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Quasi-Suslin weak duals

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Abstract

Cascales, Kąkol, and Saxon (CKS) ushered Kaplansky and Valdivia into the grand setting of Cascales/Orihuela spaces E by proving:

(K) *If* E *is countably tight, then so is the weak space* $(E,\sigma(E,E'))$, and
(V) $(E,\sigma(E,E'))$ *is countably tight iff weak dual* $(E',\sigma(E',E))$ *is* K*-analytic*.

The ensuing flow of quasi-Suslin weak duals that are not K-analytic, *a la* Valdivia's example, continues here, where we argue that locally convex spaces E with quasi-Suslin weak duals are (K, V)'s best setting: largest by far, optimal *vis-a-vis* Valdivia. The vaunted CKS setting proves *not* larger, in fact, than Kaplansky's. We refine and exploit the quasi-LB strong dual interplay.

Keywords: K-analytic; Quasi-Suslin; Quasi-LB; Quasibarrelled

1. Introduction

Originally, (K) and (V) had little in common. Kaplansky set (K) in the class $\mathcal{K}$ of locally convex spaces (lcs) whose weak duals are countable unions of compact sets [9, §24, 1(6)]. Decades later, Valdivia set (V) in the class $\mathcal{V}$ of strong duals of Fréchet spaces, omitting tightness [16, p. 66, (24)]. The common setting by CKS [2] in the Cascales–Orihuela class $\mathfrak{G}$ was a major advance measured by the bounty of $\mathfrak{G}\setminus(\mathcal{K}\cup\mathcal{V})$, which contains, e.g., all nonmetrizable (LF)-spaces that are not (DF)-spaces.

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CKS [2, Problem 2] asked "Are there nice classes other than $\mathfrak{G}$ for which (K) holds?" We answer in Theorem 1 with the class $\mathfrak{M}$ of lcs having quasi-Suslin weak duals. Valdivia evokes $\mathfrak{M}$ and CKS essentially proves the theorem. But we must prove that $\mathfrak{M}$ is nice (large) enough to make Theorem 1 appreciably better than its predecessors. It is known that $\mathfrak{M} \supset \mathcal{K}, \mathcal{V}$. To prove that $\mathfrak{M} \supset \mathfrak{G}$ (Corollary 1), we must solve [6, Problem 2]. To see that $\mathfrak{M}$ is substantially larger than $\mathfrak{G}$, we introduce a nice class $\mathfrak{M}_{\mathrm{ac}}$ and show that $\mathfrak{M} \underset{\neq}{\supsetneq} \mathfrak{M}_{\mathrm{ac}} \underset{\neq}{\supsetneq} \mathfrak{G}$.

Note that $\mathfrak{G}$ is not as nice as an understated Kaplansky might suggest (see [2]): our Example 1, a quasibarrelled space in $\mathcal{K} \setminus \mathfrak{G}$, shows that the *original* Kaplansky is *not* corollary to CKS, even if we consider only Mackey spaces.

From the start (1987) it was known that $\mathfrak{M} \neq \mathfrak{G}$: any nonseparable Hilbert space is in $\mathfrak{G}$, $\mathcal{K}$, $\mathfrak{M}$, but with its weak topology is no longer in $\mathfrak{G}$, since the weak unit ball is compact and nonmetrizable. Indeed, $\mathfrak{G}$ lacks the duality invariance of $\mathcal{K}$ and $\mathfrak{M}$ and excludes most weak topologies (Proposition 2).

We refine the study [6] of $\mathfrak{G}$. Although $\mathfrak{M}$ and $\mathfrak{G}$ share subspace stability, $\mathfrak{G}$ is also stable under the taking of countable products and countable direct sums [4, Propositions 4, 5], and $\mathfrak{M}$ likely is not: the argument in [16, p. 55, (4)] fails. We show (Corollary 2) that *uncountable* products are never in $\mathfrak{M}$, hence never in $\mathfrak{G}$. The proof is via the folkloric Lemma 1, which is vital to Example 3, which illustrates superiority of $\mathfrak{M}$ in Theorem 1, which is the main theme of the paper.

2. The best class theorem

Here we review some definitions and prove Theorem 1. We designate and define class $\mathfrak{M}$, the theorem's new setting, and argue that $\mathfrak{M}$ has the following merits:

- It incorporates into Theorem 1 all previous versions of (K) and (V).
- It restores Kaplansky's duality invariance.
- It is optimal for Valdivia's purpose.

Let us consider completely regular Hausdorff topologies only, equip the positive integers $\mathbb{N}$ with the discrete topology, $\mathbb{N}^{\mathbb{N}}$ with the product topology, and for $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$, write $\alpha \leqslant \beta$ to mean that $\alpha(i) \leqslant \beta(i)$ for each $i \in \mathbb{N}$.

A lcs E belongs to Cascales and Orihuela's *class* $\mathfrak{G}$ [4] if there is a family $\{A_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of subsets of its topological dual E' (called a $\mathfrak{G}$-*representation*) such that:

(G1) $E' = \bigcup\{A_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$;
(G2) $A_\alpha \subset A_\beta$ when $\alpha \leqslant \beta$;
(G3) in each A_α, sequences are equicontinuous.

To indicate (G2), we may simply say that $\{A_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is an *ordered family*. A $\mathfrak{G}$-representation is *closed* if every A_α is $\sigma(E', E)$-closed; *bornivorous* if every $\beta(E', E)$-bounded set is contained in some A_α. A $\mathfrak{G}$-*base* for a lcs E is a base $\{U_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of neighborhoods of the origin in E such that $U_\beta \subset U_\alpha$ for $\alpha \leqslant \beta$ in $\mathbb{N}^{\mathbb{N}}$. A large class with good stability properties, $\mathfrak{G}$ contains all (*LF*)-spaces, (*DF*)-spaces, etc., and is a preferred setting for several classic theorems.

In the prequel [6, Example 3] we proved that some lcs admit $\mathfrak{G}$-bases, and thus closed $\mathfrak{G}$-representations, but no bornivorous $\mathfrak{G}$-represention. This solves [6, Problem 1]. Yes, we confess, we overlooked our own answer! As penance we offer Corollary 1, the sacrificial solution to [6, Problem 2] that began this paper.

A topological space X is *quasi-Suslin* (respectively, K-*analytic*) if it admits a *quasi-Suslin* (respectively, K-*analytic*) *map*, i.e., a map T from $\mathbb{N}^{\mathbb{N}}$ into the family of all subsets (respectively, all compact subsets) of X such that

(K1) $\bigcup\{T(\alpha)\colon \alpha \in \mathbb{N}^{\mathbb{N}}\} = X$; and
(K2) if a sequence $\{\alpha_n\}_n$ in $\mathbb{N}^{\mathbb{N}}$ converges to α and $x_n \in T(\alpha_n)$ for all $n \in \mathbb{N}$, then $\{x_n\}_n$ has an adherent point in X contained in $T(\alpha)$.

Variant definitions [16] are reconciled by Rogers [13].

If X is covered by a *sequence* $\{C_n\}_n$ of ⟨countably compact⟩ [compact] sets, then X is ⟨quasi-Suslin⟩ [K-analytic]: a ⟨quasi-Suslin⟩ [K-analytic] map T is given by writing $T(\alpha) = C_{\alpha(1)}$ ($\alpha \in \mathbb{N}^{\mathbb{N}}$). Let $\mathfrak{M}$ (respectively, $\mathfrak{N}$) denote the class of lcs E with quasi-Suslin (respectively, K-analytic) weak dual $(E', \sigma(E', E))$. Clearly,

$$\mathfrak{M} \supset \mathfrak{N} \supset \mathcal{K}. \tag{†}$$

Valdivia [16, pp. 65–67] explicitly proved that $\mathfrak{M} \supset \mathcal{V}$ and $\mathfrak{M} \setminus \mathfrak{N} \neq \emptyset$.

Let us say that a lcs E is an *absolutely convex quasi-Suslin* (acqS) space if it admits an acqS *map*, i.e., a quasi-Suslin map T for which $\{T(\alpha)\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is an ordered family of absolutely convex sets. Let $\mathfrak{M}_{\mathrm{ac}}$ denote the class of lcs whose weak duals are acqS. Obviously, $\mathfrak{M} \supset \mathfrak{M}_{\mathrm{ac}}$.

A topological space X is: *countably tight* if, for each subset A, each closure point of A is a closure point of a countable (possibly finite) subset of A; *realcompact* if X is homeomorphic to a closed subset of a product of reals; *Lindelöf* if every open covering admits a countable subcovering. A lcs E is, respectively, *⟨quasi⟩barrelled* or ℓ^∞-*⟨quasi⟩barrelled* if every $\sigma(E', E)$-bounded ⟨$\beta(E', E)$-bounded⟩ set or sequence is equicontinuous [12]. A lcs E has *property* (C) or (qC) if, respectively, the $\sigma(E', E)$- or $\beta(E', E)$-bounded sequences have adherent points in $(E', \sigma(E', E))$ [10]. Clearly, ⟨quasi⟩barrelled ⇒ ℓ^∞-⟨quasi⟩barrelled ⇒ property (⟨q⟩C).

Let us reset [2, 4.6–4.8] and [6, Theorem 1] from $\mathfrak{G}$ to $\mathfrak{M}$ and $\mathfrak{M}_{\mathrm{ac}}$ in a single theorem. Happily, $\mathfrak{M}$, $\mathfrak{M}_{\mathrm{ac}}$ and properties (C), (qC) match the duality invariance of conditions (a), (b), ..., quite unlike $\mathfrak{G}$ in [2, 4.6], whose proof we follow. We anticipate independently proved results from later sections. Orihuela [11, Example (C) and Theorem 5] proved that each $E \in \mathfrak{M}$ is weakly angelic, so that in $(E, \sigma(E, E'))$, the closure of any relatively countably compact set A is countably tight. The conclusion holds for arbitrary A if and only if E is also in $\mathfrak{N}$ (Theorem 1($\mathbb{A}$)).

Theorem 1. *Consider the following conditions for a* lcs E.

(a) *The weak space* $(E, \sigma(E, E'))$ *is countably tight.*
(b) $(E, \mathcal{T})$ *is countably tight for some* $\langle E, E'\rangle$*-compatible topology* $\mathcal{T}$.
(c) *The weak dual* $(E', \sigma(E', E))$ *is realcompact.*
(d) $E \in \mathfrak{N}$; *i.e., the weak dual* $(E', \sigma(E', E))$ *is* K*-analytic.*
(e) *The finite product* $(E', \sigma(E', E))^n$ *is Lindelöf for every* $n = 1, 2, \ldots$.
(f) *The weak dual* $(E', \sigma(E', E))$ *is Lindelöf.*
(α) *The Mackey space* $(E, \mu(E, E'))$ *is countably tight.*
(β) *The Mackey space* $(E, \mu(E, E'))$ *is barrelled.*
(β') *The Mackey space* $(E, \mu(E, E'))$ *is quasibarrelled.*

($\mathbb{A}$) *If* $E \in \mathfrak{M}$, *then* (a), ..., (f) *are equivalent.*
($\mathbb{B}$) *If* $E \in \mathfrak{M}$ *and* E *has property* (C), *then* (a), ..., (f), (α), (β) *are equivalent.*
($\mathbb{C}$) *If* $E \in \mathfrak{M}_{\mathrm{ac}}$ *and* E *has property* (qC), *then* (a), ..., (f), (α), (β') *are equivalent.*

Proof. ($\mathbb{A}$) Assume $E \in \mathfrak{M}$.

(a) ⇒ (b). Obvious.

(b) ⇒ (c). A simple application of [16, p. 137, (6)]; see [2, 4.6].

(c) ⇒ (d). Theorem 2 covers $(E', \sigma(E', E))$ with an ordered family $\{A_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of countably compact sets, and we proceed exactly as in [2, 4.6].

(d) ⇒ (e). From [16, (9), (12) on pp. 61, 62].

Just as in [2, 4.6], we see that (e) ⇒ (a) from Arkhangel'skii's theorem, and note that [(e) ⇒ (f)] and [(f) ⇒ (c)] are trivial and well known, respectively. Thus (a), ..., (f) are equivalent.

($\mathbb{B}$) In the next three arguments assume E also has property (C).

(c) ⇒ (β). Any absolutely convex $\sigma(E', E)$-closed and bounded set is weakly countably compact by property (C), and then weakly compact by (c), hence equicontinuous under the Mackey topology.

(β) ⇒ (α). Theorem 6 implies $(E, \mu(E, E')) \in \mathfrak{G}$, and (quasi)barrelled spaces in $\mathfrak{G}$ are countably tight by either [2, 4.8] or Corollary 5.

(α) ⇒ (b). Obvious.

($\mathbb{C}$) Assume that $E \in \mathfrak{M}_{\mathrm{ac}}$ and has property (qC).

(c) ⇒ (β'). Every $\beta(E', E)$-bounded set A is contained in the bipolar $A^{\circ\circ}$ which, by properties (qC) and (c), is equicontinuous under the Mackey topology.

(β') ⇒ (α). Apply [Corollary 5, (iii) ⇒ (vii)].

(α) ⇒ (b). Always. □

Part ($\mathbb{A}$) is [2, 4.6], the full CKS version of (V), but with $\mathfrak{M}$ replacing $\mathfrak{G}$, and with (b) added and CKS's (ii), (iii) dropped for brevity's sake. Because $\mathfrak{M}$ is substantially larger than $\mathfrak{G}$ in theory and practice (Remark 1 and Example 3), part ($\mathbb{A}$) substantially improves [2, 4.6].

The same is true for [2, 4.7], i.e., for (K). Indeed, [E is countably tight] ⇒ (b) ⇒ (a), if $E \in \mathfrak{M}$, which proves (K) directly from ($\mathbb{A}$) in the larger setting $\mathfrak{M}$.

Next, consider [2, 4.8], which says that

$$[E \in \mathfrak{G} \text{ and } E \text{ is quasibarrelled}] \quad \Rightarrow \quad [E \text{ is countably tight}].$$

Since the hypothesis implies [$E \in \mathfrak{M}_{\mathrm{ac}}$, E has property (qC), and $E = (E, \mu(E, E'))$], the conclusion is immediate from [($\mathbb{C}$), (β') ⇒ (α)].

Our [6, Theorem 1(I)] says that if E has a bornivorous $\mathfrak{G}$-representation, equiv., if $E \in \mathfrak{G}$ and E is ℓ^∞-quasibarrelled (see Theorem 9), then [(a) ⇔ (α) ⇔ (β')]. This is immediate from ($\mathbb{C}$).

Two-thirds of [6, Theorem 1(II)] says that if E has a bornivorous $\mathfrak{G}$-representation, then

$$[E \text{ is quasibarrelled}] \quad \Leftrightarrow \quad [E \text{ is countably tight}].$$

This follows from [($\mathbb{C}$), (β') ⇔ (α)], since countably tight ℓ^∞-quasibarrelled spaces are Mackey [6, Proposition 4]. One featured application is [6, Corollary 4]: A Fréchet space F is distinguished if and only if its strong dual E is countably tight.

Comparison with the originals is interesting, as well. Valdivia's version of (V) [16, p. 66, (24)] says precisely that

$$\text{If } E \in \mathcal{V}, \quad \text{then } \big[(d) \ \Leftrightarrow \ (\beta)\big]. \tag{VvV}$$

Members of $\mathcal{V}$ are ℓ^∞-barrelled, hence have property (C), so that (VvV) is corollary to [($\mathbb{B}$), (d) ⇔ (β)]. Moreover, the latter result, and not the former, applies to Examples 2, 3, which add to our bounty [6, Section 5] of spaces in $\mathfrak{M} \setminus \mathfrak{N}$. Part ($\mathbb{A}$) optimally suits Valdivia's need to find lcs in $\mathfrak{M} \setminus \mathfrak{N}$, since it tests *all* members of $\mathfrak{M}$ for membership in $\mathfrak{N}$, not just those in $\mathcal{V}$.

Finally, Kaplansky's version of (K) [9, §24, 1(6)] says precisely that

$$\text{If } E \in \mathcal{K}, \quad \text{then (a) holds.} \tag{KvK}$$

From (†), [($\mathbb{A}$), (d) ⇒ (a)] simply says that *if* $E \in \mathfrak{N}$, *then* (a) *holds*, which, again by (†), betters (KvK), and considerably so, for $\mathfrak{N} \setminus \mathcal{K}$ contains, e.g., all nonmetrizable (*LF*)-spaces. (Barrelled spaces in $\mathcal{K}$ are metrizable.)

But (K), itself, does not imply (KvK), because $\mathcal{K} \setminus \mathfrak{G} \neq \emptyset$ (Example 1); also, because $\mathcal{K}$ contains uncountably tight lcs such as G of [6, Proposition 8].

3. Supporting results

Cascales [1, Theorem 2, Proposition 1] proved the following two facts.

Theorem 2 *(Cascales). X is a quasi-Suslin space if and only if X admits a quasi-Suslin map T such that $T(\alpha) \subset T(\beta)$ for all $\alpha \leqslant \beta$.*

Thus every quasi-Suslin space is covered by an ordered family $\{T(\alpha)\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of sets which, by (K2), are countably compact.

We say that $A \subset X$ is *full* if it contains all adherent points in X of sequences in A.

Theorem 3 *(Cascales). X is quasi-Suslin if it is covered by an ordered family $\{A_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of countably compact full subsets.*

For X to be quasi-Suslin, then, it suffices that it be covered by an ordered family of compact sets. For K-analyticity, the condition is necessary, but not sufficient (Talagrand; also, see [6, Example 13]). For E to be in $\mathfrak{M}$, it suffices that E admit a closed $\mathfrak{G}$-representation, since closed sets are full.

Surprisingly, *non*-closed $\mathfrak{G}$-representations also suffice.

Theorem 4. *If (E,τ) is in class $\mathfrak{G}$, its weak dual $(E',\sigma(E',E))$ admits an acqS map T such that $\{T(\alpha)\colon \alpha\in\mathbb{N}^{\mathbb{N}}\}$ is a $\mathfrak{G}$-representation for (E,τ).*

Proof. Let $\{A_\alpha\colon \alpha\in\mathbb{N}^{\mathbb{N}}\}$ satisfy (G1)–(G3). For each $\alpha\in\mathbb{N}^{\mathbb{N}}$ define

$$B_\alpha := \bigcup\left\{S^{\circ\circ}\colon\ S \text{ is a countable subset of } A_\alpha\right\}.$$

Since countable unions of countable sets are countable, each sequence R in B_α is in the bipolar $S^{\circ\circ}$ of a sequence S in A_α that is equicontinuous by (G3). By the Alaoglu–Bourbaki and bipolar theorems, the equicontinuous absolutely convex $S^{\circ\circ}$ is $\sigma(E',E)$-compact. Clearly, then, B_α is absolutely convex and R has weak adherent points, all of which are in $S^{\circ\circ}\subset B_\alpha$; i.e., B_α is absolutely convex and weakly countably compact and full. Moreover, $\{B_\alpha\colon \alpha\in\mathbb{N}^{\mathbb{N}}\}$ satisfies (G1)–(G3).

For each $\alpha\in\mathbb{N}^{\mathbb{N}}$ and each $n\in\mathbb{N}$, define

$$B_{\alpha|n} := \bigcup\left\{B_\beta\colon\ \beta\in\mathbb{N}^{\mathbb{N}} \text{ with } \beta(i)=\alpha(i) \text{ for } 1\leqslant i\leqslant n\right\}.$$

For each $\alpha\in\mathbb{N}^{\mathbb{N}}$, define

$$T(\alpha) := \bigcap_{n\in\mathbb{N}} B_{\alpha|n}.$$

By Cascales' proof of the above Theorem 3, as found in [1, Proposition 1], the map T is quasi-Suslin for $(E',\sigma(E',E))$.

Easily, each $B_{\alpha|n}$, hence $T(\alpha)$, is absolutely convex. Since $\{B_\alpha\colon \alpha\in\mathbb{N}^{\mathbb{N}}\}$ satisfies (G1), (G2), so does $\{T(\alpha)\colon \alpha\in\mathbb{N}^{\mathbb{N}}\}$. Thus T is acqS.

Suppose $\{u_n\}_n$ is a sequence in $T(\alpha)$. For each $n\in\mathbb{N}$, then, $u_n\in B_{\alpha|n}$, which means that

$$u_n\in B_{\beta_n}$$

for some $\beta_n\in\mathbb{N}^{\mathbb{N}}$ with $\beta_n(i)=\alpha(i)$ for $1\leqslant i\leqslant n$. Define $\mu\in\mathbb{N}^{\mathbb{N}}$ such that, for each $i\in\mathbb{N}$,

$$\mu(i) := \max\left\{\beta_n(i)\colon\ n\in\mathbb{N}\right\}.$$

Note that $\mu(i)$ is well defined, the maximum of at most i integers. Clearly $\beta_n\leqslant\mu$, thus $B_{\beta_n}\subset B_\mu$, and thus $u_n\in B_\mu$ for all $n\in\mathbb{N}$. By (G3), $\{u_n\}_n$ is equicontinuous. Therefore $\{T(\alpha)\colon \alpha\in\mathbb{N}^{\mathbb{N}}\}$ also satisfies (G3). □

A stronger topology on E' is useful in [5, Theorem 3].

Theorem 5. *The previous Theorem holds when $\sigma(E',E)$ is replaced by τ_p, the topology of uniform convergence on the precompact sets in (E,τ).*

Proof. Since τ_p and $\sigma(E',E)$ coincide on equicontinuous sets [9, p. 264], one merely repeats the previous proof. □

Corollary 1. $\mathfrak{M}\supset\mathfrak{M}_{\mathrm{ac}}\supset\mathfrak{G}$.

Remark 1. If $E\in\mathfrak{G}$ with $\dim(E')>\aleph_0$, then $(E,\sigma(E,E'))\in\mathfrak{M}_{\mathrm{ac}}\setminus\mathfrak{G}$ by Proposition 2, below. Example 3 is a Mackey space in $\mathfrak{M}_{\mathrm{ac}}\setminus\mathfrak{G}$, and Example 1 is in $\mathfrak{M}\setminus\mathfrak{M}_{\mathrm{ac}}$. Thus $\mathfrak{M}\supsetneqq\mathfrak{M}_{\mathrm{ac}}\supsetneqq\mathfrak{G}$, making $\mathfrak{M}$ substantially larger than $\mathfrak{G}$.

Lemma 1. *If an uncountable set H is covered by an ordered family $\{A_\alpha\colon \alpha\in\mathbb{N}^{\mathbb{N}}\}$, then $H\cap A_\mu$ is infinite for some $\mu\in\mathbb{N}^{\mathbb{N}}$.*

Proof. For all $p, n_1, \dots, n_p \in \mathbb{N}$, set

$$B_{n_1 \dots n_p} := \bigcup \{A_\beta\colon\ \beta \in \mathbb{N}^{\mathbb{N}} \text{ with } \beta(i) = n_i \text{ for } 1 \leqslant i \leqslant p\}.$$

Since $\{B_k\colon\ k \in \mathbb{N}\}$ covers H, there exists $n_1 \in \mathbb{N}$ for which $B_{n_1} \cap H$ is uncountable. Next, note that the countable collection $\{B_{n_1 k}\colon\ k \in \mathbb{N}\}$ covers B_{n_1} and hence covers the ucountable set $B_{n_1} \cap H$. Therefore there is some $n_2 \in \mathbb{N}$ such that $B_{n_1 n_2} \cap H$ is uncountable. Continuing, we fix $(n_1, n_2, \dots) \in \mathbb{N}^{\mathbb{N}}$ such that $B_{n_1 \dots n_p} \cap H$ is uncountable for $p = 1, 2, \dots$. We select a sequence $\{u_p\}_p$ of distinct points with u_p in $B_{n_1 \dots n_p} \cap H$ and find corresponding $\beta_p \in \mathbb{N}^{\mathbb{N}}$ such that, for each $p \in \mathbb{N}$,

$$u_p \in A_{\beta_p} \quad \text{and} \quad \beta_p(i) = n_i \quad \text{for } 1 \leqslant i \leqslant p.$$

As in the proof of Theorem 4, we find an upper bound μ for $\{\beta_p\colon\ p \in \mathbb{N}\}$, so that $\{u_p\}_p \subset A_\mu$. □

Proposition 1. *For E a linear space with uncountable dimension, neither $(E, \sigma(E, E^*))$ nor $(E^*, \sigma(E^*, E))$ is quasi-Suslin.*

Proof. E contains a linearly independent set $\{x_\kappa\colon\ \kappa \in \omega_1\}$, where ω_1 is the set of all countable ordinals. For each $\kappa \in \omega_1$ with $\kappa \geqslant \omega_0$, select a bijection $N_\kappa : \{\iota \in \omega_1\colon\ \iota \leqslant \kappa\} \to \mathbb{N}$. Independence ensures that, for each $\iota \in \omega_1$, there exists $u_\iota \in E^*$ defined however we please at each x_κ. We merely insist that

$$\langle x_\kappa, u_\iota \rangle = N_\kappa(\iota) \quad \text{when } \kappa \geqslant \max(\omega_0, \iota).$$

If S is any countably infinite subset of ω_1 and $\kappa := \sup S$, then

$$\{\langle x_\kappa, u_\iota \rangle = N_\kappa(\iota)\colon\ \iota \in S\}$$

is a set of distinct points necessarily unbounded in $\mathbb{N}$. Therefore $H := \{u_\iota\colon\ \iota \in \omega_1\}$ consists of $\aleph_1$ distinct points, and no infinite subset of H is $\sigma(E^*, E)$-bounded, thus none is contained in a $\sigma(E^*, E)$-countably compact set. In light of Theorem 2 and Lemma 1, then, $(E^*, \sigma(E^*, E))$ cannot be quasi-Suslin.

We may instead insist on the transpose, so that

$$\langle x_\iota, u_\kappa \rangle = N_\kappa(\iota) \quad \text{when } \kappa \geqslant \max(\omega_0, \iota),$$

which similarly shows that $(E, \sigma(E, E^*))$ is not quasi-Suslin. □

In particular, if $\dim(E) > \aleph_0$, then the lcs$(E, \sigma(E, E^*))$ is not in $\mathfrak{M}$, hence not in $\mathfrak{N}$, even though its weak dual is obviously realcompact, which shows that [(c) ⇒ (d)] fails if the hypothesis of $(\mathbb{A})$ is omitted.

Proposition 2. *If E is a* lcs *with* $\dim(E') > \aleph_0$*, then* $(E, \sigma(E, E')) \notin \mathfrak{G}$.

Proof. Otherwise, the completion $(E'^*, \sigma(E'^*, E'))$ of $(E, \sigma(E, E'))$ is also in $\mathfrak{G}$, thus in $\mathfrak{M}$; i.e., $(E', \sigma(E', E'^*))$ is quasi-Suslin, contradicting Proposition 1. □

Clearly, $\mathfrak{M}$, $\mathfrak{M}_{\mathrm{ac}}$, $\mathfrak{N}$ do not preserve completions. Perhaps $\mathfrak{M}$ and $\mathfrak{M}_{\mathrm{ac}}$ also deny finite products, as do countably compact sets, a fact overlooked in [16, pp. 55, 56]. Even $\mathfrak{G}$ rejects *uncountable* products (see [4, Propositions 4, 5]).

Corollary 2. *Let $\{E_\iota\colon\ \iota \in I\}$ be an uncountable collection of nonzero* lcs . *Neither the direct sum $S := \bigoplus_{\iota \in I} E_\iota$ nor the product $P := \prod_{\iota \in I} E_\iota$ is in class $\mathfrak{M}$.*

Proof. Each E_ι contains a 1-dimensional subspace L_ι. Therefore S contains the subspace $M := \bigoplus_{\iota \in I} L_\iota$ and P contains the subspace $N := \prod_{\iota \in I} L_\iota$. The weak duals of M and N are, respectively, $(M^*, \sigma(M^*, M))$ and, by identification, $(M, \sigma(M, M^*))$. By Proposition 1, neither M nor N is in $\mathfrak{M}$. By subspace stability, neither S nor P is in $\mathfrak{M}$. □

Theorem 6. *For an ℓ^∞-barrelled space E we have*: $E \in \mathfrak{G} \Leftrightarrow E \in \mathfrak{M}$.

Proof. Corollary 1 proves one half. When $E \in \mathfrak{M}$, Theorem 2 provides a family of weakly countably compact sets $A_\alpha := T(\alpha)$ satisfying (G1), (G2). But, in any lcs, countably compact sets are bounded. Hence A_α is weakly bounded, and equicontinuity of its sequences follows from the fact that E is ℓ^∞-barrelled; i.e., (G3) holds. □

Proposition 3. *An ℓ^∞-quasibarrelled space E is in $\mathfrak{G}$ if its strong dual is quasi-Suslin.*

Proof. In the previous proof, replace "weak(ly)" and "ℓ^∞-barrelled" with "strong(ly)" and "ℓ^∞-quasibarrelled." □

Example 2 denies the converse, Corollary 6 affirms it for $E = C_p(X)$. By $C_c(X)$ and $C_p(X)$ we mean the continuous function space $C(X)$ endowed with the compact-open and pointwise topologies, respectively.

The ℓ^∞-quasibarrelled spaces in $\mathfrak{G}$ later prove to be the class $\mathcal{F}$ defined by Cascales and Orihuela in terms of the strong dual [4, p. 371]. For the moment, we characterize $\mathcal{F}$ in terms of the weak dual.

Theorem 7. *For an ℓ^∞-quasibarrelled space E we have: $E \in \mathfrak{G} \Leftrightarrow E \in \mathfrak{M}_{\mathrm{ac}}$.*

Proof. Theorem 4 proves one part. If $E \in \mathfrak{M}_{\mathrm{ac}}$, then $(E', \sigma(E', E))$ is covered by an ordered family $\{T(\alpha)\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of absolutely convex countably compact sets. The Banach–Mackey theorem ensures that $T(\alpha)$ is strongly bounded, so that, by ℓ^∞-quasibarrelledness, every sequence in $T(\alpha)$ is equicontinuous; i.e., $\{T(\alpha)\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ satisfies (G1)–(G3). □

4. Quasi-LB strong duals

A *Banach disk* in a lcs F is an absolutely convex set B, bounded in F, whose generated normed space F_B is complete. Valdivia [15] defined a *quasi-LB representation* of a lcs F to be a family $\{B_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of Banach disks in F satisfying

(Q1) $F = \bigcup\{B_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ and
(Q2) $B_\alpha \subset B_\beta$ when $\alpha \leqslant \beta$.

If F admits a quasi-LB representation, it is a *quasi-LB space*.

Theorem 8. *Strong duals of spaces in $\mathfrak{M}_{\mathrm{ac}}$ are quasi-LB spaces. In fact, if T is any acqS map for the weak dual $(E', \sigma(E', E))$, then $\{T(\alpha)\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a quasi-LB representation of the strong dual $(E', \beta(E', E))$.*

Proof. Each $T(\alpha)$ is weakly countably compact, hence a weak Banach disk by [12, 3.2.5]. By the Banach–Mackey theorem, $T(\alpha)$ is also strongly bounded. □

To the list in [8] we may add

Corollary 3. *$C_c(X)$ is a df-space if and only if it belongs to class $\mathfrak{G}$ and its strong dual is a Baire space.*

Proof. If $C_c(X)$ is a df-space, then by a theorem of Buchwalter and Schmets, $C_c(X)$ is ℓ^∞-quasibarrelled (see [8, Corollary 3.3]), and in class $\mathfrak{G}$ (see [6, Example 2(D′)]). Moreover, the strong dual is a Fréchet space [8], and thus Baire.

Conversely, if $C_c(X)$ is in $\mathfrak{G}$ and the strong dual is Baire, it is also quasi-LB (Theorem 8), therefore Fréchet [15, Corollary 1.6], which implies that $C_c(X)$ is a df-space [8]. □

Bornivorous $\mathfrak{G}$-representations are central to [6]. We proved there that the class of spaces for which they exist lies somewhere between the quasibarrelled spaces in $\mathfrak{G}$ and the ℓ^∞-quasibarrelled spaces in $\mathfrak{G}$. We now show that it is simply the latter class.

Theorem 9. *The following five assertions are equivalent for a* lcs *E.*

(1) *E admits a bornivorous $\mathfrak{G}$-representation.*
(2) *E is ℓ^∞-quasibarrelled and $E \in \mathfrak{G}$.*
(3) *E is ℓ^∞-quasibarrelled and $E \in \mathfrak{M}_{\text{ac}}$.*
(4) *E is ℓ^∞-quasibarrelled and its strong dual E' is a quasi-LB space.*
(5) *E is ℓ^∞-quasibarrelled and its strong bidual E'' admits a $\mathfrak{G}$-base.*

Proof. Throughout the argument it is assumed that $\beta(E', E)$ is the topology on E', and that E'' is the dual of E' endowed with the $\beta(E'', E')$ topology.

(1) $\Rightarrow$ (2). This follows from the definitions.

(2) $\Leftrightarrow$ (3). Theorem 7.

(3) $\Rightarrow$ (4). Theorem 8.

(4) $\Rightarrow$ (5). Since E' is a quasi-LB space, Valdivia's [15, Proposition 2.2] provides a quasi-LB representation $\{B_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ with the additional property that every Banach disk in E' is contained in some B_α. Since E' is sequentially complete [12, 8.2.15(ii)], every bounded set in E' is contained in a Banach disk, and thus is contained in some B_α; i.e., $\{B_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a fundamental family of bounded sets in E'. With U_α set equal to the polar of B_α in E'', it is now clear that $\{U_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a $\mathfrak{G}$-base for E''.

(5) $\Rightarrow$ (1). Let $\{U_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a $\mathfrak{G}$-base for E'', and let A_α be the polar of U_α in E'. Clearly, (G1) and (G2) hold. Since U_α is a neighborhood of zero in the strong bidual, A_α is bounded in the strong dual. Therefore sequences in A_α are equicontinuous, since E is ℓ^∞-quasibarrelled; i.e., (G3) holds. Since every neighborhood of zero in E'' contains some U_α, every bounded set in E' is contained in some A_α. Therefore $\{A_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a bornivorous $\mathfrak{G}$-representation. □

Theorem 9 adds nicely to the material in [6, Proposition 2, Examples 1, 2]. Example 2 shows that (1)–(5) $\not\Rightarrow$ [E is ℓ^∞-quasibarrelled and its *strong* dual is quasi-Suslin], although the reverse implication holds by Proposition 3.

In the next three corollaries we progress from ℓ^∞-quasibarrelled spaces to quasibarrelled spaces to those of the form $C_p(X)$ and note an increasingly rich array of properties that characterize membership in $\mathfrak{G}$, including, in the last case, having a quasi-Suslin *strong* dual.

Corollary 4. *For an ℓ^∞-quasibarrelled space E, the following are equivalent.*

(i) $E \in \mathfrak{G}$.
(ii) *E admits a bornivorous $\mathfrak{G}$-representation.*
(iii) $E \in \mathfrak{M}_{\text{ac}}$.
(iv) *The strong dual E' is a quasi-LB space.*
(v) *The strong bidual E'' admits a $\mathfrak{G}$-base.*

Corollary 5. *For E quasibarrelled,* (i)–(v) *are equivalent to these conditions*:

(vi) *E admits a $\mathfrak{G}$-base.*
(vii) *$E \in \mathfrak{M}_{\text{ac}}$ and E is countably tight.*

Proof. When E is quasibarrelled, it is embedded in the strong bidual E'', so that (v) $\Rightarrow$ (vi).

Suppose E admits a $\mathfrak{G}$-base $\{U_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$. For each $\alpha \in \mathbb{N}^{\mathbb{N}}$ and each $n \in \mathbb{N}$, define

$$D_{\alpha|n} := \bigcap \bigl\{U_\beta^{\circ\circ}\colon \beta(i) = \alpha(i) \text{ for } 1 \leqslant i \leqslant n\bigr\}.$$

Let α be given and suppose there is a bounded set B with $B \not\subset nD_{\alpha|n}$ for $n = 1, 2, \ldots$. Choose $x_n \in B$ such that $n^{-1}x_n \notin D_{\alpha|n}$, choose $\beta^{(n)} \in \mathbb{N}^{\mathbb{N}}$ such that $\beta^{(n)}(i) = \alpha(i)$ for $1 \leqslant i \leqslant n$ and $n^{-1}x_n \notin U_{\beta^{(n)}}$, and set $\mu = \sup_n \beta^{(n)}$ as before. Clearly, the 0-neighborhood U_μ misses the null sequence $\{n^{-1}x_n\}_n$, a contradiction. Therefore $V_\alpha := \bigcup_n D_{\alpha|n}$ absorbs all bounded sets B. In fact, the bornivorous barrel $\overline{V}_\alpha$ is a neighborhood of the origin satisfying

$$\overline{V}_\alpha \subset (1+\varepsilon)V_\alpha \subset (1+\varepsilon)U_a^{\circ\circ}$$

for all $\varepsilon > 0$, according to [12, 8.2.27] and the definition of V_α. Hence $\{V_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a base of neighborhoods of the origin. If A is a set whose closure contains the origin, then from each of the countably many distinct nonempty intersections $D_{\alpha|n} \cap A$ choose one point. The aggregate is a countable subset of A whose closure contains the origin, thereby proving E is countably tight. Clearly, $\{U_\alpha^\circ\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ witnesses that $E \in \mathfrak{G} \subset \mathfrak{M}_{\mathrm{ac}}$. Therefore (vi) $\Rightarrow$ (vii).

Trivially, (vii) $\Rightarrow$ (iii). □

Remark 2. The countably tight argument refines earlier versions in [2,3,6]. The two corollaries efficiently contain three-fourths of, and significantly improve one-half of [3, Lemma 2]. Several times over, they conveniently describe the two Cascales/Orihuela classes $\mathcal{F}$ and $\mathcal{F}'$ [4, Corollary 2.2(i), (ii)]; most simply, as the quasibarrelled and ℓ^∞-quasibarrelled spaces in $\mathfrak{G}$, respectively.

Corollary 6. *For $E = C_p(X)$,* (i)–(vii) *are equivalent to* (viii)–(xi)*, below.*

(viii) *X is countable.*
(ix) *$C_p(X)$ is metrizable.*
(x) *The strong dual E' of $C_p(X)$ is quasi-Suslin.*
(xi) *The strong dual E' of $C_p(X)$ is K-analytic.*

Proof. Since $C_p(X)$ is always quasibarrelled [7], (i)–(vii) are equivalent for all choices of the Tichonov space X. Obviously, (viii) $\Rightarrow$ (ix) $\Rightarrow$ (i). We have (i) $\Rightarrow$ (viii) via Proposition 2, since $E = C_p(X)$ has its weak topology and $\dim(E') = |X|$. Thus (i)–(ix) are equivalent.

(viii) $\Rightarrow$ (xi). If $|X| = \dim(E') \leqslant \aleph_0$, then E' is covered by an increasing sequence $\{C_n\}_n$ of finite-dimensional compact sets.

(xi) $\Rightarrow$ (x). Obvious.

(x) $\Rightarrow$ (i). Proposition 3 applies. □

Example 1. Let X be the union of an increasing sequence $\{K_n\}_n$ of uncountable compact sets, and put $E := C_p(X)$.

- $E \notin \mathfrak{G}$, because $\neg$ (viii) $\Rightarrow \neg$(i).
- E is quasibarrelled [7], therefore Mackey.
- $E \in \mathcal{K} \setminus \mathfrak{M}_{\mathrm{ac}}$. From Theorem 7, $E \notin \mathfrak{M}_{\mathrm{ac}}$. Identifying points of X with their evaluation maps, we may think of X as a Hamel basis for E'. Since points of E are continuous, X inherits a coarser topology from $(E', \sigma(E', E))$. Hence K_n is $\sigma(E', E)$-compact. Define
$$C_n := \left\{ \sum_{i=1}^n a_i x_i \in E'\colon \sum_{i=1}^n |a_i| \leqslant n \text{ and } x_1, \ldots, x_n \in K_n \right\}.$$
By continuity of vector operations in $(E', \sigma(E', E))$, it is clear that C_n is a continuous image of the product $\Lambda_n \times K_n^n$ for an appropriate compact set Λ_n in $\mathbb{R}^n$, and thus C_n is $\sigma(E', E)$-compact. As $\{K_n\}_n$ covers X, so $\{C_n\}_n$ covers E'; i.e., $E \in \mathcal{K}$.
- $E \in \mathfrak{M} \setminus \mathfrak{M}_{\mathrm{ac}}$, because $\mathfrak{M} \supset \mathfrak{N} \supset \mathcal{K}$.

5. Two examples in $\mathfrak{M} \setminus \mathfrak{N}$

Let Λ be an indexing set of size $\mathfrak{c}$ (the continuum). The Banach space $\ell^1(\Lambda)$ has unit ball B, say, and strong dual $\ell^\infty(\Lambda)$ with unit ball $U = B^\circ = I^\Lambda$, where $I := \{c\colon c \text{ is a scalar with } |c| \leqslant 1\}$. We define
$$\ell^\infty_{cs}(\Lambda) := \{u \in \ell^\infty(\Lambda)\colon u \text{ vanishes outside a countable subset of } \Lambda\}$$
and set $U_{cs} := U \cap \ell^\infty_{cs}(\Lambda)$.

Example 2. Let G be the linear space $\ell^1(\Lambda)$, let $G' = \ell^\infty_{cs}(\Lambda)$, and give G the Mackey topology $\mu(G, G')$.

- G and $\ell^1(\Lambda)$ have the same bounded sets, since $c_0(\Lambda) \subset G'$.

- The strong dual of G is a Banach space with unit ball U_{cs}.
- G admits a bornivorous $\mathfrak{G}$-representation. Indeed, $A_\alpha := \alpha(1) \cdot U_{cs}$ easily satisfies the definition.
- The strong dual of G is not quasi-Suslin. It is metrizable and not separable, hence not Lindelöf, hence not quasi-Suslin [16, (12), (26), pp. 62, 67].
- $G \in \mathfrak{M}_{\mathrm{ac}}$. By Corollary 1. Or, more directly, $T(\alpha) := \alpha(1) \cdot U_{cs}$ defines an acqS map T for the weak dual.
- $G \notin \mathfrak{N}$. Since G has property (C) and is not quasibarrelled ($G \neq \ell^1(\Lambda)$), apply Theorem 1, part ($\mathbb{B}$) or ($\mathbb{C}$). Or use [($\mathbb{A}$), (d) $\Rightarrow$ (a)]: The origin is in the weak closure of the set A of all canonical unit vectors in G, and this is not the case for any countable subset of A.

Adding an idea from [14], we shall obtain a Mackey space F in $\mathfrak{M}_{\mathrm{ac}} \setminus (\mathfrak{N} \cup \mathfrak{G})$ whose analysis, therefore, can only rest on Theorem 1, not its predecessor [2, 4.6].

Since $\ell^1(\Lambda)$ is a Banach space, every weakly bounded sequence in the dual $\ell^\infty(\Lambda)$ has a weak adherent point. Observe that the relative $\sigma(\ell^\infty(\Lambda), \ell^1(\Lambda))$-topology coincides with the product topology on $U = I^\Lambda$; from now on, this will be the assumed topology on U. Let $\mathcal{W}$ denote the set of all countably infinite subsets of ω_1. Note that $|\mathcal{W}| = \mathfrak{c}$. We use $\mathcal{W}$ to index a partition $\{\Lambda_W\colon\ W \in \mathcal{W}\}$ of Λ into $\mathfrak{c}$ pairwise disjoint sets with each $|\Lambda_W| = \mathfrak{c}$. In choosing a set $\{u_\theta\colon\ \theta \in \omega_1\} \subset U$, we may demand, independently for each of the disjoint sets Λ_W, that the countable set of restrictions $\{u_\theta|_{\Lambda_W}\colon\ \theta \in W\}$ be any arbitrary countable subset of the uncountable product I^{Λ_W}. Since separability is $\mathfrak{c}$-multiplicative, then, there exists $\{u_\theta \in U\colon\ \theta \in \omega_1\}$ such that, for each $W \in \mathcal{W}$,

$$\text{the countable set of restrictions}\quad \{u_\theta|_{\Lambda_W}\colon\ \theta \in W\}\quad \text{is dense in the uncountable separable product } I^{\Lambda_W}. \qquad (\star)$$

Let H_0 denote the linear span of $U_{cs} \bigcup \{u_\theta\colon\ \theta \in \omega_1\}$. One easily sees that $|H_0| = \mathfrak{c}$. Inducting on the well-ordered set ω_1, we choose a family $\{H_\theta\colon\ \theta \in \omega_1\}$ of linear subspaces of $\ell^\infty(\Lambda)$ such that

each H_θ has size $\mathfrak{c}$ and contains at least one weak adherent pointof every $\sigma(\ell^\infty(\Lambda), \ell^1(\Lambda))$-bounded sequence in H_ι, for all $\iota < \theta$.

Indeed, suppose $0 < \delta \in \omega_1$ and we are given $\{H_\theta\colon\ 0 \leqslant \theta < \delta\}$ having the desired property, with H_0 fixed as above. The set $S := \bigcup\{H_\theta\colon\ 0 \leqslant \theta < \delta\}$ is a countable union of sets of size $\mathfrak{c}$, and hence has size $\mathfrak{c}$, as does, then, the collection of all sequences from S. We constitute R by choosing a weak adherent point in $\ell^\infty(\Lambda)$ for each weakly bounded sequence in S, and let H_δ be the linear span of R to complete the induction.

By considering constant sequences from H_ι, it is clear that $H_\iota \subset H_\theta$ whenever $\iota < \theta$, and therefore

$$H := \bigcup \{H_\theta\colon\ \theta \in \omega_1\}$$

is a linear subspace of $\ell^\infty(\Lambda)$ of size $\mathfrak{c} \cdot \aleph_1 = \mathfrak{c}$.

Example 3. Let F be the linear space $\ell^1(\Lambda)$, set $F' = H$, and give F the Mackey topology $\mu(F, F')$.

(I) *F and the Banach space $\ell^1(\Lambda)$ have the same bounded sets*, since $F' \supset c_0(\Lambda)$.

(II) *The strong dual of F is a Banach space with unit ball $V := U \cap F'$*. Indeed,

(III) *V is weakly countably compact*. Any sequence in V is contained in some H_θ, and has a weak adherent point in $H_{\theta+1} \cap V$.

(IV) *The strong dual of F is a quasi-LB space*. Take $B_\alpha = \alpha(1) \cdot V$.

(V) $F \in \mathfrak{M}_{\mathrm{ac}}$. Take $T(\alpha) = \alpha(1) \cdot V$.

(VI) *F is a Mackey space with property (C)*. The Banach–Steinhaus theorem puts each weakly bounded sequence in F' inside a positive multiple nV of V, and each nV is weakly countably compact (III).

(VII) $F \notin \mathfrak{G}$. Otherwise, Lemma 1 provides $W \in \mathcal{W}$ such that $\{u_\theta\colon\ \theta \in W\}$ is equicontinuous, and thus its bipolar K is a closed subset of the compact U and of the weak dual F'. Let x be an arbitrary element of I^{Λ_W}. By ($\star$), there is a net $\mathcal{N}$ in $\{u_\theta\colon\ \theta \in W\}$ whose net $\mathcal{R}$ of restrictions to Λ_W converges to x. As does every net in K, the net $\mathcal{N}$ has an adherent point $u^{(x)}$ in K. Moreover, since $\mathcal{R}$ converges to x, it is clear that $u^{(x)}|_{\Lambda_W} = x$. We have shown that $\{u|_{\Lambda_W}\colon\ u \in K\} = I^{\Lambda_W}$, a set of size $2^{\mathfrak{c}}$. Therefore

$$2^{\mathfrak{c}} \leqslant |K| \leqslant |F'| = \mathfrak{c},$$

a contradiction.

(VIII) *F is not* ℓ^∞*-quasibarrelled*. Theorem 7.

(IX) $F \notin \mathfrak{N}$. Otherwise, F is barrelled (Theorem 1), which contradicts (VIII).

Note that Examples 1–3 and Theorems 1, 4, 6 prove classes $\mathfrak{M}$, $\mathfrak{M}_{\mathrm{ac}}$, $\mathfrak{G}$, $\mathfrak{N}$ are distinct, yet identical within the purview of barrelled spaces.

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