}{2}-1}\bigl(|x|s\bigr)\,ds. \tag{4.3}$$

It is known [29, 2.12.4.28] that

$$\int_0^\infty \frac{y^{\nu+1}}{y^2+z^2} J_\nu(cy)\,dy = z^\nu K_\nu(cz), \quad -1<\nu<\frac{3}{2}, \tag{4.4}$$

where $K_\nu$ is the McDonald function. If $n \leqslant 4$, then the above restriction upon $\nu = \frac{n}{2}-1$ is satisfied, and (4.4) implies the representation

$$\widetilde{Z}(p,x) = (2\pi)^{-\frac{n}{2}} |x|^{1-\frac{n}{2}} \mathcal{K}(p)\bigl(p\mathcal{K}(p)\bigr)^{\frac{1}{2}(\frac{n}{2}-1)} K_{\frac{n}{2}-1}\bigl(|x|\sqrt{p\mathcal{K}(p)}\,\bigr). \tag{4.5}$$

We have simpler formulas in the lowest dimensions—if $n=2$, then

$$\widetilde{Z}(p,x) = \frac{1}{2\pi}\mathcal{K}(p) K_0\bigl(|x|\sqrt{p\mathcal{K}(p)}\,\bigr), \tag{4.6}$$

if $n=1$, then

$$\widetilde{Z}(p,x) = \frac{1}{2}\frac{\mathcal{K}(p)}{\sqrt{p\mathcal{K}(p)}} e^{-|x|\sqrt{p\mathcal{K}(p)}} \tag{4.7}$$

because $K_{-1/2}(z) = K_{1/2}(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z}$ (see [1]).

The function $K_\nu$ decays exponentially at infinity

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z}, \quad z\to\infty,$$

while $K_\nu(z) \sim Cz^{-\nu}$, as $z\to 0$ (if $\nu>0$), and $K_0(z)\sim -\log z$. We see that the function on the right in (4.5) belongs to $L_1(\mathbb{R}^n)$ in $x$ for any $n$, not only for $n\leqslant 4$. Using the identity

$$\int_0^\infty r J_\nu(br) K_\nu(cr)\,dr = b^\nu c^{-\nu}\bigl(b^2+c^2\bigr)^{-1}$$

[29, 2.16.21.1] we check that the inverse Fourier transform of the right-hand side of (4.5) coincides with $\mathcal{K}(p)(p\mathcal{K}(p)+|\xi|^2)^{-1}$. Therefore the formula (4.5) is valid for any $n$.

Let us consider estimates of the function $Z$ and its derivatives. Qualitatively, the behavior of $Z$ is similar to that of the fundamental solution of the Cauchy problem for the fractional diffusion equation (1.1) (see [9,10,19]). In addition to the singularity at $t=0$, $Z(t,x)$ has, if $n>1$, a singularity at $x=0$ (a logarithmic singularity, if $n=2$, and a power one, if $n\geqslant 3$). As usual $Z(t,x)\to\delta(x)$, as $t\to 0$. This means that the singularity at $t=0$ becomes "visible" near the origin in $x$. In fact, we obtain separate estimates for a small $|x|$, showing the character of singularities in $t$ and $x$, and for a large $|x|$. In addition, subsequent applications of the fundamental solutions require estimates of $\mathbb{D}^{(\mu)}Z$, applicable simultaneously for all $x\neq 0$, and uniform in $t$. Of course, estimates for $\mathbb{D}^{(\mu)}Z$ at the origin and infinity can be obtained from the relation $\mathbb{D}^{(\mu)}Z = \Delta Z$.

All the above estimates deal with a finite time interval, $t\in(0,T]$, and it is this kind of estimates, that is needed to study the Cauchy problem. Separately we will give some estimates of $Z$ for large values of $t$, just to see the qualitative behavior of $Z$.

**Theorem 4.1.** *Suppose that $\mu\in C^2[0,1]$, $\mu(\alpha)=\alpha^\nu\mu_1(\alpha)$, $\mu_1(\alpha)\geqslant\rho>0$, $0\leqslant\alpha\leqslant 1$, $\nu\geqslant 0$. Denote by $\varepsilon$ a small positive number. The function $Z$ is infinitely differentiable for $t\neq 0$ and $x\neq 0$. The following estimates hold for $0<t\leqslant T$.*

*If $n=1$, then*

$$\bigl|D_x^m Z(t,x)\bigr| \leqslant Ct^{-\frac{m+1}{2}}, \quad |x|\leqslant\varepsilon,\ 0\leqslant m\leqslant 3. \tag{4.8}$$

*If $n=2$, then*

$$\bigl|Z(t,x)\bigr| \leqslant Ct^{-1}\log|x|^{-1}, \quad |x|\leqslant\varepsilon, \tag{4.9}$$

$$\bigl|D_x^m Z(t,x)\bigr| \leqslant Ct^{-1}|x|^{-m}, \quad |x|\leqslant\varepsilon,\ 1\leqslant m\leqslant 3. \tag{4.10}$$

*If $n \geqslant 3$, then*

$$\left|D_x^m Z(t,x)\right| \leqslant Ct^{-1}|x|^{-n+2-m}, \quad |x| \leqslant \varepsilon,\ 0 \leqslant m \leqslant 3. \tag{4.11}$$

*In all cases,*

$$\left|D_x^m Z(t,x)\right| \leqslant Ce^{-a|x|} \quad (a>0),\ |x| \geqslant \varepsilon^{-1}. \tag{4.12}$$

*The estimate of $\mathbb{D}^{(\mu)}Z$, uniform in $t$, is as follows*:

$$\left|\left(\mathbb{D}^{(\mu)}Z\right)(t,x)\right| \leqslant C|x|^{-n-2}e^{-a|x|} \quad (a>0),\ |x| \neq 0. \tag{4.13}$$

*If $|x| \leqslant \varepsilon$, then*

$$\left|\left(\mathbb{D}^{(\mu)}Z\right)(t,x)\right| \leqslant Ct^{-2}|x|^{-n+2}. \tag{4.13'}$$

**Proof.** As before, using Jordan's lemma we write

$$Z(t,x) = (2\pi)^{-\frac{n}{2}}|x|^{1-\frac{n}{2}} \int\limits_{S_{\gamma,\omega}} e^{pt}\mathcal{K}(p)\left(p\mathcal{K}(p)\right)^{\frac{1}{2}(\frac{n}{2}-1)} K_{\frac{n}{2}-1}\left(|x|\sqrt{p\mathcal{K}(p)}\right)dp, \quad x \neq 0. \tag{4.14}$$

The integral in (4.14) consists of the ones on $T_{\gamma,\omega}$ and $\Gamma^{\pm}_{\gamma,\omega}$. Let us begin with the first of them, denoted by $Z^0(t,x)$. Below we assume that $\gamma > e$.

If $p \in T_{\gamma,\omega}$, then $p = \gamma e^{i\varphi}$, $|\varphi| \leqslant \omega\pi$, $\frac{1}{2} \leqslant \omega < 1$. Under our assumptions,

$$p\mathcal{K}(p) = \int\limits_0^1 \gamma^\alpha e^{i\alpha\varphi}\alpha^\nu \mu_1(\alpha)\,d\alpha.$$

Let us consider the location of values of $p\mathcal{K}(p)$, $p \in T_{\gamma,\omega}$. If $|\varphi| \leqslant \pi/2$, then $\operatorname{Re} p\mathcal{K}(p) \geqslant 0$. Suppose that $\frac{\pi}{2} < \varphi \leqslant \omega\pi$. Then $\operatorname{Re} p\mathcal{K}(p) \geqslant R\cos(\omega\pi)$, $R = \int_0^1 \gamma^\alpha\alpha^\nu\mu_1(\alpha)\,d\alpha$,

$$\operatorname{Im} p\mathcal{K}(p) \geqslant \rho\int\limits_0^1 \alpha^\nu \sin(\alpha\varphi)\,d\alpha = \rho\varphi^{-1-\nu}\int\limits_0^\varphi \beta^\nu\sin\beta\,d\beta \geqslant \rho(\omega\pi)^{-1-\nu}\int\limits_0^{\pi/2}\beta^\nu\sin\beta\,d\beta > 0,$$

so that $0 \leqslant \arg p\mathcal{K}(p) < \pi$, as $p$ belongs to the part of $T_{\gamma,\omega}$ lying in the upper half-plane. Similarly, $-\pi < \arg p\mathcal{K}(p) \leqslant 0$ for the part from the lower half-plane. Thus, $|\arg p\mathcal{K}(p)| < \pi$, and since $T_{\gamma,\omega}$ is compact, we have $|\arg p\mathcal{K}(p)| \leqslant \varphi_0 < \pi$, $p \in T_{\gamma,\omega}$.

This means that

$$\operatorname{Re}\sqrt{p\mathcal{K}(p)} \geqslant \cos\frac{\varphi_0}{2}\cdot \inf_{p\in T_{\gamma,\omega}}\left|\int\limits_0^1 p^\alpha\mu(\alpha)\,d\alpha\right| \stackrel{\text{def}}{=} r_0 > 0$$

because $\operatorname{Im}\int_0^1 p^\alpha\mu(\alpha)\,d\alpha = 0$ with $p \in T_{\gamma,\omega}$ only if $p=\gamma$, and there $\operatorname{Re}\int_0^1 \gamma^\alpha\mu(\alpha)\,d\alpha \neq 0$.

Therefore, using the above-mentioned asymptotics of the McDonald function, we find that

$$\left|Z^0(t,x)\right| \leqslant Ce^{-a|x|} \quad (a>0),\ |x| \geqslant \varepsilon^{-1}. \tag{4.15}$$

As $|x| \leqslant \varepsilon$, we get

$$\left|Z^0(t,x)\right| \leqslant \begin{cases} C, & \text{if } n=1,\\ C\log|x|^{-1}, & \text{if } n=2,\\ C|x|^{-n+2}, & \text{if } n \geqslant 3.\end{cases} \tag{4.16}$$

Let $Z^{\pm}(t,x)$ be the parts of $Z(t,x)$ corresponding to the integration over $\Gamma^{\pm}_{\gamma,\omega}$. If, for example, $n \geqslant 3$, then

$$\left|Z^{\pm}(t,x)\right| \leqslant C|x|^{\frac{1-n}{2}}\int\limits_\gamma^\infty e^{\operatorname{tr}\cos(\omega\pi)}\frac{1}{\log r}\left(\frac{r}{\log r}\right)^{\frac{n-3}{4}} e^{-a|x|(\frac{r}{\log r})^{1/2}}\,dr, \quad a>0. \tag{4.17}$$

Let us make the change of variables $z = z(r) = (\frac{r}{\log r})^{1/2}$. In order to express (asymptotically) $r$ as a function of $z$, we denote $s = \log r$, so that $s^{-1}e^s = z^2$ where $s \to \infty$ and $z \to \infty$. Taking the logarithm of both parts of the last equality we get $s - \log s = 2\log z$. It is known [12, p. 50] that

$$s = 2\log z + O(\log\log z), \quad z \to \infty.$$

Therefore $r = r(z)$ satisfies the inequalities

$$z^2(\log z)^{-b} \leqslant r(z) \leqslant (\log z)^b \tag{4.18}$$

for some $b \geqslant 0$.

For $|x| \geqslant \varepsilon^{-1}$, the factor $e^{\mathrm{tr}\cos(\omega\pi)}$ in (4.17) can be estimated by 1, and after the use of (4.18) the power terms, as well as the logarithmic ones, are dominated by the exponential factor (the integral in $z$ is taken over $(\gamma_1, \infty)$, $\gamma_1 > 0$), so that

$$\left|Z^{\pm}(t,x)\right| \leqslant Ce^{-a'|x|}, \quad a' > 0,$$

and, together with (4.15), this implies (4.12) for $n \geqslant 3$, $m = 0$.

For $|x| < \varepsilon$, the factor $e^{-a|x|(\frac{r}{\log r})^{1/2}}$ is estimated by 1, and an elementary estimate gives that

$$\left|Z^{\pm}(t,x)\right| \leqslant Ct^{-1}|x|^{-n+2},$$

which implies the required estimate of $Z$.

The bounds for the derivatives, as well as the estimates (4.8)–(4.10) for $n = 1$ and $n = 2$ are obtained in a similar way. Some of the estimates can in fact be slightly refined (using the asymptotic formulas for the Laplace integrals with logarithmic factors [32]), involving $\frac{t^{-1}}{\log t^{-1}}$ for small values of $t$, instead of $t^{-1}$.

Let us prove (4.13). Let $n \geqslant 3$; the cases $n = 2$ and $n = 1$ are similar. The estimates of the McDonald function for small and large arguments can be combined as follows:

$$\left|K_{\frac{n}{2}-1}(z)\right| \leqslant C|z|^{-\frac{n}{2}+1}e^{-a|z|}, \quad z \neq 0, \tag{4.19}$$

with a possibly different choice of the constant $a > 0$.

Next, let us write an integral representation of $\mathbb{D}^{(\mu)}Z$. If $u(t) = e^{pt}$, then

$$\left(\mathbb{D}^{(\mu)}u\right)(t) = p\int_0^t k(t-\tau)e^{p\tau}\,d\tau = pe^{pt}\int_0^t k(s)e^{-ps}\,ds.$$

Using the expression (2.4) for $k(s)$, the identity

$$\int_0^t s^{-\alpha}e^{-ps}\,ds = p^{\alpha-1}\gamma(1-\alpha, pt)$$

[28, 1.3.2.3], where $\gamma$ is an incomplete gamma function, we find that

$$\left(\mathbb{D}^{(\mu)}u\right)(t) = e^{pt}\int_0^1 p^{\alpha}\frac{\mu(\alpha)}{\Gamma(1-\alpha)}\gamma(1-\alpha, pt)\,d\alpha.$$

It is known that

$$\gamma(1-\alpha, z) \sim \frac{1}{1-\alpha}z^{1-\alpha}, \quad z \to 0,$$

$$\gamma(1-\alpha, z) \sim \Gamma(1-\alpha) - z^{-\alpha}e^{-z}, \quad z \to \infty$$

(see Chapter 9 in [1]). This implies the inequality

$$\left|\frac{\gamma(1-\alpha, z)}{\Gamma(1-\alpha)}\right| \leqslant C$$

valid, in particular, for all $z = pt$, $p \in S_{\gamma,\omega}$, $t \in (0, T]$. Recalling also that

$$\int_0^1 p^\alpha \mu(\alpha)\,d\alpha = p\mathcal{K}(p)$$

we see that the application of $\mathbb{D}^{(\mu)}$ to the integral representing $Z$ leads to the appearance of the factor $|p\mathcal{K}(p)|$ in the estimates of $\mathbb{D}^{(\mu)}Z$, compared to those of $Z$. Using also (4.19) and estimating by 1 the decaying exponential involving $t$ (in the integrals over $\Gamma^{\pm}_{\gamma,\omega}$) we come to the inequality (4.13).

The proof of the inequality (4.13′) is similar to those for estimates of the derivatives in spatial variables. □

## 4.2. *Subordination and positivity*

Let us find a connection between $Z$ and the fundamental solution of the heat equation. Our approach follows [4] where the case $n = 1$ was considered (without a full rigor).

Let us consider the function

$$g(u,p) = \mathcal{K}(p)e^{-up\mathcal{K}(p)}, \quad u > 0,\ \operatorname{Re} p > 0.$$

Let $p = \gamma + i\tau$, $\gamma > 0$, $\tau \in \mathbb{R}$. As $|\tau| \to \infty$,

$$\mathcal{K}(p) \sim \frac{C}{\log\sqrt{\gamma^2+\tau^2} + i\arg p}, \quad \arg p \to \pm\frac{\pi}{2}.$$

It follows that

$$\operatorname{Re}\bigl(p\mathcal{K}(p)\bigr) \sim C\left\{\frac{\gamma}{\log\sqrt{\gamma^2+\tau^2}} + \frac{\pi}{2}\frac{|\tau|}{(\log\sqrt{\gamma^2+\tau^2})^2}\right\}, \quad |\tau| \to \infty,$$

whence

$$\bigl|e^{-up\mathcal{K}(p)}\bigr| \leqslant C\exp\left\{-au\left(\frac{\gamma}{\log(\gamma^2+\tau^2)} + \frac{|\tau|}{(\log(\gamma^2+\tau^2))^2}\right)\right\}, \quad a > 0. \tag{4.20}$$

Writing $\log(\gamma^2+\tau^2) \leqslant C(\gamma^2+\tau^2)^\varepsilon$, $0 < \varepsilon < 1/4$, we find from (4.20) that

$$\begin{aligned}
&\int_{-\infty}^{\infty} \bigl|g(u,\gamma+i\tau)\bigr|\,d\tau \\
&\quad\leqslant C\int_0^\infty \exp\left(-au\left(\frac{\gamma}{(\gamma^2+\tau^2)^\varepsilon} + \frac{\tau}{(\gamma^2+\tau^2)^{2\varepsilon}}\right)\right)d\tau \leqslant C\int_0^1 e^{-au\frac{\gamma}{(\gamma^2+\tau^2)^\varepsilon}}\,d\tau + C\int_1^\infty e^{-au\frac{\tau}{(\gamma^2+\tau^2)^{2\varepsilon}}}\,d\tau \\
&\quad\leqslant Ce^{-au\frac{\gamma}{(\gamma^2+1)^\varepsilon}} + C\gamma\int_{\gamma^{-1}}^\infty e^{-a'u\gamma^{1-4\varepsilon}y^{1-4\varepsilon}}\,dy \leqslant C + C\int_0^\infty e^{-a'uz^{1-4\varepsilon}}\,dz
\end{aligned}$$

($a' > 0$), whence

$$\sup_{\gamma\geqslant 1}\int_{-\infty}^{\infty}\bigl|g(u,\gamma+i\tau)\bigr|\,d\tau < \infty. \tag{4.21}$$

It follows from (4.21) (see [7]) that $g(u,p)$ is the Laplace transform of some locally integrable function $G(u,t)$:

$$g(u,p) = \int_0^\infty e^{-pt}G(u,t)\,dt, \tag{4.22}$$

and the integral in (4.22) is absolutely convergent if $\operatorname{Re} p \geqslant 1$.

On the other hand, the function $p\mathcal{K}(p)$ is positive and has a completely monotone derivative, so that $e^{-up\mathcal{K}(p)}$ is completely monotone. Since $\mathcal{K}(p)$ is completely monotone, we find that $g$ is completely monotone in $p$ (we have used criterions 1 and 2 of the complete monotonicity; see [13]), so that $G(u,t) \geqslant 0$ by Bernstein's theorem.

**Theorem 4.2.**

(i) *The fundamental solution $Z(t,x)$ satisfies the subordination identity*

$$Z(t,x) = \int_0^\infty G(u,t)(4\pi u)^{-n/2} e^{-\frac{|x|^2}{4u}}\, du, \quad x \neq 0,\ t > 0, \tag{4.23}$$

*where $G(u,t) \geqslant 0$ and*

$$\int_{\mathbb{R}^n} G(u,t)\, du = 1. \tag{4.24}$$

(ii) *For all $t > 0$, $x \neq 0$, $Z(t,x)$ is non-negative, and*

$$\int_{\mathbb{R}^n} Z(t,x)\, dx = 1. \tag{4.25}$$

**Proof.** In order to prove (4.24), we integrate (4.22) in $p$ using Fubini's theorem. We get

$$\int_0^\infty e^{-pt}\, dt \int_0^\infty G(u,t)\, du = \frac{1}{p},$$

which implies (4.24).

Let us prove (4.23). The convergence of the integral at infinity follows from (4.24), while near the origin the function $u \mapsto (4\pi u)^{-n/2} e^{-\frac{|x|^2}{4u}}$ decays exponentially. Let $v(t,x)$ be the right-hand side of (4.23). Multiplying by $e^{-pt}$ and integrating in $t$ we find that

$$\int_0^\infty e^{-pt} v(t,x)\, dt = \mathcal{K}(p) \int_0^\infty e^{-up\mathcal{K}(p)} (4\pi u)^{-n/2} e^{-\frac{|x|^2}{4u}}\, du.$$

By the formula 2.3.16.1 from [28], the right-hand side coincides with the one from (4.5), so that $v(t,x) = Z(t,x)$. Now the non-negativity of $Z$ is a consequence of (4.23), and the identity (4.25) follows from (4.23), (4.24) and Fubini's theorem. □

### *4.3. Long time behavior*

Let us give a rigorous proof of the asymptotics of the mean square displacement, basic for applications of the distributed order calculus. We also give some long time estimates of the fundamental solution $Z$.

**Theorem 4.3.**

(i) *Let*

$$m(t) = \int_{\mathbb{R}^n} |x|^2 Z(t,x)\, dx.$$

*If $\mu(0) \neq 0$, then*

$$m(t) \sim C \log t, \quad t \to \infty. \tag{4.26}$$

*If*

$$\mu(\alpha) \sim a\alpha^{\nu}, \quad \alpha \to 0, \quad a, \nu > 0, \tag{4.27}$$

*then*

$$m(t) \sim C(\log t)^{1+\nu}, \quad t \to \infty. \tag{4.28}$$

(ii) *Suppose that* (4.27) *holds with* $\nu > 1$, *if* $n = 1$, *and with an arbitrary* $\nu > 0$, *if* $n \geqslant 2$. *Then for* $|x| \leqslant \varepsilon$, $\varepsilon > 0$, *and* $t > \varepsilon^{-1}$,

$$Z(t,x) \leqslant \begin{cases} C(\log t)^{-\frac{\nu-1}{2}}, & \text{if } n = 1, \\ C|\log|x||(\log t)^{-\nu}\log(\log t), & \text{if } n = 2, \\ C|x|^{-n+2}(\log t)^{-\nu-1}, & \text{if } n \geqslant 3. \end{cases} \tag{4.29}$$

**Proof.** (i) It follows from the Plancherel identity for the Fourier transform that

$$m(t) = -(2\pi)^n \big(\Delta_\xi \widehat{Z}(t,\xi)\big)\big|_{\xi=0}.$$

Applying the Laplace transform in $t$ we find that

$$\widetilde{m}(p) = -(2\pi)^n \left\{\Delta_\xi\left(\frac{1}{p\mathcal{K}(p) + |\xi|^2}\right)\right\}\bigg|_{\xi=0},$$

and after an easy calculation we get

$$\widetilde{m}(p) = \frac{2n\cdot(2\pi)^n}{p^2\mathcal{K}(p)},$$

whence

$$m(t) = 2n\cdot(2\pi)^n \int_0^t \varkappa(\tau)\,d\tau,$$

where $\varkappa$ was introduced in Section 3.1. Now the relations (4.26) and (4.28) are consequences of Karamata's Tauberian theorem [13].

(ii) As before, we proceed from the integral representation (4.14) where the contour $S_{\gamma,\omega}$ consists of a finite part $T_{\gamma,\omega}$ and the rays $\Gamma^{\pm}_{\gamma,\omega}$. Let $n = 1$. Then (4.14) takes the form

$$Z(t,x) = \frac{1}{2}\int_{S_{\gamma,\omega}} e^{pt}\frac{\mathcal{K}(p)}{\sqrt{p\mathcal{K}(p)}} e^{-|x|\sqrt{p\mathcal{K}(p)}}\,dp. \tag{4.30}$$

As $p \to 0$, $\sqrt{p\mathcal{K}(p)} \sim C(\log p^{-1})^{-\frac{1+\nu}{2}}$, $\frac{\mathcal{K}(p)}{\sqrt{p\mathcal{K}(p)}} \sim Cp^{-1}(\log p^{-1})^{-\frac{1+\nu}{2}}$, where $\frac{1+\nu}{2} > 1$. These asymptotic relations make it possible to pass to the limit in (4.30), as $\gamma \to 0$, substantiating simultaneously the convergence to 0 of the integral over $T_{\gamma,\omega}$ and the existence of the integrals over the rays starting at the origin.

Thus,

$$Z(t,x) \leqslant C\int_0^\infty e^{rt\cos(\omega\pi)} r^{-1}|\log r|^{-\frac{1+\nu}{2}} e^{-a|x||\log r|^{-\frac{1+\nu}{2}}}\,dr. \tag{4.31}$$

Let us decompose the integral in (4.31) into the sum of the integrals over $(0, 1/2)$ and $(1/2, \infty)$. Estimating the latter we drop the factor containing $|x|$ and obtain easily the exponential decay, as $t \to \infty$. The integral over $(0, 1/2)$ is estimated via the function

$$M(t) = \int_0^{1/2} e^{-art} r^{-1}\left(\log\frac{1}{r}\right)^{-\frac{1+\nu}{2}}\,dr.$$

Integrating by parts we see that

$$M(t) \leqslant C\left(e^{-\frac{at}{2}} + t\int_0^{1/2} e^{-art}\left(\log\frac{1}{r}\right)^{\frac{1-\nu}{2}} dr\right).$$

It is known (see (18.52) in [33] or (32.11) in [34]) that

$$\int_0^{1/2} e^{-art}\left(\log\frac{1}{r}\right)^{\frac{1-\nu}{2}} dr \leqslant Ct^{-1}(\log t)^{\frac{1-\nu}{2}}$$

for large values of $t$. This implies the first inequality of (4.29).

Let $n = 2$. Then

$$\widetilde{Z}(p,x) = \frac{1}{2\pi}\mathcal{K}(p)K_0\big(|x|\sqrt{p\mathcal{K}(p)}\,\big),$$

so that we have, for $|x| < \varepsilon$ and small $|p|$, that

$$\big|\widetilde{Z}(p,x)\big| \leqslant C|p|^{-1}\big|\log|p|\big|^{-1-\nu}\big(\log|x|^{-1} + \log\log|p|^{-1}\big).$$

This estimate is sufficient (for $\nu > 0$) to substantiate passing to the limit, as $\gamma \to 0$. The above argument gives, as the main part of the upper estimate of $Z(t,x)$ for a large $t$, the expression

$$C\big(\log|x|^{-1}\big)\int_0^{1/2} e^{-art}r^{-1}|\log r|^{-1-\nu}\big(\log\log r^{-1}\big)\,dr$$
$$= C_1\big(\log|x|^{-1}\big)\int_0^{1/2} e^{-art}\big(\log\log r^{-1}\big)\frac{d}{dr}\big(\log r^{-1}\big)^{-\nu}\,dr.$$

Integrating by parts we reduce the investigation of the above integral in $r$ to that of two integrals,

$$\int_0^{1/2} e^{-art}r^{-1}\big(\log r^{-1}\big)^{-1-\nu}\,dr$$

(it has been estimated above), and

$$\int_0^{1/2} e^{-art}\big(\log r^{-1}\big)^{-\nu}\log\log r^{-1}\,dr \sim Ct^{-1}(\log t)^{-\nu}\log(\log t), \quad t \to \infty$$

[34, 32.11]. This results in the second estimate from (4.29). The third one is derived similarly. □

The relations (4.26) and (4.28) for the case where $n = 1$ and $\mu(\alpha) \equiv \text{const}$ or $\mu(\alpha) \equiv \text{const}\cdot\alpha^\nu$ were proved in [4].

## 5. The Cauchy problem

### *5.1. The homogeneous equation*

Let us consider Eq. (4.1) with the initial condition

$$u(0,x) = \varphi(x), \quad x \in \mathbb{R}^n, \tag{5.1}$$

where $\varphi$ is a locally Hölder continuous function of the sub-exponential growth: for any $b > 0$,

$$\big|\varphi(x)\big| \leqslant C_b e^{b|x|}. \tag{5.2}$$

We will assume that the weight function $\mu$ defining the distributed order derivative $\mathbb{D}^{(\mu)}$ satisfies the conditions of Theorem 4.1.

**Theorem 5.1.**

(i) *The function*

$$u(t,x)=\int_{\mathbb{R}^n} Z(t,x-\xi)\varphi(\xi)\,d\xi \tag{5.3}$$

*is a classical solution of the Cauchy problem* (4.1)–(5.1)*, that is the function* (5.3) *is twice continuously differentiable in $x$ for each $t>0$, for each $x\in\mathbb{R}^n$ it is continuous in $t>0$, the function*

$$t\mapsto\int_0^t k(t-\tau)u(\tau,x)\,d\tau,\quad t>0,$$

*is continuously differentiable, Eq.* (4.1) *is satisfied, and*

$$u(t,x)\to\varphi(x),\quad \text{as } t\to 0, \tag{5.4}$$

*for all $x\in\mathbb{R}^n$.*

(ii) *On each finite time interval $(0,T]$, the solution $u(t,x)$ satisfies the inequality*

$$\left|u(t,x)\right|\leqslant Ce^{d|x|},\quad x\in\mathbb{R}^n, \tag{5.5}$$

*with some constants $C,d>0$. If $\varphi$ is bounded, then*

$$\left|u(t,x)\right|\leqslant C,\quad x\in\mathbb{R}^n,\ 0<t\leqslant T. \tag{5.6}$$

(iii) *For each $x\in\mathbb{R}^n$, there exists such an $\varepsilon>0$ that*

$$\left|\mathbb{D}^{(\mu)}u(t,x)\right|\leqslant C_x t^{-1+\varepsilon},\quad 0<t\leqslant T. \tag{5.7}$$

**Proof.** Using (4.25) we can write

$$u(t,x)=\int_{\mathbb{R}^n} Z(t,x-\xi)\bigl[\varphi(\xi)-\varphi(x)\bigr]\,d\xi+\varphi(x). \tag{5.8}$$

Let us fix $x$ and prove (5.4), that is prove that the integral in (5.8) (denoted by $u_0(t,x)$) tends to 0.

Let $n=1$. Then

$$u_0(t,x)=\frac{1}{4\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\frac{\mathcal{K}(p)}{\sqrt{p\mathcal{K}(p)}}H(p,x)\,dp, \tag{5.9}$$

where $\gamma>0$,

$$H(p,x)=\int_{-\infty}^{\infty} e^{-|x-\xi|\sqrt{p\mathcal{K}(p)}}\bigl[\varphi(\xi)-\varphi(x)\bigr]\,d\xi$$

(the change of the order of integration leading to (5.9) will be justified when we prove the decay of $H(p,x)$, as $p\to\gamma\pm i\infty$; see below).

By our assumption,

$$\left|\varphi(x)-\varphi(\xi)\right|\leqslant C_x|x-\xi|^\lambda,\quad \lambda>0,\ |x-\xi|\leqslant 1.$$

Let $\rho=\sqrt{p\mathcal{K}(p)}$, $p=\gamma+i\tau$. As $|\tau|\to\infty$, $\rho\sim C(\frac{|\tau|}{\log|\tau|})^{1/2}$. We have

$$\begin{aligned}\left|H(p,x)\right|&\leqslant C\int_{|x-\xi|\leqslant 1} e^{-\rho|x-\xi|}|x-\xi|^\lambda\,d\xi+C\int_{|x-\xi|>1} e^{-\rho|x-\xi|+b|\xi|}\,d\xi+\left|\varphi(x)\right|\int_{|x-\xi|>1} e^{-\rho|x-\xi|}\,d\xi\\&\leqslant C\rho^{-1-\lambda}+Ce^{b|x|}\int_{|z|>1} e^{(b-\rho)|z|}\,dz+2\left|\varphi(x)\right|\int_1^\infty e^{-\rho z}\,dz\leqslant C\rho^{-1-\lambda},\end{aligned}$$

if $b$ is taken such that $\rho > b$. Therefore the absolute value of the integrand in (5.9) does not exceed

$$Ce^{\gamma t}|\tau|^{-1-\frac{\lambda}{2}}\big(\log|\tau|\big)^{\lambda/2},$$

so that the integral in (5.9) exists and possesses a limit, as $t \to 0$, equal to

$$u_0(0,x) = \frac{1}{4\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} \frac{\mathcal{K}(p)}{\sqrt{p\mathcal{K}(p)}} H(p,x)\,dp. \tag{5.10}$$

The integrand in (5.10) is analytic in $p$ on the half-plane $\operatorname{Re} p \geqslant \gamma$. Let us consider (within that half-plane) a contour consisting of an interval $\{p\colon \operatorname{Re} p = \gamma, |p| \leqslant R\}$ and the arc $\{p\colon \operatorname{Re} p > \gamma, |p| = R\}$, $R > \gamma$. The absolute value of the integral over the arc (with the same integrand as in (5.10)) does not exceed $CR^{-\lambda/2}(\log R)^{\lambda/2} \to 0$, as $R \to \infty$. This means that $u_0(0,x) = 0$, and we have proved (5.4) for $n = 1$. The scheme of proof is completely similar for $n > 1$ too; one has only to use the asymptotics of the McDonald function.

If we perform the above estimates, not ignoring the dependence on $x$ but, on the contrary, taking it into account, then we obtain the estimates (5.5) and (5.6).

Due to the estimates of $Z$ given in Theorem 4.1, we may differentiate once in the spatial variables in (5.3) under the sign of integral. Using also the identity (4.25) we get, for a fixed $x^0$, the formula

$$\frac{\partial u(t,x^0)}{\partial x_k} = \int\limits_{\mathbb{R}^n} \frac{\partial Z(t,x^0-\xi)}{\partial x_k}\big[\varphi(\xi) - \varphi\big(x^0\big)\big]\,d\xi. \tag{5.11}$$

Let us decompose the domain of integration in (5.11) into the union of

$$\Omega_1 = \big\{\xi \in \mathbb{R}^n\colon \big|x^0 - \xi\big| \geqslant 1\big\}$$

and $\Omega_2 = \mathbb{R}^n \setminus \Omega_1$. Correspondingly, the integral becomes a sum of two functions, $w_1(t,x) + w_2(t,x)$. If $x$ is in a small neighbourhood of $x^0$, while $\xi \in \Omega_1$, then $|x - \xi|$ is separated from zero. Therefore

$$\frac{\partial w_1(t,x^0)}{\partial x_k} = \int\limits_{\Omega_1} \frac{\partial^2 Z(t,x^0-\xi)}{\partial x_k^2}\big[\varphi(\xi) - \varphi\big(x^0\big)\big]\,d\xi. \tag{5.12}$$

Let $d$ be a small positive number, $\tilde{d} = (0,\dots,d,\dots)$, with $d$ being at the $k$th place. Then

$$\begin{aligned}
&\frac{1}{d}\big[w_2\big(t,x^0+\tilde{d}\big) - w_2\big(t,x^0\big)\big] - \int\limits_{\Omega_2} \frac{\partial^2 Z(t,x^0-\xi)}{\partial x_k^2}\big[\varphi(\xi) - \varphi\big(x^0\big)\big]\,d\xi \\
&= \frac{1}{d}\int\limits_{|x^0-\xi|\leqslant 2d} \frac{\partial Z(t,x^0+\tilde{d}-\xi)}{\partial x_k}\big[\varphi(\xi) - \varphi\big(x^0\big)\big]\,d\xi - \frac{1}{d}\int\limits_{|x^0-\xi|\leqslant 2d} \frac{\partial Z(t,x^0-\xi)}{\partial x_k}\big[\varphi(\xi) - \varphi\big(x^0\big)\big]\,d\xi \\
&\quad - \int\limits_{|x^0-\xi|\leqslant 2d} \frac{\partial^2 Z(t,x^0-\xi)}{\partial x_k^2}\big[\varphi(\xi) - \varphi\big(x^0\big)\big]\,d\xi \\
&\quad + \int\limits_{2d\leqslant|x^0-\xi|\leqslant 1} \left\{\frac{1}{d}\left[\frac{\partial Z(t,x^0+\tilde{d}-\xi)}{\partial x_k} - \frac{\partial Z(t,x^0-\xi)}{\partial x_k}\right] - \frac{\partial^2 Z(t,x^0-\xi)}{\partial x_k^2}\right\}\big[\varphi(\xi) - \varphi\big(x^0\big)\big]\,d\xi.
\end{aligned} \tag{5.13}$$

The integrals converge due to the local Hölder continuity of $\varphi$.

We have (if $n \geqslant 2$)

$$\begin{aligned}
&\frac{1}{d}\int\limits_{|x^0-\xi|\leqslant 2d} \frac{\partial Z(t,x^0+\tilde{d}-\xi)}{\partial x_k}\big[\varphi(\xi) - \varphi\big(x^0\big)\big]\,d\xi \\
&\leqslant Ct^{-1}d^{-1}\int\limits_{|x^0-\xi|\leqslant 2d} \big|x^0+\tilde{d}-\xi\big|^{-n+1}\big|\xi - x^0\big|^{\lambda}\,d\xi = Ct^{-1}d^{-1}\int\limits_{|\eta|\leqslant 2d} |\eta+\tilde{d}|^{-n+1}|\eta|^{\lambda}\,d\eta \\
&\leqslant Ct^{-1}d^{\lambda} \to 0, \quad d \to 0
\end{aligned}$$

(the change of variables $\eta = d\zeta$ was made in the last integral). In a similar way we obtain estimates of other integrals over the set $\{|x^0 - \xi| \leqslant 2d\}$.

In the integral over its complement, we use the Taylor formula:

$$\frac{1}{d}\left[\frac{\partial Z(t, x^0 + \tilde{d} - \xi)}{\partial x_k} - \frac{\partial Z(t, x^0 - \xi)}{\partial x_k}\right] - \frac{\partial^2 Z(t, x^0 - \xi)}{\partial x_k^2} = \frac{d}{2}\frac{\partial^3 Z(t, x^0 + \theta\tilde{d} - \xi)}{\partial x_k^3}, \quad 0 < \theta < 1.$$

If $|x^0 - \xi| \geqslant 2d$, then

$$\left|x^0 + \theta\tilde{d} - \xi\right| \geqslant \left|\xi - x^0\right| - d \geqslant \frac{1}{2}\left|\xi - x^0\right|.$$

Using the inequality for the third derivative of $Z$ from Theorem 4.1 we find that the last integral in (5.13) does not exceed

$$C\,dt^{-1} \int\limits_{2d \leqslant |x^0 - \xi| \leqslant 1} \left|\xi - x^0\right|^{-n-1+\lambda} d\xi \leqslant Ct^{-1}d^{\lambda} \to 0,$$

as $d \to 0$.

It follows from (5.12), (5.13) and the above estimates that

$$\frac{\partial^2 u(t, x^0)}{\partial x_k^2} = \int\limits_{\mathbb{R}^n} \frac{\partial^2 Z(t, x^0 - \xi)}{\partial x_k}\left[\varphi(\xi) - \varphi\left(x^0\right)\right] d\xi. \tag{5.14}$$

If $n = 1$, then the formula (5.14) is obtained by a straightforward differentiation under the sign of integral.

Let us consider the distributed order derivative $\mathbb{D}^{(\mu)}u$. First of all we check the identity

$$\mathbb{D}^{(\mu)} Z(t, x) = \Delta Z(t, x), \quad t > 0,\ x \neq 0. \tag{5.15}$$

A direct calculation based on identities for the derivatives of the McDonald function [1] shows that $\Delta\widetilde{Z}(p, x) = p\mathcal{K}(p)\widetilde{Z}(p, x)$. On the other hand, if $x \neq 0$, then $Z(t, x) \to 0$, as $t \to 0$. This fact follows from the integral representation of $Z$ in a manner similar to the above proof of (5.4). Therefore the Laplace transform of $\mathbb{D}^{(\mu)} Z(t, x)$, $x \neq 0$, equals $p\mathcal{K}(p)\widetilde{Z}(p, x)$, which implies (5.15).

Now, having the estimates of the derivatives of $Z$ in spatial variables given in Theorem 4.1, from (5.15) we get estimates for $\mathbb{D}^{(\mu)} Z$ sufficient to justify the distributed differentiation in (5.8). Thus we come to the formula

$$\left(\mathbb{D}^{(\mu)} u\right)\left(t, x^0\right) = \int\limits_{\mathbb{R}^n} \left(\mathbb{D}^{(\mu)} Z\right)\left(t, x^0 - \xi\right)\left[\varphi(\xi) - \varphi\left(x^0\right)\right] d\xi. \tag{5.16}$$

Together with (5.14) and (5.15), this proves that $u(t, x)$ is a solution of Eq. (4.1).

In order to prove (5.7), we use the inequalities (4.13), (4.13′), and the assumption (5.2) with $b < a$. Substituting into (5.16) we get, for a fixed $x^0$, that

$$\begin{aligned}
\left|\left(\mathbb{D}^{(\mu)} u\right)\left(t, x^0\right)\right| &\leqslant Ct^{-2} \int\limits_{|x^0 - \xi| < t^{1/2}} \left|x^0 - \xi\right|^{-n+2+\lambda} d\xi + C \int\limits_{t^{1/2} \leqslant |x^0 - \xi| \leqslant 1} \left|x^0 - \xi\right|^{-n-2+\lambda} e^{-a|x^0 - \xi|} d\xi \\
&\quad + C \int\limits_{|x^0 - \xi| > 1} \left|x^0 - \xi\right|^{-n-2} e^{-a|x^0 - \xi|}\left(e^{b|\xi|} + e^{b|x^0|}\right) d\xi \\
&\leqslant Ct^{-2} \int\limits_0^{t^{1/2}} r^{1+\lambda}\, dr + C \int\limits_{t^{1/2}}^1 r^{-3+\lambda} e^{-ar}\, dr + C \leqslant Ct^{-1+\frac{\lambda}{2}}
\end{aligned}$$

for small values of $t$, as desired. □

### 5.2. The inhomogeneous equation

Let us consider the Cauchy problem

$$\big(\mathbb{D}_t^{(\mu)}u\big)(t,x) - \Delta u(t,x) = f(t,x), \quad x \in \mathbb{R}^n,\ t > 0, \tag{5.17}$$

$$u(0,x) = 0. \tag{5.18}$$

We assume that the function $f$ is continuous in $t$, bounded and locally Hölder continuous in $x$, uniformly with respect to $t$. Our task in this section is to obtain a solution of (5.17)–(5.18) in the form of a "heat potential"

$$u(t,x) = \int_0^t d\tau \int_{\mathbb{R}^n} E(t-\tau, x-y) f(\tau,y)\,dy. \tag{5.19}$$

In contrast to the classical theory of parabolic equations [14], the kernel $E$ in (5.19) does not coincide with the fundamental solution $Z$—just as this happens for fractional diffusion equations [9,10]. However the behavior of the function $E$ is very similar to that of $Z$. Applying formally the Laplace transform in $t$ and the Fourier transform in $x$ we find that

$$\widehat{\widetilde{E}}(p,\xi) = \frac{1}{p\mathcal{K}(p) + |\xi|^2}$$

whence

$$\widetilde{E}(p,x) = (2\pi)^{-\frac{n}{2}} |x|^{1-\frac{n}{2}} \big(p\mathcal{K}(p)\big)^{\frac{1}{2}(\frac{n}{2}-1)} K_{\frac{n}{2}-1}\big(|x|\sqrt{p\mathcal{K}(p)}\,\big), \tag{5.20}$$

which differs from (4.5) only by the absense of the factor $\mathcal{K}(p)$ with a logarithmic behavior at infinity. Therefore the function $E(t,x)$ obtained from (5.20) via contour integration, satisfies the same estimates (see Theorem 4.1) as the function $Z$, except the estimates for large values of $t$.

The function $E(t,x)$ is non-negative. Indeed, the function $p \mapsto p^{\nu/2}K_\nu(a\sqrt{p})$, $a > 0$, is the Laplace transform of the function

$$t \mapsto \frac{a^\nu}{(2t)^{\nu+1}} e^{-\frac{a^2}{4t}}$$

(see [7]). This means that the above function in $p$ is completely monotone. Since the function $p \mapsto p\mathcal{K}(p)$ is positive and has a completely monotone derivative, we find that $\widetilde{E}(p,x)$ is completely monotone in $p$, so that $E(t,x) \geqslant 0$, $x \neq 0$.

The counterparts of the estimates (4.29) (proved just as in Theorem 4.3) are as follows. If (4.27) holds with $\nu \geqslant 0$, then for $|x| \leqslant \varepsilon$, $\varepsilon > 0$, and $t > \varepsilon^{-1}$

$$E(t,x) \leqslant \begin{cases} Ct^{-1}(\log t)^{\frac{1+\nu}{2}}, & \text{if } n = 1, \\ Ct^{-1}\log\log t \log|x|^{-1}, & \text{if } n = 2, \\ Ct^{-1}|x|^{-n+2}, & \text{if } n \geqslant 3. \end{cases} \tag{5.21}$$

The function $E$ has (in $x$) an exponential decay at infinity.

In fact, for the analysis of the potential (5.19) we need estimates of $E$ and its derivatives, uniform in $t \in (0,T]$.

**Proposition 5.2.** *Let $\mu$ satisfy the conditions of Theorem 4.1. Then, uniformly in $t \in (0,T]$,*

$$\big|D_x^j E(t,x)\big| \leqslant C|x|^{-j-n}\big|1 + |\log|x||\big|^\beta e^{-a|x|}, \quad x \neq 0,\ j \geqslant 0, \tag{5.22}$$

$$\big|\mathbb{D}_t^{(\mu)} E(t,x)\big| \leqslant C|x|^{-n-2}\big|1 + |\log|x||\big|^\beta e^{-a|x|}, \quad x \neq 0, \tag{5.23}$$

*where $C, a, \beta$ are positive constants.*

**Proof.** Let, for example, $n \geqslant 3$ (other cases are considered in a similar way). As usual, we write the Laplace inversion formula and deform the contour of integration to $S_{\gamma,\omega}$. The integral over $T_{\gamma,\omega}$ gives an exponentially decaying contribution without local singularities. In the integrals over the rays $\Gamma^{\pm}_{\gamma,\omega}$ we use the upper bound

$$\big|K_{\frac{n}{2}-1}(z)\big| \leqslant C|z|^{-\frac{n}{2}+1}e^{-a|z|}, \quad z \neq 0,$$

($a > 0$) obtained from the asymptotics of the McDonald function near the origin and infinity. As in the proof of Theorem 4.1, we perform the change of variable $z = (\frac{r}{\log r})^{1/2}$ and use the inequality (4.18) for the dependence of $r$ on $z$.

As a result, for the integrals over $\Gamma^{\pm}_{\gamma,\omega}$ we obtain the upper bound

$$C|x|^{-n+2}\int_{\gamma_1}^{\infty} z(\log z)^{\beta} e^{-a|x|z}\,dz \leqslant C|x|^{-n}\big(\big|\log|x|\big|+1\big)^{\beta} e^{-a'|x|}$$

with some positive constants, and we come to the estimate (5.22), $j=0$. The estimates of the derivatives in spatial variables are proved similarly.

The proof of (5.23) is completely analogous to that of the inequality (4.13) for $\mathbb{D}^{(\mu)}Z$. □

As we have noticed,

$$\widetilde{E}(p,x) = \frac{1}{\mathcal{K}(p)}\widetilde{Z}(p,x),$$

and since

$$\int_{\mathbb{R}^n} \widetilde{Z}(p,x)\,dx = \frac{1}{p},$$

we have

$$\int_{\mathbb{R}^n} \widetilde{E}(p,x)\,dx = \frac{1}{p\mathcal{K}(p)},$$

so that we come to an interesting identity

$$\int_{\mathbb{R}^n} E(t,x)\,dx = \varkappa(t). \tag{5.24}$$

The existence of the integral in (5.24) follows from the above estimates or from the fact that $\varkappa \in L_1^{\mathrm{loc}}$ (see (3.2)) and Fubini's theorem.

**Theorem 5.3.** *Under the above assumptions regarding $f$, and the assumptions of Theorem* 4.1 *regarding $\mu$, the function* (5.19) *is a classical solution of the problem* (5.17)–(5.18)*, bounded near the origin in $t$ for each $x \in \mathbb{R}^n$.*

**Proof.** The initial condition (5.18) is evidently satisfied. Just as for the kernel $Z$ above, we prove that $\mathbb{D}^{(\mu)}E - \Delta E = 0$ for $x \neq 0$. Next, we may differentiate once in (5.19) under the sign if integral. Indeed (here and below we make estimates for $n \geqslant 3$; other cases are similar),

$$\int_0^t d\tau \int_{\mathbb{R}^n} \left|\frac{\partial E(t-\tau,x-y)}{\partial x_j}\right| dy$$
$$\leqslant C\int_0^t d\tau \int_{|x|>\sqrt{\tau}} |x|^{-n-1}\big(1+\big|\log|x|\big|\big)^{\beta} e^{-a|x|}\,dx + C\int_0^t \tau^{-1}\,d\tau \int_{|x|\leqslant\sqrt{\tau}} |x|^{-n+1}\,dx$$
$$\leqslant C\int_0^t \tau^{-1/2}\,d\tau \int_{|y|>1} |y|^{-n-1}\left(1+\log|y|+\frac{1}{2}\log\tau^{-1}\right)dy + C\int_0^t \tau^{-1/2}\,d\tau \int_{|y|\leqslant 1} |y|^{-n+1}\,dy < \infty.$$

In order to calculate the second order derivatives, note that the function

$$u_h(t,x) = \int_0^{t-h} d\tau \int_{\mathbb{R}^n} E(t-\tau,x-y)f(\tau,y)\,dy, \quad t > h > 0,$$

may be differentiated twice, and that

$$\int_{\mathbb{R}^n} \frac{\partial^2}{\partial x_i^2} E(t-\tau, x-y)\,dy = 0,$$

whence

$$\frac{\partial^2 u_h(t,x)}{\partial x_i^2} = \int_0^{t-h} d\tau \int_{\mathbb{R}^n} \frac{\partial^2}{\partial x_i^2} E(t-\tau, x-y)\big[f(\tau,y) - f(\tau,x)\big]\,dy. \tag{5.25}$$

Using the local Hölder continuity and boundedness of $f$, we perform estimates as above and prove the possibility to pass to the limit in (5.25), as $h \to 0$, so that

$$\Delta u(t,x) = \int_0^t d\tau \int_{\mathbb{R}^n} \Delta E(t-\tau, x-y)\big[f(\tau,y) - f(\tau,x)\big]\,dy. \tag{5.26}$$

To calculate $\mathbb{D}^{(\mu)} u$, we use (5.24) and write

$$u(t,x) = \int_0^t d\tau \int_{\mathbb{R}^n} E(t-\tau, x-y)\big[f(\tau,y) - f(\tau,x)\big]\,dy + \int_0^t \varkappa(t-\tau) f(\tau,x)\,d\tau \stackrel{\text{def}}{=} u_1(t,x) + u_2(t,x).$$

Recall that $u_2(t,x) = (\mathbb{I}^{(\mu)} f)(t,x)$, so that $\mathbb{D}^{(\mu)} u_2 = f$ (see Section 3).

Let us consider $u_1$. First we estimate $\frac{\partial E}{\partial t}$. As before, we use the contour integral representation of $E$ and note that the differentiation in $t$ leads to an additional factor $p$ in the integrals. This results in the estimates

$$\left|\frac{\partial E(t,x)}{\partial t}\right| \leqslant C t^{-2} |x|^{-n+2}, \quad |x| \leqslant \varepsilon, \tag{5.27}$$

$$\left|\frac{\partial E(t,x)}{\partial t}\right| \leqslant C |x|^{-n-2}\big(\big|\log|x|\big| + 1\big)^{\beta} e^{-a|x|}, \quad |x| \neq 0. \tag{5.28}$$

As the first step of computing $\mathbb{D}^{(\mu)} u_1$, we compute $\frac{\partial u_1}{\partial t}$. Note that

$$\int_{\mathbb{R}^n} E(t-\tau, x-y)\big[f(\tau,y) - f(\tau,x)\big]\,dy = \frac{1}{(2\pi)^{n/2+1} i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{p(t-\tau)} \big(p\mathcal{K}(p)\big)^{\frac{1}{2}(\frac{n}{2}-1)} L_n(p,x,\tau)\,dp, \tag{5.29}$$

where

$$L_n(p,x,\tau) = \int_{\mathbb{R}^n} |x-y|^{1-\frac{n}{2}} \big[f(\tau,y) - f(\tau,x)\big] K_{\frac{n}{2}-1}\big(|x-y|\sqrt{p\mathcal{K}(p)}\,\big)\,dy.$$

The role of the function $L_n$ is quite similar to that of the function $H$ introduced in the proof of Theorem 5.1 (where the case $n = 1$ was considered in detail). Using, as it was done there, the local Hölder continuity and boundedness of $f$ we find that

$$\big|L_n(p,x,\tau)\big| \leqslant C \big|\sqrt{p\mathcal{K}(p)}\,\big|^{-\frac{n}{2}-\lambda-1}.$$

As in the proof of Theorem 5.1, we deform the contour of integration to the right of the line in (5.29) and show that

$$\lim_{\tau \to t} \int_{\mathbb{R}^n} E(t-\tau, x-y)\big[f(\tau,y) - f(\tau,x)\big]\,dy = 0. \tag{5.30}$$

On the other hand, using (5.27) and (5.28) we get

$$\int_{\mathbb{R}^n} \left|\frac{\partial E(t-\tau, x-y)}{\partial t}\right| \left|f(\tau, y) - f(\tau, x)\right| dy$$

$$\leqslant C \int_{|y|\geqslant\sqrt{t-\tau}} |y|^{-n-2+\lambda}\left(\left|\log|y|\right|+1\right)^{\beta} e^{-a|y|}\, dy + C(t-\tau)^{-2} \int_{|y|<\sqrt{t-\tau}} |y|^{-n+2+\lambda}\, dy$$

$$= 2C \int_{\sqrt{t-\tau}}^{\infty} r^{-3+\lambda}\left(|\log r|+1\right)^{\beta} e^{-ar}\, dr + 2C(t-\tau)^{-2} \int_0^{\sqrt{t-\tau}} r^{1+\lambda}\, dr$$

$$\leqslant C(t-\tau)^{-1+\lambda/2}\left(\left|\log(t-\tau)\right|+1\right)^{\beta}.$$

Together with (5.30), this implies the equality

$$\frac{\partial u_1}{\partial t} = \int_0^t d\tau \int_{\mathbb{R}^n} \frac{\partial E(t-\tau, x-y)}{\partial t}\left[f(\tau, y) - f(\tau, x)\right] dy.$$

Now we compute $\mathbb{D}^{(\mu)}u_1$ using the formula (2.3), the fact that $k \in L_1^{\mathrm{loc}}$ (following from (2.4)) and Fubini's theorem:

$$\left(\mathbb{D}^{(\mu)}u_1\right)(t,x) = \int_0^t d\tau \int_{\mathbb{R}^n} \left(\mathbb{D}^{(\mu)}E\right)(t-\tau, x-y)\left[f(\tau, y) - f(\tau, x)\right] dy.$$

Together with (5.26), this means that $\Delta u = \mathbb{D}^{(\mu)}u_1 = \mathbb{D}^{(\mu)}u - f$, as desired. □

In Theorem 5.3 we constructed a solution $u$ of the problem (5.17)–(5.18), such that $u = u_1 + u_2$, $u_1(0,x) = u_2(0,x) = 0$, $u_1$ is absolutely continuous in $t$, and $u_2 = \mathbb{I}^{(\mu)} f$. On this solution $u$,

$$\mathbb{I}^{(\mu)}\mathbb{D}^{(\mu)}u = \mathbb{I}^{(\mu)}\left(k * u_1'\right) + \mathbb{I}^{(\mu)} f = \varkappa * k * u_1' + u_2 = u_1 + u_2 = u$$

($u'$ means the derivative in $t$). Applying $\mathbb{I}^{(\mu)}$ to both sides of Eq. (5.17) we find that

$$u(t,x) - \int_0^t \varkappa(t-s)\Delta u(s,x)\, ds = (\varkappa * f)(t,x). \tag{5.31}$$

Equation (5.31) can be interpreted as an abstract Volterra equation

$$u + \varkappa * (Au) = \varphi, \tag{5.32}$$

if we assume that $u$ belongs to some Banach space $X$ (in the variable $x$), and $A$ is the operator $-\Delta$ on $X$. The operator $-A$ generates a contraction semigroup if, for example, $X = L_2(\mathbb{R}^n)$ or $X = C_\infty(\mathbb{R}^n)$ (the space of continuous functions decaying at infinity; see Section X.8 in [31]). Now the existence of a solution in $L_1(0,T;X)$ can be obtained from a general theory of Eqs. (5.32) developed in [6]; it is essential that $\varkappa$ is completely monotone (conditions of some other papers devoted to Eqs. (5.32) do not cover our situation). Of course, our "classical" approach gives a much more detailed information about solutions, while the abstract method is applicable to more general equations.

## 6. Uniqueness theorems

### *6.1. Bounded solutions*

In this section we consider a more general equation

$$\left(\mathbb{D}^{(\mu)}u\right)(t,x) = Lu(t,x), \quad x \in \mathbb{R}^n,\ 0 < t \leqslant T, \tag{6.1}$$

with the zero initial condition

$$u(0,x) = 0. \tag{6.2}$$

Here $L$ is an elliptic second order differential operator with bounded continuous real-valued coefficients:

$$Lu = \sum_{i,j=1}^{n} a_{ij}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(t,x)\frac{\partial u}{\partial x_j} + c(t,x)u,$$

$$\sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j > 0, \quad 0 \neq \xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n.$$

We assume that $\mu \in C^3[0,1]$, $\mu(1) \neq 0$.

We will consider classical solutions $u(t,x)$, such that $(\mathbb{D}^{(\mu)}u)(t,x)$ belongs, for each fixed $x$, to $L_p(0,T)$ with some $p > 1$. As we saw in Theorems 5.1 and 5.3, the solutions for the case $L = \Delta$ obtained via the fundamental solution and heat potential possess the last property making it possible to represent $\mathbb{D}^{(\mu)}u$ in the Marchaud form (3.9).

It is often convenient to transform Eq. (6.1) setting

$$u(t,x) = u_\lambda(t)w(t,x)$$

where $\lambda > 0$, and $u_\lambda$ is the solution of the equation $\mathbb{D}^{(\mu)}u_\lambda = \lambda u_\lambda$ constructed in Section 2.3. It is easy to check that the function $w$ satisfies the equation

$$(A_\lambda w)(t,x) = (L-\lambda)w(t,x)$$

where

$$(A_\lambda w)(t,x) = \frac{1}{u_\lambda(t)}\left\{k(t)w(t,x) + \lim_{\varepsilon\to 0}\int_0^{t-\varepsilon} u_\lambda(\tau)k'(t-\tau)\big[w(\tau,x) - w(t,x)\big]\,d\tau\right\}. \tag{6.3}$$

The operator (6.3) is very similar in its properties to the distributed order derivative $\mathbb{D}^{(\mu)}$.

**Theorem 6.1.** *If $u(t,x)$ is a bounded classical solution of the problem* (6.1)–(6.2)*, such that for each $x \in \mathbb{R}^n$, $\mathbb{D}^{(\mu)}u \in L_p(0,T)$ for some $p > 1$, then $u(t,x) \equiv 0$.*

**Proof.** Let $M = \sup|u(t,x)|$. Consider the function

$$F_R(t,x) = \frac{M}{R^2}\left(|x|^2 + \sigma\int_0^t \varkappa(s)\,ds + 1\right),$$

with $R, \sigma > 0$. It follows from (3.2) that $\int_0^t \varkappa(s)\,ds \to 0$, as $t \to 0$. As we have seen (Section 3.2), $\mathbb{D}_t^{(\mu)}(\int_0^t \varkappa(s)\,ds) = 1$, so that

$$\big(\mathbb{D}^{(\mu)}F_R\big)(t,x) = \frac{\sigma M}{R^2}.$$

Let $c_0 = \sup|c(t,x)|$, $d > 0$, $\lambda = c_0 + d$. Since $u_\lambda$ is non-decreasing (Theorem 2.3), and $k'(s) \leqslant 0$, we have

$$(A_\lambda F_R)(t,x) \geqslant \frac{(\mathbb{D}^{(\mu)}F_R)(t,x)}{u_\lambda(T)} = \frac{\sigma M}{R^2 u_\lambda(T)}.$$

On the other hand,

$$(LF_R)(t,x) \leqslant \frac{2M}{R^2}\left(\sum_{i=1}^{n} a_{ii}(x) + \sum_{j=1}^{n} b_j(x)x_j\right) \leqslant \frac{2M}{R^2}\big(C_1 + C_2|x|\big), \quad C_1, C_2 > 0,$$

so that taking $d > 0$ we get

$$\big(\big(A_\lambda - (L-\lambda)\big)F_R\big)(t,x) \geqslant \frac{M}{R^2}\left(\frac{\sigma}{u_\lambda(T)} - 2C_1 - 2C_2|x| + d|x|^2 + d\right) \geqslant 0$$

for all $x \in \mathbb{R}^n$, $t \in (0,T)$, if $\sigma$ is taken sufficiently big.

Denote $v(t,x) = u(t,x) - F_R(t,x)$. By the above inequalities,

$$(A_\lambda v)(t,x) - (L-\lambda)v(t,x) \leqslant 0. \tag{6.4}$$

If $|x| = R$, then $v(t,x) = u(t,x) - M - R^{-2}(\sigma \int_0^t \varkappa(s)\,ds + 1) < 0$. Next, $v(0,x) = -F_R(0,x) < 0$ for all $x$. This means that $v(t,x) \leqslant 0$ for $|x| < R$, $t \in [0,T]$. Indeed, otherwise the function $v$ would possess a point of the global maximum $(t^0, x^0)$ on the set $\{(t,x) \mid 0 < t \leqslant T,\ |x| < R\}$, such that $v(t^0,x^0) > 0$. Then

$$(L-\lambda)v\big(t^0,x^0\big) \leqslant 0$$

(see the proof of the maximum principle for a second order parabolic differential equation [14,20]), so that $(A_\lambda v)(t^0,x^0) \leqslant 0$, due to (6.4). However it follows from (6.3) that $(A_\lambda v)(t^0,x^0) > 0$, and we have come to a contradiction.

Thus, we have proved that

$$u(t,x) \leqslant \frac{M}{R^2}\left(|x|^2 + \sigma \int_0^t \varkappa(s)\,ds + 1\right), \quad |x| \leqslant T.$$

Since $R$ is arbitrary, we find that $u(t,x) \leqslant 0$ for all $t \in [0,T]$, $x \in \mathbb{R}^n$. Considering $-u(t,x)$ instead of $u(t,x)$ we prove that $u(t,x) \equiv 0$. □

The above proof was based on standard "maximum principle" arguments. In fact, it is easy to prove, for Eq. (6.1), an analog of the maximum principle itself. Namely, let $c(t,x) - \lambda \leqslant 0$ for $t \in [0,T]$, $x \in G$, where $G \subset \mathbb{R}^n$ is a bounded domain. Suppose that

$$(L-\lambda)u(t,x) - (A_\lambda u)(t,x) \geqslant 0, \quad (t,x) \in [0,T] \times \overline{G}.$$

Then, if $\sup_{[0,T]\times\overline{G}} u > 0$, then

$$\sup_{[0,T]\times\overline{G}} u = \sup_{[0,T]\times\partial G} u.$$

The proof is similar to the classical one [20].

### *6.2. Solutions of subexponential growth*

In this section we will prove a more exact uniqueness theorem for the case where $n = 1$, $L = \frac{\partial^2}{\partial x^2}$.

**Theorem 6.2.** *Suppose that $u(t,x)$ is a classical solution of the problem* (6.1)–(6.2) *with $n = 1$, $L = \frac{\partial^2}{\partial x^2}$, such that for any $a > 0$,*

$$\big|u(t,x)\big| \leqslant C_a e^{a|x|}, \quad 0 < t \leqslant T,\ x \in \mathbb{R}^1,$$

*and $\mathbb{D}_t^{(\mu)} u \in L_p(0,T)$, $p > 1$, in $t$, for any fixed $x$. Then $u(t,x) \equiv 0$.*

**Proof.** This time we choose the comparison function as

$$F_R^{(1)}(t,x) = Me^{aR}\big[Z(t,x-R) + Z(t,x+R)\big], \quad |x| \leqslant R, \tag{6.5}$$

where $Z$ is the above fundamental solution of the Cauchy problem (Section 4), $M$ and $a$ are positive constants to be specified later. We will need the following auxiliary result.

**Lemma 6.3.** *For any $T > 0$, there exists such a constant $\rho_0 > 0$ that*

$$Z(t,0) \geqslant \rho_0, \quad 0 < t \leqslant T.$$

**Proof.** By (4.7),

$$\widetilde{Z}(p,0)=\frac{1}{2}\sqrt{p^{-1}\mathcal{K}(p)},$$

so that

$$\widetilde{Z}(p,0)\sim\frac{\sqrt{\mu(1)}}{2}(p\log p)^{-1/2},\quad p\to\infty. \tag{6.6}$$

Note that in Section 4 we used the Laplace inversion formula for $Z(t,x)$ only for $x\neq 0$. Here the task is just the opposite, and we use the inversion formula from [11] involving the derivative of the Laplace image. In our case

$$\frac{\partial\widetilde{Z}(p,0)}{\partial p}=\frac{1}{2p}\left(\frac{d}{dp}\sqrt{\mathcal{K}(p)}\cdot\sqrt{p}-\frac{\sqrt{\mathcal{K}(p)}}{2\sqrt{p}}\right),\quad \frac{d}{dp}\sqrt{\mathcal{K}(p)}=\frac{\mathcal{K}'(p)}{2\sqrt{\mathcal{K}(p)}},$$

$$\mathcal{K}'(p)=\int_0^1(\alpha-1)p^{\alpha-2}\mu(\alpha)\,d\alpha=o\left(p^{-1}\right),\quad p\to\infty,$$

so that

$$\left|\frac{\partial\widetilde{Z}(p,0)}{\partial p}\right|\leqslant C_\varepsilon|p|^{-\frac{3}{2}+\varepsilon},\quad \operatorname{Re}p\geqslant 1,$$

for any $\varepsilon>0$. This is sufficient for the inversion formula

$$Z(t,0)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\widetilde{Z}(p,0)e^{pt}\,dp,\quad t\neq 0, \tag{6.7}$$

where $\gamma\geqslant 1$. Using (6.6), (6.7) and an asymptotic theorem for the Laplace inversion (see (22.115) and (22.114) in [33]) we find the asymptotics

$$Z(t,0)=Ct^{-1/2}\left(\log\frac{1}{t}\right)^{-3/2},\quad t\to +0. \tag{6.8}$$

For our purpose, it is sufficient to derive from (6.8) that $Z(t,0)\to +\infty$, as $t\to +0$.

On the other hand, it follows from the subordination identity (4.23) and the fact that $G(u,t)>0$ for each $t$ on the set of a positive measure in $u$ (see (4.24)), that $Z(t,0)>0$ for each $t>0$. Together with (6.8), this implies the required inequality. □

**Proof of Theorem 6.2** *(continued).* If $|x|=R$, that is $x=\pm R$, then by (6.5) and Lemma 6.3,

$$F_R^{(1)}(t,x)=Me^{aR}\left[Z(t,0)+Z(t,\pm 2R)\right]\geqslant M\rho_0e^{aR}.$$

We have $u(t,x)\leqslant F_R^{(1)}(t,x)$, if $C_a\leqslant M\rho_0$ ($a$ has not yet been chosen), that is if $M$ is chosen in such a way that $M\geqslant C_a\rho_0^{-1}$.

The function $w(t,x)=F_R^{(1)}(t,x)-u(t,x)$ satisfies, if $|x|\leqslant R$, $t\in(0,T)$, the equation $\mathbb{D}_t^{(\mu)}w=\frac{\partial^2 w}{\partial x^2}$. If $|x|=R$, then $w(t,x)\geqslant 0$. In addition,

$$\liminf_{t\to 0}w(t,x)\geqslant 0,$$

if $|x|<R$. It follows that $w(t,x)\geqslant 0$ for $t\in(0,T]$, $|x|\leqslant R$.

Indeed, if $w(t,x)<0$ for some $t$ and $x$, then there exist $t^0\in(0,T]$, $x^0\in\mathbb{R}^1$, $|x^0|<R$, such that

$$w\left(t^0,x^0\right)=\inf_{\substack{|x|\leqslant R\\ t\in(0,T]}}w(t,x)<0.$$

If $|x| < R$, then the function $F_R^{(1)}$ is infinitely differentiable in $t$, with the derivative being continuous on $[0, T]$. Therefore we may write $\mathbb{D}^{(\mu)}w$ in the Marchaud form, so that

$$k(t)w(t,x) + \lim_{\varepsilon\to 0}\int_{\varepsilon}^{t} k'(\tau)\bigl[w(t-\tau,x) - w(t,x)\bigr]\,d\tau = \frac{\partial^2 w(t,x)}{\partial x^2}. \tag{6.9}$$

For $(t,x) = (t^0, x^0)$, we see that the left-hand side of (6.9) is negative, while the right-hand side is non-negative, and we get a contradiction. Thus, we have proved that

$$u(t,x) \leqslant Me^{aR}\bigl[Z(t,x-R) + Z(t,x+R)\bigr], \quad 0 < t \leqslant T,\ |x| \leqslant R. \tag{6.10}$$

Let us fix $x$ and consider the limit $R \to \infty$. For large values of $R$ we have

$$Z(t, x \pm R) \leqslant Be^{-bR}, \quad t \in (0, T],$$

where $b > 0$ depends only on $T$, $B > 0$ depends on $T$ and $x$, and not on $R$. By (6.10),

$$u(t,x) \leqslant BMe^{(a-b)R}. \tag{6.11}$$

Now we choose $a$ in such a way that $a < b$, and (see above) fix $M \geqslant C_a\rho_0^{-1}$. Obviously, $M$ does not depend on $R$. Passing to the limit in (6.11), as $R \to \infty$, we see that $u(t,x) \leqslant 0$ for arbitrary $t$ and $x$. Similarly, taking $-u(t,x)$ instead of $u(t,x)$, we find that $u(t,x) \geqslant 0$. □

## References


[1] H. Bateman, A. Erdélyi, Higher Transcendental Functions, vol. 2, McGraw–Hill, New York, 1953.
[2] M. Caputo, Mean fractional-order derivatives, differential equations and filters, Ann. Univ. Ferrara Sez. VII Sci. Mat. 41 (1995) 73–84.
[3] A.V. Chechkin, R. Gorenflo, I.M. Sokolov, Retarding subdiffusion and accelerating superdiffusion governed by distributed order fractional diffusion equations, Phys. Rev. E (3) 66 (046129) (2002) 1–7.
[4] A.V. Chechkin, R. Gorenflo, I.M. Sokolov, V.Yu. Gonchar, Distributed order fractional diffusion equation, Fract. Calc. Appl. Anal. 6 (2003) 259–279.
[5] A.V. Chechkin, J. Klafter, I.M. Sokolov, Fractional Fokker–Planck equation for ultraslow kinetics, Europhys. Lett. 63 (2003) 326–332.
[6] Ph. Clément, J.A. Nohel, Abstract linear and nonlinear Volterra equations preserving positivity, SIAM J. Math. Anal. 10 (1979) 365–388.
[7] V.A. Ditkin, A.P. Prudnikov, Integral Transforms and Operational Calculus, Pergamon Press, Oxford, 1965.
[8] Yu.A. Dubinskij, The algebra of pseudo-differential operators with analytic symbols and its applications to mathematical physics, Russian Math. Surveys 37 (1982) 109–153.
[9] S.D. Eidelman, S.D. Ivasyshen, A.N. Kochubei, Analytic Methods in the Theory of Differential and Pseudo-Differential Equations of Parabolic Type, Birkhäuser, Basel, 2004.
[10] S.D. Eidelman, A.N. Kochubei, Cauchy problem for fractional diffusion equations, J. Differential Equations 199 (2004) 211–255.
[11] M.A. Evgrafov, Analytic Functions, Saunders, Philadelphia, 1966.
[12] M.V. Fedoryuk, Asymptotics. Integrals and Series, Nauka, Moscow, 1987 (in Russian).
[13] W. Feller, An Introduction to Probability Theory and Its Applications, vol. 2, Wiley, New York, 1971.
[14] A. Friedman, Partial Differential Equations of Parabolic Type, Prentice Hall, Englewood Cliffs, NJ, 1964.
[15] R. Gorenflo, F. Mainardi, Simply and multiply scaled diffusion limits for continuous time random walks, J. Phys. Conf. Ser. 7 (2005) 1–16.
[16] R. Gorenflo, F. Mainardi, Fractional relaxation of distributed order, in: M.M. Novak (Ed.), Complex Mundi. Emergent Patterns in Nature, World Scientific, Singapore, 2006, pp. 33–42.
[17] L.Ya. Kobelev, Ya.L. Kobelev, The fractional derivatives with orders as functions depending on the variable of integration, in: Proc. FDA'04: Fractional Differentiation and Its Applications, Bordeaux, France, July 19–21, 2004, pp. 132–136.
[18] A.N. Kochubei, A Cauchy problem for evolution equations of fractional order, Differ. Equ. 25 (1989) 967–974.
[19] A.N. Kochubei, Fractional-order diffusion, Differ. Equ. 26 (1990) 485–492.
[20] E.M. Landis, Second Order Equations of Elliptic and Parabolic Type, Amer. Math. Soc., Providence, RI, 1998.
[21] C.F. Lorenzo, T.T. Hartley, Variable order and distributed order fractional operators, Nonlinear Dynam. 29 (2002) 57–98.
[22] M.M. Meerschaert, H.-P. Scheffler, Stochastic model for ultraslow diffusion, Stochastic Process. Appl. 116 (2006) 1215–1235.
[23] R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: A fractional dynamics approach, Phys. Rep. 339 (2000) 1–77.
[24] R. Metzler, J. Klafter, The restaurant at the end of the random walk: Recent developments in the description of anomalous transport by fractional dynamics, J. Phys. A 37 (2004) R161–R208.
[25] M. Naber, Distributed order fractional subdiffusion, Fractals 12 (2004) 23–32.
[26] A.M. Nakhushev, Fractional Calculus and Its Applications, Fizmatlit, Moscow, 2003 (in Russian).
[27] F.W.J. Olver, Asymptotics and Special Functions, Academic Press, New York, 1974.
[28] A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, Integrals and Series. vol. 1: Elementary Functions, Gordon and Breach, New York, 1986.
[29] A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, Integrals and Series, vol. 2: Special Functions, Gordon and Breach, New York, 1986.

[30] A.V. Pskhu, Partial Differential Equations of Fractional Order, Nauka, Moscow, 2005 (in Russian).
[31] M. Reed, B. Simon, Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-Adjointness, Academic Press, New York, 1975.
[32] E.Ya. Riekstynsh (Riekstiņš), Asymptotic Expansions of Integrals, vol. 1, Zinatne, Riga, 1974 (in Russian).
[33] E.Ya. Riekstynsh (Riekstiņš), Asymptotic Expansions of Integrals, vol. 2, Zinatne, Riga, 1977 (in Russian).
[34] E.Ya. Riekstynsh (Riekstiņš), Asymptotic Expansions of Integrals, vol. 3, Zinatne, Riga, 1981 (in Russian).
[35] S.G. Samko, R.P. Cardoso, Sonine integral equations of the first kind in $L_p(0,b)$, Fract. Calc. Appl. Anal. 6 (2003) 235–258.
[36] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, New York, 1993.
[37] W.R. Schneider, W. Wyss, Fractional diffusion and wave equations, J. Math. Phys. 30 (1989) 134–144.
[38] I.M. Sokolov, A.V. Chechkin, J. Klafter, Distributed-order fractional kinetics, Acta Phys. Polon. B 35 (2004) 1323–1341.
[39] S. Umarov, R. Gorenflo, Cauchy and nonlocal multi-point problems for distributed order pseudo-differential equations, Z. Anal. Anwend. 24 (2005) 449–466.