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Analysis of piecewise linear approximations to the generalized Stokes problem in the velocity–stress–pressure formulation ☆

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Abstract

In this paper we study continuous piecewise linear polynomial approximations to the generalized Stokes equations in the velocity–stress–pressure first-order system formulation by using a cell vertex finite volume/least-squares scheme. This method is composed of a direct cell vertex finite volume discretization step and an algebraic least-squares step, where the least-squares procedure is applied after the discretization process is accomplished. This combined approach has the advantages of both finite volume and least-squares approaches. An error estimate in the H^1 product norm for continuous piecewise linear approximating functions is derived. It is shown that, with respect to the order of approximation for H^2-regular exact solutions, the method exhibits an optimal rate of convergence in the H^1 norm for all unknowns, velocity, stress, and pressure. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Continuous piecewise linear polynomials may be the most natural and simple approximating functions in the numerical schemes, especially for large-scale computations, for solving partial differential equations. This paper deals with the problem of continuous piecewise linear polynomial approximations to the generalized Stokes problem. We consider the generalized Stokes equations supplemented

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with the homogeneous Dirichlet velocity boundary condition in the velocity–stress–pressure first-order system formulation by using a cell vertex finite volume/least-squares numerical scheme. It is shown that, with respect to the order of approximation for H^2-regular exact solutions, this combined scheme exhibits an optimal rate of convergence in the H^1 norm for all unknowns (velocity, stress, and pressure).

This study is motivated by some significant observations from the numerical experiments of least-squares finite element methods [8]. Over the past decade, least-squares finite element methods have become increasingly popular for the approximate solution of first-order systems of partial differential equations; a small sample of the recent literature is given in [3]. The specific features of the least-squares finite element approach that make it potentially advantageous compared with, e.g., mixed finite element methods [4,15] are as follows:

- It leads to a minimization problem; it is not subject to the Ladyzhenskaya–Babuska–Brezzi condition (cf. [4,15]).
- A single continuous piecewise polynomial space can be used for the approximation of all unknowns.
- The resulting linear system is symmetric and positive definite with condition number $O(h^{-2})$, where h denotes the mesh size.
- The value of the homogeneous least-squares functional of the approximate solution provides a practical and sharp a posteriori error estimator at no additional cost.

Although the least-squares finite element approach exhibits many advantageous features, the practicality of this approach is still not fully documented due to lack of study of the behavior of the approach in the presence of difficulties arising from, for example, the application to convection-dominated problems, the need to conserve some global physical quantity such as mass (mass conservation is one of major concerns of users of computational fluid dynamic algorithms). The latter issue is the main subject of the investigation [8]. In [8], it was reported that least-squares finite element methods do a very poor job at conserving mass, and then a remedy was proposed. Unfortunately, this remedy loses the positive definiteness resulting from the least-squares formulation and is led to indefinite problems similar to those that arise in mixed finite element methods. See [8] for the numerical experiments.

On the other hand, in recent years finite volume methods have been widely and successfully used for the numerical solution of conservation laws (cf. [7,10–12,17–19]). The popularity of this class of schemes stems from their structural simplicity and the presence of conservation properties inherited from the differential equations. Depending on the location of the points in the computational cells where the unknowns are kept, the most common finite volume methods can be classified as cell vertex, cell center, or vertex-based methods. In [7], a combined cell vertex finite volume/least squares for first-order elliptic systems without the "reaction" term in the plane has been proposed and analyzed. This method is composed of a cell vertex finite volume discretization step and an algebraic least-squares step, where the least-squares procedure is applied after the discretization process is effected. Hence, this combined approach has the advantages of both finite volume and least-squares finite element methods.

The purpose of this paper is to extend this methodology to the generalized stationary Stokes equations with the homogeneous Dirichlet velocity boundary condition in two- and three-dimensional bounded domains. Aiming to develop stabilized methods for viscoelastic flows, people usually first treat Newtonian flows formulated in terms of velocity, stress, and pressure. In the recent literature, one can find a number of methods proposed to solve the equations in terms of these variables (see, e.g. [1,2,6,9,16,20,21]). Among the possible velocity–stress–pressure Stokes formulations, we follow here the one introduced in the work of Cai et al. [6]. Introducing the velocity flux variable (we call stress here), defined as the vector of gradients of the Stokes velocities, we can recast the generalized Stokes problem into an equivalent first-order system formulation. The combined cell vertex finite volume/least-squares method is then applied and analyzed. It can be shown that the combined approach achieves an optimal rate of convergence in the H^1 product norm for piecewise linear approximating functions for all unknowns, velocity, stress, and pressure.

The paper is organized as follows. In Section 2, we briefly review the velocity–stress–pressure formulation for the generalized Stokes equations and an a priori estimate for the first-order system. In Section 3, the cell vertex finite volume/least-squares scheme is described. In Section 4, an error estimate in the H^1 product norm for continuous piecewise linear approximating functions is derived. In Section 5, an H^1-equivalent a posteriori error estimator is discussed. Finally, in Section 6, some concluding remarks are given.

2. Problem formulation

Let Ω be a bounded, open, and connected domain in $\mathbf{R}^d$, $d = 2$ or 3, with Lipschitz boundary $\partial\Omega$. Let $\mathbf{f} = (f_1, \ldots, f_d)^{\mathrm{T}} \in [L^2(\Omega)]^d$ be a given vector function representing the density of body force. The stationary incompressible Navier–Stokes equations supplemented with the homogeneous Dirichlet velocity boundary condition can be posed as [14,15]

$$\begin{aligned} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0 \quad & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} \quad & \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where the symbols Δ, ∇, and $\nabla\cdot$ stand for the Laplacian, gradient, and divergence operators, respectively; $\mathbf{u} = (u_1, \ldots, u_d)^{\mathrm{T}}$ is the velocity; p is the pressure satisfying the zero mean condition $\int_\Omega p \, \mathrm{d}\Omega = 0$; $0 < \nu \leqslant 1$ is the viscosity constant. All of variables are assumed to be nondimensionalized.

According to the theory of Brezzi–Rappaz–Raviart [5], the formulation and analysis of discretization methods for the Stokes problem are critical for the understanding of similar methods for the Navier–Stokes problem. Thus, we restrict our attention to the Stokes flows. Neglecting the nonlinear term in the above system, we have

$$\begin{aligned} -\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0 \quad & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} \quad & \text{on } \partial\Omega. \end{aligned} \tag{2.2}$$

Further, for the sake of generality, we consider the following generalized Stokes equations supplemented with the homogeneous Dirichlet velocity boundary condition

$$\begin{aligned} -\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} + \delta p = g \quad & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} \quad & \text{on } \partial\Omega, \end{aligned} \tag{2.3}$$

where $g \in H^1(\Omega)$ is a given scalar function with $\int_\Omega g \, d\Omega = 0$; δ is a fixed nonnegative constant bounded uniformly in ν. When $\delta = 0$, $g = 0$ we have the Stokes problem, and the case of $\delta > 0$ arises from the linear elasticity problem. See Remark 2.1 below for the details.

When $d = 2$, we define the curl operator "$\nabla \times$" for smooth two-component vector function $\mathbf{v} = (v_1, v_2)^{\mathrm{T}}$ by

$$\nabla \times \mathbf{v} = \partial_1 v_2 - \partial_2 v_1.$$

When $d = 3$, we define the curl of a smooth three-component vector function $\mathbf{w} = (w_1, w_2, w_3)^{\mathrm{T}}$ by

$$\nabla \times \mathbf{w} = (\partial_2 w_3 - \partial_3 w_2, \partial_3 w_1 - \partial_1 w_3, \partial_1 w_2 - \partial_2 w_1)^{\mathrm{T}}.$$

Now, introducing the auxiliary $d \times d$ new independent variables

$$\mathbf{U} = \nabla \mathbf{u}^{\mathrm{T}} = (\nabla u_1, \ldots, \nabla u_d)$$

called stresses here, we can transform (2.3) into the following so-called velocity–stress–pressure first-order system formulation

$$\begin{aligned} -\nu(\nabla \cdot \mathbf{U})^{\mathrm{T}} + \nabla p = \mathbf{f} \quad & \text{in } \Omega, \\ \mathbf{U} - \nabla \mathbf{u}^{\mathrm{T}} = \mathbf{0} \quad & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} + \delta p = g \quad & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} \quad & \text{on } \partial\Omega. \end{aligned} \tag{2.4}$$

Note that the definition of $\mathbf{U}$, the "continuity" equation $\nabla \cdot \mathbf{u} + \delta p = g$ in Ω, and the Dirichlet boundary condition $\mathbf{u} = \mathbf{0}$ on $\partial\Omega$ imply the respective properties

$$\begin{aligned} \nabla \times \mathbf{U} = \mathbf{0} \quad & \text{in } \Omega, \\ \mathrm{tr}\mathbf{U} + \delta p = g \quad & \text{in } \Omega, \\ \mathbf{n} \times \mathbf{U} = \mathbf{0} \quad & \text{on } \partial\Omega, \end{aligned} \tag{2.5}$$

where the trace operator "tr" is defined as $\mathrm{tr}\mathbf{U} = \mathbf{U}_{11} + \cdots + \mathbf{U}_{dd}$. Thus, an equivalent extended system for (2.4) is given by

$$\begin{aligned} -\nu(\nabla \cdot \mathbf{U})^{\mathrm{T}} + \nabla p = \mathbf{f} \quad & \text{in } \Omega, \\ \mathbf{U} - \nabla \mathbf{u}^{\mathrm{T}} = \mathbf{0} \quad & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} + \delta p = g \quad & \text{in } \Omega, \\ \nabla \mathrm{tr}\mathbf{U} + \delta \nabla p = \nabla g \quad & \text{in } \Omega, \\ \nabla \times \mathbf{U} = \mathbf{0} \quad & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} \quad & \text{on } \partial\Omega, \\ \mathbf{n} \times \mathbf{U} = \mathbf{0} \quad & \text{on } \partial\Omega. \end{aligned} \tag{2.6}$$

Refer to Table 1 for the numbers of unknowns, equations, and boundary conditions of problem (2.6) in different dimensions.

Table 1
Generalized Stokes system (2.6)

Dim.	No. of unknowns	No. of equations	No. of boundary conditions
	d^2+d+1	$3d^2-d+1$	$2d^2-2d$
$d=2$	7	11	4
$d=3$	13	25	12

Remark 2.1. An important aspect of the pressure-perturbed form of the generalized Stokes problem (2.3) is that it allows our results to apply to the Dirichlet problem for the linear elasticity equations. In particular, consider the following Dirichlet problem

$$\begin{aligned} -\mu\Delta\mathbf{u} - (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) &= \mathbf{f} \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \partial\Omega, \end{aligned}$$

where $\mathbf{u}$ now represents displacements and μ, λ are the positive Lamé constants. This system can be recast in form (2.3) by introducing the artificial pressure variable

$$p = -\frac{\lambda}{2\mu}(\nabla\cdot\mathbf{u})$$

by rescaling $\mathbf{f}$, and by letting $g=0$, $\delta = 2\mu/\lambda$, and $\nu = \lambda/(2(\lambda+\mu))$.

We use the standard notation and definition for the Sobolev spaces $H^m(\Omega)$, $m \geqslant 0$, with inner products $(\cdot,\cdot)_{m,\Omega}$ and norms $\|\cdot\|_{m,\Omega}$. As usual, $L^2(\Omega) = H^0(\Omega)$, and $L^2_0(\Omega)$ denotes the subspace of $L^2(\Omega)$ that consists of square integrable functions with zero mean. Let $[H^m(\Omega)]^d$ denote the corresponding product spaces, and the inner products and norms will be still denoted by $(\cdot,\cdot)_{m,\Omega}$ and $\|\cdot\|_{m,\Omega}$, respectively, when there is no chance of confusion. We will be interested in the following three function spaces with respect to the unknown functions, velocity $\mathbf{u}$, stress $\mathbf{U}$, and pressure p

$$\mathscr{V} = \{\mathbf{v}:\ \mathbf{v}\in[H^1(\Omega)]^d \text{ and } \mathbf{v}=\mathbf{0} \text{ on } \partial\Omega\}, \tag{2.7}$$

$$\mathscr{S} = \{\mathbf{V}:\ \mathbf{V}\in[H^1(\Omega)]^{d^2} \text{and } \mathbf{n}\times\mathbf{V}=\mathbf{0} \text{ on } \partial\Omega\}, \tag{2.8}$$

$$\mathscr{Q} = \{q:\ q\in H^1(\Omega) \text{ and } (q,1)_{0,\Omega}=0\}. \tag{2.9}$$

The existence, uniqueness and smoothness of the weak solution $(\mathbf{u},p)$ to problem (2.3) are well-known. Problem (2.3) has a unique weak solution $(\mathbf{u},p)\in[H^1_0(\Omega)]^d\times L^2_0(\Omega)$ for any $\mathbf{f}\in[L^2(\Omega)]^d$ and $g\in H^1(\Omega)$ (see, e.g., [4,14,15]). Moreover, if the boundary $\partial\Omega$ of the domain Ω is $C^{1,1}$, then the following H^2 regularity result holds

$$\nu\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega} \leqslant C(\|\mathbf{f}\|_{0,\Omega} + \nu\|g\|_{1,\Omega}). \tag{2.10}$$

When the domain Ω is a convex polyhedron, a priori estimate (2.10) is still valid when $\delta > 0$. For the case $\delta = 0$, one has only a weaker estimate (cf. [6]). Throughout this paper, in any estimate or inequality, the quantity C with or without subscripts will denote a generic positive constant always independent of ν and the mesh parameter h, which will be introduced later, and need not be the same constant in different occurrences.

We conclude this section with the following a priori estimate for the first-order system (2.6) that will play a crucial role in the error estimates for our continuous piecewise linear approximation scheme. The proof can be found in [6]. Define the notation

$$\mathscr{K}((\mathbf{v},\mathbf{V},q)) \equiv \| -\nu(\nabla\cdot\mathbf{V})^{\mathrm{T}} + \nabla q\|_{0,\Omega}^2 + \nu^2\|\mathbf{V} - \nabla\mathbf{v}^{\mathrm{T}}\|_{0,\Omega}^2 + \nu^2\|\nabla\cdot\mathbf{v} + \delta q\|_{0,\Omega}^2 + \nu^2\|\nabla \mathrm{tr}\mathbf{V} + \delta\nabla q\|_{0,\Omega}^2 + \nu^2\|\nabla\times\mathbf{V}\|_{0,\Omega}^2.$$

Theorem 2.1. *Assume that the domain Ω is a bounded convex polyhedron or has $C^{1,1}$ boundary $\partial\Omega$ and that regularity estimate* (2.10) *holds. Then there exists a positive constant C independent of ν such that for any $\mathbf{v}\in\mathscr{V}$, $\mathbf{V}\in\mathscr{S}$, and $q\in\mathscr{Q}$, we have*

$$\frac{1}{C}(\nu^2\|\mathbf{v}\|_{1,\Omega}^2 + \nu^2\|\mathbf{V}\|_{1,\Omega}^2 + \|q\|_{1,\Omega}^2) \leqslant \mathscr{K}((\mathbf{v},\mathbf{V},q)) \leqslant C(\nu^2\|\mathbf{v}\|_{1,\Omega}^2 + \nu^2\|\mathbf{V}\|_{1,\Omega}^2 + \|q\|_{1,\Omega}^2). \tag{2.11}$$

Throughout the remainder of this paper, we will always assume that the conditions in Theorem 2.1 hold such that we have the a priori estimate (2.11).

3. The cell vertex finite volume scheme

For finite volume approximation, there are generally a pair of discretizations of the domain Ω, a primal partition and a dual partition. However, these two discretizations of Ω are the same in the cell vertex finite volume computations. Let $\{\mathscr{T}_h\}$ be a family of regular triangulations of the domain Ω (cf. [4,13]), where

$$\mathscr{T}_h = \{\Omega_k^h:\ k = 1,\ldots,T(h)\},$$

$h=\max\{\mathrm{diam}(\Omega_k^h):\ \Omega_k^h\in\mathscr{T}_h\}$ denotes the grid size, and $T(h)$ denotes the number of triangles (cells). Let σ_k^h denote the area (volume) of the cell Ω_k^h, i.e., $\sigma_k^h=|\Omega_k^h|$. Let $P_1(\Omega_k^h)$ denote the space of linear polynomials defined over Ω_k^h. Define the following three continuous piecewise linear approximating function spaces with respect to the triangulation $\mathscr{T}_h$,

$$\mathscr{V}_h = \{\mathbf{v}_h\in\mathscr{V}:\ \mathbf{v}_h|_{\Omega_k^h}\in[P_1(\Omega_k^h)]^d \text{ for } k=1,\ldots,T(h)\}, \tag{3.1}$$

$$\mathscr{S}_h = \{\mathbf{V}_h\in\mathscr{S}:\ \mathbf{V}_h|_{\Omega_k^h}\in[P_1(\Omega_k^h)]^{d^2} \text{ for } k=1,\ldots,T(h)\}, \tag{3.2}$$

$$\mathscr{Q}_h = \{q_h\in\mathscr{Q}:\ q_h|_{\Omega_k^h}\in P_1(\Omega_k^h) \text{ for } k=1,\ldots,T(h)\}. \tag{3.3}$$

Then the finite-dimensional function spaces $\mathscr{V}_h$, $\mathscr{S}_h$, and $\mathscr{Q}_h$ satisfy the following approximation properties: for any $\mathbf{v}\in\mathscr{V}\cap[H^2(\Omega)]^d$, $\mathbf{V}\in\mathscr{S}\cap[H^2(\Omega)]^{d^2}$, and $q\in\mathscr{Q}\cap H^2(\Omega)$, there exist $\mathbf{v}_h\in\mathscr{V}_h$,

$\mathbf{V}_h \in \mathscr{S}_h$, and $q_h \in \mathscr{Q}_h$ such that

$$\|\mathbf{v} - \mathbf{v}_h\|_{1,\Omega} \leqslant Ch^1 \|\mathbf{v}\|_{2,\Omega}, \tag{3.4}$$

$$\|\mathbf{V} - \mathbf{V}_h\|_{1,\Omega} \leqslant Ch^1 \|\mathbf{V}\|_{2,\Omega}, \tag{3.5}$$

$$\|q - q_h\|_{1,\Omega} \leqslant Ch^1 \|q\|_{2,\Omega}, \tag{3.6}$$

where C is a positive constant independent of $\mathbf{v}$, $\mathbf{V}$, q, and h.

We now present the cell vertex finite volume/least-squares algorithm. The first step is to form an overdetermined linear system of equations

Determine $(\mathbf{u}_h, \mathbf{U}_h, p_h) \in \mathscr{V}_h \times \mathscr{S}_h \times \mathscr{Q}_h$ such that

$$\frac{1}{\sqrt{\sigma_k^h}} \int_{\Omega_k^h} (-\nu(\nabla \cdot \mathbf{U}_h)^{\mathrm{T}} + \nabla p_h)\, \mathrm{d}\Omega = \frac{1}{\sqrt{\sigma_k^h}} \int_{\Omega_k^h} \mathbf{f}\, d\Omega,$$

$$\frac{\nu}{\sqrt{\sigma_k^h}} \int_{\Omega_k^h} (\mathbf{U}_h - \nabla \mathbf{u}_h^{\mathrm{T}})\, \mathrm{d}\Omega = \mathbf{0},$$

$$\frac{\nu}{\sqrt{\sigma_k^h}} \int_{\Omega_k^h} (\nabla \cdot \mathbf{u}_h + \delta p_h)\, \mathrm{d}\Omega = \frac{\nu}{\sqrt{\sigma_k^h}} \int_{\Omega_k^h} g\, \mathrm{d}\Omega,$$

$$\frac{\nu}{\sqrt{\sigma_k^h}} \int_{\Omega_k^h} (\nabla \mathrm{tr} \mathbf{U}_h + \delta \nabla p_h)\, \mathrm{d}\Omega = \frac{\nu}{\sqrt{\sigma_k^h}} \int_{\Omega_k^h} \nabla g\, \mathrm{d}\Omega,$$

$$\frac{\nu}{\sqrt{\sigma_k^h}} \int_{\Omega_k^h} (\nabla \times \mathbf{U}_h)\, \mathrm{d}\Omega = \mathbf{0} \tag{3.7}$$

for $k = 1, \ldots, T(h)$.

The equations in (3.7) are simply formed by integrating the components of the differential system (2.6) over each of the cells $\Omega_k^h \in \mathscr{T}_h$. The scaling factors $1/\sqrt{\sigma_k^h}$ and $\nu/\sqrt{\sigma_k^h}$ are not important in practice, but are convenient for the analysis below.

The system (3.7) consists of $(3d^2 - d + 1)T(h)$ equations; the number of unknowns is less than $(d^2 + d + 1)N(h)$, where $N(h)$ is the number of vertices in the triangulation $\mathscr{T}_h$. Since, in general, $T(h)$ is roughly equal to $2N(h)$ for $d = 2$ and $N(h)$ for $d = 3$, we see that $(3d^2 - d + 1)T(h)$ is larger than $(d^2 + d + 1)N(h)$. Thus, once a basis for $\mathscr{V}_h \times \mathscr{S}_h \times \mathscr{Q}_h$ is chosen, problem (3.7) is equivalent to an overdetermined linear system of the form

$$\mathscr{A}X = R. \tag{3.8}$$

The second step is to solve the overdetermined problem by the following algebraic least-squares problem

Determine $\tilde{X}$ such that

$$\|\mathscr{A}\tilde{X} - R\|_2 = \min_X \|\mathscr{A}X - R\|_2 \tag{3.9}$$

or equivalently, to solve the normal equations
Determine $\tilde{X}$ such that

$$\mathscr{A}^{\mathrm{T}}\mathscr{A}\tilde{X} = \mathscr{A}^{\mathrm{T}}R. \tag{3.10}$$

4. Error analysis

This section is devoted to the error analysis for the continuous piecewise linear approximate solution $(\mathbf{u}_h, \mathbf{U}_h, p_h)$ obtained by solving problem (3.10). We will show that problem (3.10) is equivalent to a minimization problem on which the error analysis is based. To this end, for each fixed triangulation $\mathscr{T}_h$, we define the $\mathscr{T}_h$-dependent nonlinear functional $\mathscr{F}_h : \mathscr{V} \times \mathscr{S} \times \mathscr{Q} \to \mathbf{R}$ by

$$\begin{aligned}\mathscr{F}_h((\mathbf{v}, \mathbf{V}, q); \mathbf{f}, g) = \sum_{k=1}^{T(h)} \Bigg\{ & \left| \frac{1}{\sqrt{\sigma_k^h}} \int_{\Omega_k^h} (-v(\nabla \cdot \mathbf{V})^{\mathrm{T}} + \nabla q - \mathbf{f})\, \mathrm{d}\Omega \right|^2 \\ & + \left| \frac{v}{\sqrt{\sigma_k^h}} \int_{\Omega_k^h} (\mathbf{V} - \nabla \mathbf{v}^{\mathrm{T}} - \mathbf{0})\, \mathrm{d}\Omega \right|^2 \\ & + \left| \frac{v}{\sqrt{\sigma_k^h}} \int_{\Omega_k^h} (\nabla \cdot \mathbf{v} + \delta q - g)\, \mathrm{d}\Omega \right|^2 \\ & + \left| \frac{v}{\sqrt{\sigma_k^h}} \int_{\Omega_k^h} (\nabla \mathrm{tr} \mathbf{V} + \delta \nabla q - \nabla g)\, \mathrm{d}\Omega \right|^2 \\ & + \left| \frac{v}{\sqrt{\sigma_k^h}} \int_{\Omega_k^h} (\nabla \times \mathbf{V} - \mathbf{0})\, \mathrm{d}\Omega \right|^2 \Bigg\}.\end{aligned}$$

Note that if $(\mathbf{u}, \mathbf{U}, p) \in \mathscr{V} \times \mathscr{S} \times \mathscr{Q}$ is the exact solution of problem (2.6), then

$$\mathscr{F}_h((\mathbf{u}, \mathbf{U}, p); \mathbf{f}, g) = 0.$$

Therefore, we define an approximate solution $(\tilde{\mathbf{u}}_h, \tilde{\mathbf{U}}_h, \tilde{p}_h)$ to $(\mathbf{u}, \mathbf{U}, p)$ as follows:
Determine $(\tilde{\mathbf{u}}_h, \tilde{\mathbf{U}}_h, \tilde{p}_h) \in \mathscr{V}_h \times \mathscr{S}_h \times \mathscr{Q}_h$ such that

$$\mathscr{F}_h((\tilde{\mathbf{u}}_h, \tilde{\mathbf{U}}_h, \tilde{p}_h); \mathbf{f}, g) = \min_{(\mathbf{v}_h, \mathbf{V}_h, q_h) \in \mathscr{V}_h \times \mathscr{S}_h \times \mathscr{Q}_h} \mathscr{F}_h((\mathbf{v}_h, \mathbf{V}_h, q_h)); \mathbf{f}, g). \tag{4.1}$$

Since $\mathscr{F}_h((\tilde{\mathbf{u}}_h, \tilde{\mathbf{U}}_h, \tilde{p}_h) + \varepsilon(\mathbf{v}_h, \mathbf{V}_h, q_h); \mathbf{f}, g)$ is a nonnegative quadratic functional in $\varepsilon \in \mathbf{R}$, for any given $(\mathbf{v}_h, \mathbf{V}_h, q_h) \in \mathscr{V}_h \times \mathscr{S}_h \times \mathscr{Q}_h$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \mathscr{F}_h((\tilde{\mathbf{u}}_h, \tilde{\mathbf{U}}_h, \tilde{p}_h) + \varepsilon(\mathbf{v}_h, \mathbf{V}_h, q_h); \mathbf{f}, g)|_{\varepsilon=0} = 0 \tag{4.2}$$

which is equivalent to

Determine $(\tilde{\mathbf{u}}_h, \tilde{\mathbf{U}}_h, \tilde{p}_h) \in \mathscr{V}_h \times \mathscr{S}_h \times \mathscr{Q}_h$ such that

$$\mathscr{B}_h((\tilde{\mathbf{u}}_h, \tilde{\mathbf{U}}_h, \tilde{p}_h); (\mathbf{v}_h, \mathbf{V}_h, q_h)) = \mathscr{L}_h((\mathbf{v}_h, \mathbf{V}_h, q_h)) \tag{4.3}$$

for all $(\mathbf{v}_h, \mathbf{V}_h, q_h) \in \mathscr{V}_h \times \mathscr{S}_h \times \mathscr{Q}_h$,

where the bilinear form $\mathscr{B}_h(\cdot;\cdot)$ and the linear form $\mathscr{L}_h(\cdot)$ are respectively defined as follows:

$$\begin{aligned}
&\mathscr{B}_h((\mathbf{v}, \mathbf{V}, q); (\mathbf{w}, \mathbf{W}, r)) \\
&= \sum_{k=1}^{T(h)} \Bigg\{ \left(\frac{1}{\sqrt{\sigma_k^h}} \int_{\Omega_k^h} (-\nu(\nabla \cdot \mathbf{V})^{\mathrm{T}} + \nabla q) \,\mathrm{d}\Omega \right) \cdot \left(\frac{1}{\sqrt{\sigma_k^h}} \int_{\Omega_k^h} (-\nu(\nabla \cdot \mathbf{W})^{\mathrm{T}} + \nabla r) \,\mathrm{d}\Omega \right) \\
&\quad + \left(\frac{\nu}{\sqrt{\sigma_k^h}} \int_{\Omega_k^h} (\mathbf{V} - \nabla \mathbf{v}^{\mathrm{T}}) \,\mathrm{d}\Omega \right) \cdot \left(\frac{\nu}{\sqrt{\sigma_k^h}} \int_{\Omega_k^h} (\mathbf{W} - \nabla \mathbf{w}^{\mathrm{T}}) \,\mathrm{d}\Omega \right) \\
&\quad + \left(\frac{\nu}{\sqrt{\sigma_k^h}} \int_{\Omega_k^h} (\nabla \cdot \mathbf{v} + \delta q) \,\mathrm{d}\Omega \right) \times \left(\frac{\nu}{\sqrt{\sigma_k^h}} \int_{\Omega_k^h} (\nabla \cdot \mathbf{w} + \delta r) \,\mathrm{d}\Omega \right) \\
&\quad + \left(\frac{\nu}{\sqrt{\sigma_k^h}} \int_{\Omega_k^h} (\nabla \mathrm{tr} \mathbf{V} + \delta \nabla q) \,\mathrm{d}\Omega \right) \cdot \left(\frac{\nu}{\sqrt{\sigma_k^h}} \int_{\Omega_k^h} (\nabla \mathrm{tr} \mathbf{W} + \delta \nabla r) \,\mathrm{d}\Omega \right) \\
&\quad + \left(\frac{\nu}{\sqrt{\sigma_k^h}} \int_{\Omega_k^h} (\nabla \times \mathbf{V}) \,\mathrm{d}\Omega \right) \cdot \left(\frac{\nu}{\sqrt{\sigma_k^h}} \int_{\Omega_k^h} (\nabla \times \mathbf{W}) \,\mathrm{d}\Omega \right) \Bigg\},
\end{aligned}$$

$$\begin{aligned}
\mathscr{L}_h((\mathbf{v}, \mathbf{V}, q)) = \sum_{k=1}^{T(h)} \Bigg\{ &\left(\frac{1}{\sqrt{\sigma_k^h}} \int_{\Omega_k^h} (-\nu(\nabla \cdot \mathbf{V})^{\mathrm{T}} + \nabla q) \,\mathrm{d}\Omega \right) \cdot \left(\frac{1}{\sqrt{\sigma_k^h}} \int_{\Omega_k^h} \mathbf{f} \, d\Omega \right) \\
&+ \left(\frac{\nu}{\sqrt{\sigma_k^h}} \int_{\Omega_k^h} (\nabla \cdot \mathbf{v} + \delta q) \,\mathrm{d}\Omega \right) \times \left(\frac{\nu}{\sqrt{\sigma_k^h}} \int_{\Omega_k^h} g \,\mathrm{d}\Omega \right) \\
&+ \left(\frac{\nu}{\sqrt{\sigma_k^h}} \int_{\Omega_k^h} (\nabla \mathrm{tr} \mathbf{V} + \delta \nabla q) \,\mathrm{d}\Omega \right) \cdot \left(\frac{\nu}{\sqrt{\sigma_k^h}} \int_{\Omega_k^h} \nabla g \,\mathrm{d}\Omega \right) \Bigg\}
\end{aligned}$$

for all $(\mathbf{v},\mathbf{V},q),(\mathbf{w},\mathbf{W},r)\in\mathscr{V}\times\mathscr{S}\times\mathscr{Q}$. The unique solvability for problem (4.3) will be proved in Theorem 4.1 later.

Now, it is easily seen, by choosing a basis for $\mathscr{V}_h\times\mathscr{S}_h\times\mathscr{Q}_h$, that problem (4.3) is equivalent to problem (3.10). That is, they have the same linear system problem $\mathscr{A}^{\mathrm{T}}\mathscr{A}\tilde{X}=\mathscr{A}^{\mathrm{T}}R$ and $(\mathbf{u}_h,\mathbf{U}_h,p_h)=(\tilde{\mathbf{u}}_h,\tilde{\mathbf{U}}_h,\tilde{p}_h)$. Therefore, throughout the remaining part of this section, we shall derive a priori error estimates for the approximate solution $(\mathbf{u}_h,\mathbf{U}_h,p_h)$ based on the variational formulation (4.3).

The next several lemmas will be useful in the proof of our main result, Theorem 4.2. Using the fact that $\mathscr{V}_h$, $\mathscr{S}_h$, and $\mathscr{Q}_h$ are continuous piecewise linear function spaces (all the second derivatives of functions in them vanish on Ω_k^h), we have the following coercivity estimate for the bilinear form $\mathscr{B}_h(\cdot;\cdot)$, which plays an important role in the error analysis.

Lemma 4.1. *The bilinear form* $\mathscr{B}_h(\cdot;\cdot)$ *satisfies the following coercivity property*:

$$\begin{aligned}
&|\mathscr{B}_h((\mathbf{v},\mathbf{V},q);(\mathbf{v}_h,\mathbf{V}_h,q_h))|\\
&\quad\geqslant |(-\nu(\nabla\cdot\mathbf{V})^{\mathrm{T}}+\nabla q,\ -\nu(\nabla\cdot\mathbf{V}_h)^{\mathrm{T}}+\nabla q_h)_{0,\Omega}\\
&\qquad+(\nu(\mathbf{V}-\nabla\mathbf{v}^{\mathrm{T}}),\ \nu(\mathbf{V}_h-\nabla\mathbf{v}_h^{\mathrm{T}}))_{0,\Omega}\\
&\qquad+(\nu(\nabla\cdot\mathbf{v}+\delta q),\ \nu(\nabla\cdot\mathbf{v}_h+\delta q_h))_{0,\Omega}\\
&\qquad+(\nu(\nabla\mathrm{tr}\mathbf{V}+\delta\nabla q),\ \nu(\nabla\mathrm{tr}\mathbf{V}_h+\delta\nabla q_h))_{0,\Omega}\\
&\qquad+(\nu(\nabla\times\mathbf{V}),\ \nu(\nabla\times\mathbf{V}_h))_{0,\Omega}|\\
&\qquad-Ch(\nu\|\mathbf{v}_h\|_{1,\Omega}+\nu\|\mathbf{V}_h\|_{1,\Omega}+\|q_h\|_{1,\Omega})\\
&\qquad\times(\|-\nu(\nabla\cdot\mathbf{V})^{\mathrm{T}}+\nabla q\|_{0,\Omega}+\|\nu(\mathbf{V}-\nabla\mathbf{v}^{\mathrm{T}})\|_{0,\Omega}\\
&\qquad+\|\nu(\nabla\cdot\mathbf{v}+\delta q)\|_{0,\Omega}+\|\nu(\nabla\mathrm{tr}\mathbf{V}+\delta\nabla q)\|_{0,\Omega}+\|\nu(\nabla\times\mathbf{V})\|_{0,\Omega}),
\end{aligned}\tag{4.4}$$

for all $(\mathbf{v},\mathbf{V},q)\in\mathscr{V}\times\mathscr{S}\times\mathscr{Q}$ *and* $(\mathbf{v}_h,\mathbf{V}_h,q_h)\in\mathscr{V}_h\times\mathscr{S}_h\times\mathscr{Q}_h$.

Proof. The proof is simple but long. The basic idea is based on the mean value theorem. Given $\Omega_k^h\in\mathscr{T}_h$, assume $\varphi:\ \Omega_k^h\subset\mathbf{R}^d\to\mathbf{R}$ is a C^1 mapping. Then the mean value theorem asserts that for any $x,y\in\Omega_k^h$, there exists $z\in\overline{xy}$, $z=tx+(1-t)y$ for some $0<t<1$, such that

$$\varphi(x)-\varphi(y)=\nabla\varphi(z)\cdot(x-y).$$

Taking integration with respect to y, we have

$$\int_{\Omega_k^h}\varphi(x)-\varphi(y)\,\mathrm{d}\Omega_y=\int_{\Omega_k^h}\nabla\varphi(z)\cdot(x-y)\,\mathrm{d}\Omega_y$$

which implies

$$\varphi(x)\int_{\Omega_k^h} 1\,\mathrm{d}\Omega_y = \int_{\Omega_k^h} \varphi(y)\,\mathrm{d}\Omega_y + \int_{\Omega_k^h} \nabla\varphi(z)\cdot(x-y)\,\mathrm{d}\Omega_y$$

and hence

$$\varphi(x) = \frac{1}{\sigma_k^h}\int_{\Omega_k^h} \varphi(y)\,\mathrm{d}\Omega_y + \frac{1}{\sigma_k^h}\int_{\Omega_k^h} \nabla\varphi(z)\cdot(x-y)\,\mathrm{d}\Omega_y. \tag{4.5}$$

Now let $\alpha = 3d^2 - d + 1$, $\Psi = (\psi_1,\ldots,\psi_\alpha)^{\mathrm{T}}$, and $\mathscr{Z} = (\zeta_1,\ldots,\zeta_\alpha)^{\mathrm{T}}$ be two vector-valued mappings defined over $\bar{\Omega}$, where $\psi_i|_{\Omega_k^h} \in L^2(\Omega_k^h)$ and $\zeta_i|_{\Omega_k^h} \in C^1(\Omega_k^h)$ for all $i = 1,\ldots,\alpha$ and $k = 1,\ldots,T(h)$. Then, using (4.5), we have

$$\begin{aligned}
(\Psi,\mathscr{Z})_{0,\Omega} &= \sum_{k=1}^{T(h)}\left\{\sum_{j=1}^{\alpha}\int_{\Omega_k^h} \psi_j(x)\zeta_j(x)\,\mathrm{d}\Omega_x\right\}\\
&= \sum_{k=1}^{T(h)}\frac{1}{\sigma_k^h}\left\{\sum_{j=1}^{\alpha}\int_{\Omega_k^h}\psi_j(x)\left(\int_{\Omega_k^h}\zeta_j(y)\,\mathrm{d}\Omega_y + \int_{\Omega_k^h}\nabla\zeta_j(z_{j,k})\cdot(x-y)\,\mathrm{d}\Omega_y\right)\mathrm{d}\Omega_x\right\}\\
&= \sum_{k=1}^{T(h)}\frac{1}{\sigma_k^h}\left\{\sum_{j=1}^{\alpha}\int_{\Omega_k^h}\psi_j(x)\,\mathrm{d}\Omega_x\int_{\Omega_k^h}\zeta_j(y)\,\mathrm{d}\Omega_y\right\}\\
&\quad + \sum_{k=1}^{T(h)}\frac{1}{\sigma_k^h}\left\{\sum_{j=1}^{\alpha}\int_{\Omega_k^h}\psi_j(x)\left(\int_{\Omega_k^h}\nabla\zeta_j(z_{j,k})\cdot(x-y)\,\mathrm{d}\Omega_y\right)\mathrm{d}\Omega_x\right\}.
\end{aligned}$$

Now taking absolute values on the both sides of the above equation, we can conclude that

$$\begin{aligned}
|(\Psi,\mathscr{Z})_{0,\Omega}| &\leqslant \left|\sum_{k=1}^{T(h)}\frac{1}{\sigma_k^h}\left(\int_{\Omega_k^h}\Psi\,\mathrm{d}\Omega_x\cdot\int_{\Omega_k^h}\mathscr{Z}\,\mathrm{d}\Omega_y\right)\right|\\
&\quad + \sum_{k=1}^{T(h)}\frac{1}{\sigma_k^h}\left\{\sum_{j=1}^{\alpha}\int_{\Omega_k^h}|\psi_j(x)|\left(\int_{\Omega_k^h}|\nabla\zeta_j(z_{j,k})\cdot(x-y)|\,\mathrm{d}\Omega_y\right)\mathrm{d}\Omega_x\right\},
\end{aligned}\tag{4.6}$$

where $z_{j,k} = t_{j,k}x + (1-t_{j,k})y \in \Omega_k^h$ for some $t_{j,k}\in(0,1)$, $j = 1,\ldots,\alpha$.

We are now in the position to prove the lemma. First of all, we set

$$\Psi = \left(-\nu(\nabla\cdot\mathbf{V})^{\mathrm{T}} + \nabla q, \nu(\mathbf{V} - \nabla\mathbf{v}^{\mathrm{T}}), \nu(\nabla\cdot\mathbf{v} + \delta q), \nu(\nabla\mathrm{tr}\mathbf{V} + \delta\nabla q), \nu(\nabla\times\mathbf{V})\right)$$

for $(\mathbf{v},\mathbf{V},q)\in\mathscr{V}\times\mathscr{S}\times\mathscr{Q}$. Then $\Psi|_{\Omega_k^h}\in[L^2(\Omega_k^h)]^\alpha$ for all $k = 1,\ldots,T(h)$. Similarly, we set

$$\mathscr{Z} = \left(-\nu(\nabla\cdot\mathbf{V}_h)^{\mathrm{T}} + \nabla q_h, \nu(\mathbf{V}_h - \nabla\mathbf{v}_h^{\mathrm{T}}), \nu(\nabla\cdot\mathbf{v}_h + \delta q_h), \nu(\nabla\mathrm{tr}\mathbf{V}_h + \delta\nabla q_h), \nu(\nabla\times\mathbf{V}_h)\right)$$

for $(\mathbf{v}_h,\mathbf{V}_h,q_h)\in\mathscr{V}_h\times\mathscr{S}_h\times\mathscr{Q}_h$. Then $\mathscr{Z}|_{\Omega_k^h}\in[C^1(\Omega_k^h)]^\alpha$ for all $k = 1,\ldots,T(h)$. Recall that $\mathbf{v}_h$, $\mathbf{V}_h$, and q_h are continuous piecewise linear polynomials on Ω, and $\mathrm{diam}(\Omega_k^h)\leqslant h$. Thus we have the following estimates: for $x, y \in \Omega_k^h$, and $z_{j,k} = t_{j,k}x + (1 - t_{j,k})y$ for some $t_{j,k} \in (0,1)$,

$$\begin{aligned}
I_{1j}(x) &\equiv \int_{\Omega_k^h} |\nabla((-v(\nabla\cdot\mathbf{V}_h)^{\mathrm{T}} + \nabla q_h)_j)(z_{j,k})\cdot(x-y)|\,\mathrm{d}\Omega_y \\
&\leqslant \int_{\Omega_k^h} |\nabla((-v(\nabla\cdot\mathbf{V}_h)^{\mathrm{T}} + \nabla q_h)_j)(z_{j,k})||(x-y)|\,\mathrm{d}\Omega_y \\
&\leqslant h\int_{\Omega_k^h} |\nabla((-v(\nabla\cdot\mathbf{V}_h)^{\mathrm{T}} + \nabla q_h)_j)(z_{j,k})|\,\mathrm{d}\Omega_y \\
&= 0, \qquad (4.7)
\end{aligned}$$

$$\begin{aligned}
I_{2j}(x) &\equiv \int_{\Omega_k^h} |\nabla(v(\mathbf{V}_h - \nabla\mathbf{v}_h^{\mathrm{T}})_j)(z_{j,k})\cdot(x-y)|\,\mathrm{d}\Omega_y \\
&\leqslant \int_{\Omega_k^h} |\nabla(v(\mathbf{V}_h - \nabla\mathbf{v}_h^{\mathrm{T}})_j)(z_{j,k})||(x-y)|\,\mathrm{d}\Omega_y \\
&\leqslant h\int_{\Omega_k^h} |\nabla(v(\mathbf{V}_h - \nabla\mathbf{v}_h^{\mathrm{T}})_j)(z_{j,k})|\,\mathrm{d}\Omega_y \\
&\leqslant h\sqrt{\sigma_k^h}\; v\|\nabla\mathbf{V}_h\|_{0,\Omega_k^h}, \qquad (4.8)
\end{aligned}$$

$$\begin{aligned}
I_3(x) &\equiv \int_{\Omega_k^h} |\nabla(v(\nabla\cdot\mathbf{v}_h + \delta q_h))(z_{j,k})\cdot(x-y)|\,\mathrm{d}\Omega_y \\
&\leqslant h\int_{\Omega_k^h} |\nabla(v(\nabla\cdot\mathbf{v}_h + \delta q_h))(z_{j,k})|\,\mathrm{d}\Omega_y \\
&\leqslant \delta h\sqrt{\sigma_k^h}\; v\|\nabla q_h\|_{0,\Omega_k^h}, \qquad (4.9)
\end{aligned}$$

$$\begin{aligned}
I_{4j}(x) &\equiv \int_{\Omega_k^h} |\nabla(v(\nabla\mathrm{tr}\mathbf{V}_h + \delta\nabla q_h)_j)(z_{j,k})\cdot(x-y)|\,\mathrm{d}\Omega_y \\
&\leqslant \int_{\Omega_k^h} |\nabla(v(\nabla\mathrm{tr}\mathbf{V}_h + \delta\nabla q_h)_j)(z_{j,k})||(x-y)|\,\mathrm{d}\Omega_y \\
&\leqslant h\int_{\Omega_k^h} |\nabla(v(\nabla\mathrm{tr}\mathbf{V}_h + \delta\nabla q_h)_j)(z_{j,k})|\,\mathrm{d}\Omega_y \\
&= 0, \qquad (4.10)
\end{aligned}$$

$$\begin{aligned}
I_{5j}(x) &\equiv \int_{\Omega_k^h} |\nabla(v(\nabla\times\mathbf{V}_h)_j)(z_{j,k})\cdot(x-y)|\,\mathrm{d}\Omega_y \\
&\leqslant \int_{\Omega_k^h} |\nabla(v(\nabla\times\mathbf{V}_h)_j)(z_{j,k})||(x-y)|\,\mathrm{d}\Omega_y
\end{aligned}$$

$$\leqslant h \int_{\Omega_k^h} |\nabla(v(\nabla \times \mathbf{V}_h)_j)(z_{j,k})| \, \mathrm{d}\Omega_y$$
$$= 0. \tag{4.11}$$

Substituting (4.7)–(4.11) into (4.6) then yields

$$\begin{aligned}
&\big|(-v(\nabla \cdot \mathbf{V})^{\mathrm{T}} + \nabla q,\ -v(\nabla \cdot \mathbf{V}_h)^{\mathrm{T}} + \nabla q_h)_{0,\Omega} \\
&\quad + (v(\mathbf{V} - \nabla \mathbf{v}^{\mathrm{T}}),\ v(\mathbf{V}_h - \nabla \mathbf{v}_h^{\mathrm{T}}))_{0,\Omega} \\
&\quad + (v(\nabla \cdot \mathbf{v} + \delta q),\ v(\nabla \cdot \mathbf{v}_h + \delta q_h))_{0,\Omega} \\
&\quad + (v(\nabla \mathrm{tr} \mathbf{V} + \delta \nabla q),\ v(\nabla \mathrm{tr} \mathbf{V}_h + \delta \nabla q_h))_{0,\Omega} \\
&\quad + (v(\nabla \times \mathbf{V}),\ v(\nabla \times \mathbf{V}_h))_{0,\Omega}| \\
&\leqslant |\mathscr{B}_h((\mathbf{v}, \mathbf{V}, q); (\mathbf{v}_h, \mathbf{V}_h, q_h))| \\
&\quad + \sum_{k=1}^{T(h)} \frac{1}{\sigma_k^h} \Bigg\{ \sum_{j=1}^{d} \int_{\Omega_k^h} |(-v(\nabla \cdot \mathbf{V})^{\mathrm{T}} + \nabla q)_j(x)| I_{1j}(x) \, \mathrm{d}\Omega_x \\
&\qquad + \sum_{j=1}^{d^2} \int_{\Omega_k^h} |v(\mathbf{V} - \nabla \mathbf{v}^{\mathrm{T}})_j(x)| I_{2j}(x) \, \mathrm{d}\Omega_x \\
&\qquad + \int_{\Omega_k^h} |v(\nabla \cdot \mathbf{v} + \delta q)(x)| I_3(x) \, \mathrm{d}\Omega_x \\
&\qquad + \sum_{j=1}^{d} \int_{\Omega_k^h} |v(\nabla \mathrm{tr} \mathbf{V} + \delta \nabla q)_j(x)| \ I_{4j}(x) \, \mathrm{d}\Omega_x \\
&\qquad + \sum_{j=1}^{2d^2-3d} \int_{\Omega_k^h} |v(\nabla \times \mathbf{V})_j(x)| I_{5j}(x) \, \mathrm{d}\Omega_x \Bigg\} \\
&\leqslant |\mathscr{B}_h((\mathbf{v}, \mathbf{V}, q); (\mathbf{v}_h, \mathbf{V}_h, q_h))| \\
&\quad + \sum_{k=1}^{T(h)} \Big\{ \Big(hv \|v(\mathbf{V} - \nabla \mathbf{v}^{\mathrm{T}})\|_{0,\Omega_k^h} \|\nabla \mathbf{V}_h\|_{0,\Omega_k^h} \Big) \\
&\quad + \ \Big(\delta hv \|v(\nabla \cdot \mathbf{v} + \delta q)\|_{0,\Omega_k^h} \|\nabla q_h\|_{0,\Omega_k^h} \Big) \Big\} \\
&\leqslant |\mathscr{B}_h((\mathbf{v}, \mathbf{V}, q); (\mathbf{v}_h, \mathbf{V}_h, q_h))| \\
&\quad + \sum_{k=1}^{T(h)} Ch \Big\{ \Big(v\|\mathbf{v}_h\|_{1,\Omega_k^h} + v\|\mathbf{V}_h\|_{1,\Omega_k^h} + \|q_h\|_{1,\Omega_k^h} \Big)
\end{aligned}$$

$$\times \Big(\| - \nu(\nabla\cdot\mathbf{V})^{\mathrm{T}} + \nabla q\|_{0,\Omega_k^h} + \|\nu(\mathbf{V}-\nabla\mathbf{v}^{\mathrm{T}})\|_{0,\Omega_k^h} + \|\nu(\nabla\cdot\mathbf{v}+\delta q)\|_{0,\Omega_k^h}$$

$$+ \|\nu(\nabla \mathrm{tr}\mathbf{V} + \delta\nabla q)\|_{0,\Omega_k^h} + \|\nu(\nabla\times\mathbf{V})\|_{0,\Omega_k^h}\Big)\Big\}$$

$$\leqslant |\mathscr{B}_h((\mathbf{v},\mathbf{V},q);(\mathbf{v}_h,\mathbf{V}_h,q_h))| + Ch\left\{\left(\nu\|\mathbf{v}_h\|_{1,\Omega} + \nu\|\mathbf{V}_h\|_{1,\Omega} + \|q_h\|_{1,\Omega}\right)\right.$$

$$\times \left(\| - \nu(\nabla\cdot\mathbf{V})^{\mathrm{T}} + \nabla q\|_{0,\Omega} + \|\nu(\mathbf{V}-\nabla\mathbf{v}^{\mathrm{T}})\|_{0,\Omega} + \|\nu(\nabla\cdot\mathbf{v}+\delta q)\|_{0,\Omega}\right.$$

$$\left.\left. + \|\nu\left(\nabla \mathrm{tr}\mathbf{V} + \delta\nabla q\right)\|_{0,\Omega} + \|\nu(\nabla\times\mathbf{V})\|_{0,\Omega}\right)\right\}.$$

This completes the proof. □

Lemma 4.2. *There exists an $h_0 > 0$ such that, for sufficiently small $h < h_0$,*

$$\mathscr{B}_h((\mathbf{v}_h,\mathbf{V}_h,q_h);(\mathbf{v}_h,\mathbf{V}_h,q_h)) \geqslant C(\nu^2\|\mathbf{v}_h\|_{1,\Omega}^2 + \nu^2\|\mathbf{V}_h\|_{1,\Omega}^2 + \|q_h\|_{1,\Omega}^2), \tag{4.12}$$

for all $(\mathbf{v}_h,\mathbf{V}_h,q_h) \in \mathscr{V}_h \times \mathscr{S}_h \times \mathscr{Q}_h$.

Proof. Combining (4.4) with (2.11), we get for all $(\mathbf{v}_h,\mathbf{V}_h,q_h) \in \mathscr{V}_h \times \mathscr{S}_h \times \mathscr{Q}_h$,

$$\mathscr{B}_h((\mathbf{v}_h,\mathbf{V}_h,q_h);(\mathbf{v}_h,\mathbf{V}_h,q_h)) \geqslant C(\nu^2\|\mathbf{v}_h\|_{1,\Omega}^2 + \nu^2\|\mathbf{V}_h\|_{1,\Omega}^2 + \|q_h\|_{1,\Omega}^2)$$

$$- Ch(\nu^2\|\mathbf{v}_h\|_{1,\Omega}^2 + \nu^2\|\mathbf{V}_h\|_{1,\Omega}^2 + \|q_h\|_{1,\Omega}^2).$$

Thus, for sufficiently small h, we obtain (4.12). This completes the proof. □

The coercivity property (4.12) ensures the unique solvability of problem (4.3) or equivalent problem (3.10), and also gives an estimate of the condition number of the matrix $\mathscr{A}^{\mathrm{T}}\mathscr{A}$ in (3.10).

Theorem 4.1. *There exists an $h_0 > 0$ such that problem* (4.3), *or equivalent problem* (3.10), *has a unique solution whenever $h < h_0$. In this case, the symmetric matrix $\mathscr{A}^{\mathrm{T}}\mathscr{A}$ appearing in* (3.10) *is positive definite. Moreover, if the family $\{\mathscr{T}_h\}$ of regular triangulations of the domain Ω is quasi-uniform, then the condition number of $\mathscr{A}^{\mathrm{T}}\mathscr{A}$ is* $O(\nu^{-2}h^{-2})$.

Proof. For sufficiently small $h < h_0$, by (4.12), we have seen the bilinear form $\mathscr{B}_h(\cdot;\cdot)$ is coercive on the continuous piecewise linear function space $\mathscr{V}_h \times \mathscr{S}_h \times \mathscr{Q}_h$. The positive definiteness of the matrix $\mathscr{A}^{\mathrm{T}}\mathscr{A}$ appearing in problem (3.10) follows immediately. Thus, the unique solvability of problem (4.3) is also ensured.

Next, we give an estimate for the condition number of the linear system arising from problem (3.10). Recall that the condition number for a symmetric and positive definite $\ell \times \ell$ matrix $\mathscr{M}$ is defined by

$$\text{condition number of } \mathscr{M} \equiv \frac{\lambda_{\max}}{\lambda_{\min}} = \frac{\max \mathscr{R}(\Xi)}{\min \mathscr{R}(\Xi)},$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the largest and smallest eigenvalues of $\mathscr{M}$, respectively, and $\mathscr{R}(\Xi)$ is the Rayleigh quotient

$$\mathscr{R}(\Xi) \equiv \frac{\Xi^{\mathrm{T}}\mathscr{M}\Xi}{\Xi^{\mathrm{T}}\Xi} \quad \text{for all } \Xi = (\xi_1,\dots,\xi_\ell)^{\mathrm{T}} \in \mathbf{R}^\ell,\ \Xi \notin \mathbf{0}.$$

Let $\mathbf{v}^1,\dots,\mathbf{v}^m$, $\mathbf{V}^1,\dots,\mathbf{V}^n$, and $q^1,\dots,q^k$ be bases for $\mathscr{V}_h$, $\mathscr{S}_h$, and $\mathscr{Q}_h$, respectively. Since the family $\{\mathscr{T}_h\}$ of triangulations is regular [13], there exist positive constants Λ_1, Λ_2, Θ_1, Θ_2, Υ_1, and Υ_2 such that, for any $\mathbf{v}_h \in \mathscr{V}_h$, $\mathbf{V}_h \in \mathscr{S}_h$, and $q_h \in \mathscr{Q}_h$ with

$$\mathbf{v}_h = \sum_{i=1}^m \xi_i \mathbf{v}^i, \quad \mathbf{V}_h = \sum_{i=1}^n \eta_i \mathbf{V}^i, \quad \text{and} \quad q_h = \sum_{i=1}^k \zeta_i q^i,$$

we have

$$\Lambda_1 h^d \sum_{i=1}^m \xi_i^2 \leqslant \|\mathbf{v}_h\|_{0,\Omega}^2 \leqslant \Lambda_2 h^d \sum_{i=1}^m \xi_i^2, \tag{4.13}$$

$$\Theta_1 h^d \sum_{i=1}^n \eta_i^2 \leqslant \|\mathbf{V}_h\|_{0,\Omega}^2 \leqslant \Theta_2 h^d \sum_{i=1}^m \eta_i^2, \tag{4.14}$$

$$\Upsilon_1 h^d \sum_{i=1}^k \zeta_i^2 \leqslant \|q_h\|_{0,\Omega}^2 \leqslant \Upsilon_2 h^d \sum_{i=1}^k \zeta_i^2. \tag{4.15}$$

If, in addition, the corresponding regular family $\{\mathscr{T}_h\}$ of triangulations of $\bar{\Omega}$ is quasi-uniform [13], i.e., there exists a positive constant C independent of h such that

$$h \leqslant C\,\mathrm{diam}(\Omega_k^h) \quad \text{for all } \Omega_k^h \in \mathscr{T}_h,\ \mathscr{T}_h \in \{\mathscr{T}_h\} \tag{4.16}$$

then we have the following inverse estimates [13]

$$\|\mathbf{v}_h\|_{1,\Omega} \leqslant Ch^{-1}\|\mathbf{v}_h\|_{0,\Omega}, \tag{4.17}$$

$$\|\mathbf{V}_h\|_{1,\Omega} \leqslant Ch^{-1}\|\mathbf{V}_h\|_{0,\Omega}, \tag{4.18}$$

$$\|q_h\|_{1,\Omega} \leqslant Ch^{-1}\|q_h\|_{0,\Omega}, \tag{4.19}$$

where C is a positive constant independent of h.

For any $\mathbf{v}_h = \sum_{i=1}^m \xi_i \mathbf{v}^i \in \mathscr{V}_h$, $\mathbf{V}_h = \sum_{i=1}^n \eta_i \mathbf{V}^i \in \mathscr{S}_h$, and $q_h = \sum_{i=1}^k \zeta_i q^i \in \mathscr{Q}_h$, by (2.11), (4.13), (4.14), and (4.15), we have

$$\begin{aligned}\mathscr{B}((\mathbf{v}_h,\mathbf{V}_h,q_h);(\mathbf{v}_h,\mathbf{V}_h,q_h)) &\geqslant C(v^2\|\mathbf{v}_h\|_{1,\Omega}^2 + v^2\|\mathbf{V}_h\|_{1,\Omega}^2 + \|q_h\|_{1,\Omega}^2)\\ &\geqslant C(v^2\|\mathbf{v}_h\|_{0,\Omega}^2 + v^2\|\mathbf{V}_h\|_{0,\Omega}^2 + \|q_h\|_{0,\Omega}^2)\\ &\geqslant C\min\{\Lambda_1,\Theta_1,\Upsilon_1\}v^2h^d\left(\sum_{i=1}^m \xi_i^2 + \sum_{i=1}^n \eta_i^2 + \sum_{i=1}^k \zeta_i^2\right).\end{aligned}$$

On the other hand, it follows from (2.11), (4.17), (4.18), and (4.19) that

$$\begin{aligned}\mathscr{B}((\mathbf{v}_h,\mathbf{V}_h,q_h);(\mathbf{v}_h,\mathbf{V}_h,q_h)) &\leqslant C(v^2\|\mathbf{v}_h\|_{1,\Omega}^2+v^2\|\mathbf{V}_h\|_{1,\Omega}^2+\|q_h\|_{1,\Omega}^2)\\ &\leqslant Ch^{-2}(v^2\|\mathbf{v}_h\|_{0,\Omega}^2+v^2\|\mathbf{V}_h\|_{0,\Omega}^2+\|q_h\|_{0,\Omega}^2)\\ &\leqslant C\max\{\Lambda_2,\Theta_2,\Upsilon_2\}(1+v^2)h^{d-2}\left(\sum_{i=1}^{m}\xi_i^2+\sum_{i=1}^{n}\eta_i^2+\sum_{i=1}^{k}\zeta_i^2\right).\end{aligned}$$

Thus, $\lambda_{\max}\leqslant C\max\{\Lambda_2,\Theta_2,\Upsilon_2\}(1+v^2)h^{d-2}$ and $\lambda_{\min}\geqslant C\min\{\Lambda_1,\Theta_1,\Upsilon_1\}v^2h^d$, and so the condition number of $\mathscr{A}^{\mathrm{T}}\mathscr{A}$ is $\mathrm{O}(v^{-2}h^{-2})$. This completes the proof. □

In what follows, let $(\mathbf{u},\mathbf{U},p)\in\mathscr{V}\times\mathscr{S}\times\mathscr{Q}$ and $(\mathbf{u}_h,\mathbf{U}_h,p_h)\in\mathscr{V}_h\times\mathscr{S}_h\times\mathscr{Q}_h$ be the solutions of problems (2.6) and (4.3), respectively.

Lemma 4.3. *We have the following error equation: for all* $(\mathbf{v}_h,\mathbf{V}_h,q_h)\in\mathscr{V}_h\times\mathscr{S}_h\times\mathscr{Q}_h$,

$$\mathscr{B}_h((\mathbf{u},\mathbf{U},p)-(\mathbf{u}_h,\mathbf{U}_h,p_h);(\mathbf{v}_h,\mathbf{V}_h,q_h))=0. \tag{4.20}$$

Proof. It can be easily seen that the exact solution $(\mathbf{u},\mathbf{U},p)$ of problem (2.6) satisfies

$$\mathscr{B}_h((\mathbf{u},\mathbf{U},p);(\mathbf{v}_h,\mathbf{V}_h,q_h))=\mathscr{L}_h((\mathbf{v}_h,\mathbf{V}_h,q_h)) \tag{4.21}$$

for all $(\mathbf{v}_h,\mathbf{V}_h,q_h)\in\mathscr{V}_h\times\mathscr{S}_h\times Q_h$ which, combining with (4.3), implies (4.20). □

Lemma 4.4. *The bilinear form* $\mathscr{B}_h(\cdot;\cdot)$ *satisfies the following inequalities*

$$|\mathscr{B}_h((\mathbf{v},\mathbf{V},q);(\mathbf{w},\mathbf{W},r))|\leqslant\mathscr{K}^{1/2}((\mathbf{v},\mathbf{V},q))\mathscr{K}^{1/2}((\mathbf{w},\mathbf{W},r)) \tag{4.22}$$

for all $(\mathbf{v},\mathbf{V},q)$, $(\mathbf{w},\mathbf{W},r)\in\mathscr{V}\times\mathscr{S}\times\mathscr{Q}$.

Proof. Using the definition of the bilinear form $\mathscr{B}_h(\cdot;\cdot)$, the Hölder inequality, and

$$\sum_{i=1}^{m}(L_iM_i)^{1/2}\leqslant\left(\sum_{i=1}^{m}L_i\right)^{1/2}\left(\sum_{i=1}^{m}M_i\right)^{1/2}$$

for all $L_1,\ldots,L_m,M_1,\ldots,M_m\in\mathbf{R}^+$, we have

$$\begin{aligned}&\mathscr{B}_h((\mathbf{v},\mathbf{V},q);(\mathbf{w},\mathbf{W},r))\\ &\quad=\sum_{k=1}^{T(h)}\left\{\frac{1}{\sigma_k^h}\left(\int_{\Omega_k^h}(-v(\nabla\cdot\mathbf{V})^{\mathrm{T}}+\nabla q)\,\mathrm{d}\Omega\right)\cdot\left(\int_{\Omega_k^h}(-v(\nabla\cdot\mathbf{W})^{\mathrm{T}}+\nabla r)\,\mathrm{d}\Omega\right)\right.\\ &\qquad+\frac{v^2}{\sigma_k^h}\left(\int_{\Omega_k^h}(\mathbf{V}-\nabla\mathbf{v}^{\mathrm{T}})\,\mathrm{d}\Omega\right)\cdot\left(\int_{\Omega_k^h}(\mathbf{W}-\nabla\mathbf{w}^{\mathrm{T}})\,\mathrm{d}\Omega\right)\\ &\qquad+\frac{v^2}{\sigma_k^h}\left(\int_{\Omega_k^h}(\nabla\cdot\mathbf{v}+\delta q)\,\mathrm{d}\Omega\right)\times\left(\int_{\Omega_k^h}(\nabla\cdot\mathbf{w}+\delta r)\,\mathrm{d}\Omega\right)\end{aligned}$$

$$
\begin{aligned}
&+\frac{v^2}{\sigma_k^h}\left(\int_{\Omega_k^h}(\nabla \mathrm{tr}\mathbf{V}+\delta\nabla q)\,\mathrm{d}\Omega\right)\cdot\left(\int_{\Omega_k^h}(\nabla \mathrm{tr}\mathbf{W}+\delta\nabla r)\,\mathrm{d}\Omega\right)\\
&+\frac{v^2}{\sigma_k^h}\left(\int_{\Omega_k^h}(\nabla\times\mathbf{V})\,\mathrm{d}\Omega\right)\cdot\left(\int_{\Omega_k^h}(\nabla\times\mathbf{W})\,\mathrm{d}\Omega\right)\Bigg\},\\
&\leqslant\sum_{k=1}^{T(h)}\Bigg\{\frac{1}{\sigma_k^h}\sum_{i=1}^{d}\Bigg(\int_{\Omega_k^h}|(-v(\nabla\cdot\mathbf{V})^{\mathrm{T}}+\nabla q)_i|\times 1\,\mathrm{d}\Omega\\
&\qquad\qquad\times\int_{\Omega_k^h}|(-v(\nabla\cdot\mathbf{W})^{\mathrm{T}}+\nabla r)_i|\times 1\,\mathrm{d}\Omega\Bigg)\\
&+\frac{v^2}{\sigma_k^h}\sum_{i=1}^{d^2}\left(\int_{\Omega_k^h}|(\mathbf{V}-\nabla\mathbf{v}^{\mathrm{T}})_i|\times 1\,\mathrm{d}\Omega\int_{\Omega_k^h}|(\mathbf{W}-\nabla\mathbf{w}^{\mathrm{T}})_i|\times 1\,\mathrm{d}\Omega\right)\\
&+\frac{v^2}{\sigma_k^h}\left(\int_{\Omega_k^h}|(\nabla\cdot\mathbf{v}+\delta q)|\times 1\,\mathrm{d}\Omega\int_{\Omega_k^h}|(\nabla\cdot\mathbf{w}+\delta r)|\times 1\,\mathrm{d}\Omega\right)\\
&+\frac{v^2}{\sigma_k^h}\sum_{i=1}^{d}\left(\int_{\Omega_k^h}|(\nabla \mathrm{tr}\mathbf{V}+\delta\nabla q)_i|\times 1\,\mathrm{d}\Omega\int_{\Omega_k^h}|(\nabla \mathrm{tr}\mathbf{W}+\delta\nabla r)_i|\times 1\,\mathrm{d}\Omega\right)\\
&+\frac{v^2}{\sigma_k^h}\sum_{i=1}^{2d^2-3d}\left(\int_{\Omega_k^h}|(\nabla\times\mathbf{V})_i|\times 1\,\mathrm{d}\Omega\int_{\Omega_k^h}|(\nabla\times\mathbf{W})_i|\times 1\,\mathrm{d}\Omega\right)\Bigg\}\\
&\leqslant\sum_{k=1}^{T(h)}\Bigg\{\sum_{i=1}^{d}\|(-v(\nabla\cdot\mathbf{V})^{\mathrm{T}}+\nabla q)_i\|_{0,\Omega_k^h}\|(-v(\nabla\cdot\mathbf{W})^{\mathrm{T}}+\nabla r)_i\|_{0,\Omega_k^h}\\
&+\sum_{i=1}^{d^2}v^2\|(\mathbf{V}-\nabla\mathbf{v}^{\mathrm{T}})_i\|_{0,\Omega_k^h}\|(\mathbf{W}-\nabla\mathbf{w}^{\mathrm{T}})_i\|_{0,\Omega_k^h}\\
&+v^2\|\nabla\cdot\mathbf{v}+\delta q\|_{0,\Omega_k^h}\|\nabla\cdot\mathbf{w}+\delta r\|_{0,\Omega_k^h}\\
&+\sum_{i=1}^{d}v^2\|(\nabla \mathrm{tr}\mathbf{V}+\delta\nabla q)_i\|_{0,\Omega_k^h}\|(\nabla \mathrm{tr}\mathbf{W}+\delta\nabla r)_i\|_{0,\Omega_k^h}\\
&+\sum_{i=1}^{2d^2-3d}v^2\|(\nabla\times\mathbf{V})_i\|_{0,\Omega_k^h}\|(\nabla\times\mathbf{W})_i\|_{0,\Omega_k^h}\Bigg\}\\
&\leqslant\sum_{k=1}^{T(h)}\Big\{\|-v(\nabla\cdot\mathbf{V})^{\mathrm{T}}+\nabla q\|_{0,\Omega_k^h}\|-v(\nabla\cdot\mathbf{W})^{\mathrm{T}}+\nabla r\|_{0,\Omega_k^h}
\end{aligned}
$$

$$
\begin{aligned}
&+ \|\nu(\mathbf{V} - \nabla \mathbf{v}^{\mathrm{T}})\|_{0,\Omega_k^h} \|\nu(\mathbf{W} - \nabla \mathbf{w}^{\mathrm{T}})\|_{0,\Omega_k^h} \\
&+ \|\nu(\nabla \cdot \mathbf{v} + \delta q)\|_{0,\Omega_k^h} \|\nu(\nabla \cdot \mathbf{w} + \delta r)\|_{0,\Omega_k^h} \\
&+ \|\nu(\nabla \mathrm{tr} \mathbf{V} + \delta \nabla q)\|_{0,\Omega_k^h} \|\nu(\nabla \mathrm{tr} \mathbf{W} + \delta \nabla r)\|_{0,\Omega_k^h} \\
&+ \|\nu(\nabla \times \mathbf{V})\|_{0,\Omega_k^h} \|\nu(\nabla \times \mathbf{W})\|_{0,\Omega_k^h} \Big\} \\
\leqslant\ & \mathscr{K}^{1/2}((\mathbf{v}, \mathbf{V}, q)) \mathscr{K}^{1/2}((\mathbf{w}, \mathbf{W}, r)).
\end{aligned}
$$

This completes the proof. □

We are now in the position to prove our main result. Using Lemmas 4.2, 4.3, and 4.4 with the standard procedure similar to that in the usual finite element analysis, we have the following H^1-optimal error estimate

Theorem 4.2. *Let* $(\mathbf{u}, \mathbf{U}, p) \in [\mathscr{V} \times \mathscr{S} \times \mathscr{Q}] \cap [H^2(\Omega)]^{d^2+d+1}$ *and* $(\mathbf{u}_h, \mathbf{U}_h, p_h) \in \mathscr{V}_h \times \mathscr{S}_h \times \mathscr{Q}_h$, $h < h_0$, *be the solutions of problems* (2.6) *and* (4.3), *respectively. Then we have*

$$\nu\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \nu\|\mathbf{U} - \mathbf{U}_h\|_{1,\Omega} + \|p - p_h\|_{1,\Omega} \leqslant Ch(\nu\|\mathbf{u}\|_{2,\Omega} + \nu\|\mathbf{U}\|_{2,\Omega} + \|p\|_{2,\Omega}). \tag{4.23}$$

Proof. From the orthogonality property (4.20), we can assert that

$$\mathscr{B}_h((\mathbf{u}, \mathbf{U}, p) - (\mathbf{v}_h, \mathbf{V}_h, q_h) + (\mathbf{v}_h, \mathbf{V}_h, q_h) - (\mathbf{u}_h, \mathbf{U}_h, p_h); (\mathbf{w}_h, \mathbf{W}_h, r_h)) = 0$$

for all $(\mathbf{v}_h, \mathbf{V}_h, q_h)$, $(\mathbf{w}_h, \mathbf{W}_h, r_h) \in \mathscr{V}_h \times \mathscr{S}_h \times \mathscr{Q}_h$. Now choose

$$(\mathbf{w}_h, \mathbf{W}_h, r_h) = (\mathbf{u}_h, \mathbf{U}_h, p_h) - (\mathbf{v}_h, \mathbf{V}_h, q_h) \in \mathscr{V}_h \times \mathscr{S}_h \times \mathscr{Q}_h$$

so that

$$
\begin{aligned}
&\mathscr{B}_h((\mathbf{u}_h, \mathbf{U}_h, p_h) - (\mathbf{v}_h, \mathbf{V}_h, q_h); (\mathbf{u}_h, \mathbf{U}_h, p_h) - (\mathbf{v}_h, \mathbf{V}_h, q_h)) \\
&\quad = \mathscr{B}_h((\mathbf{u}, \mathbf{U}, p) - (\mathbf{v}_h, \mathbf{V}_h, q_h); (\mathbf{u}_h, \mathbf{U}_h, p_h) - (\mathbf{v}_h, \mathbf{V}_h, q_h))
\end{aligned}
$$

for all $(\mathbf{v}_h, \mathbf{V}_h, q_h) \in \mathscr{V}_h \times \mathscr{S}_h \times \mathscr{Q}_h$. Then, for $h < h_0$, it follows from (4.12), (4.22), and (2.11) that

$$
\begin{aligned}
&\nu^2\|\mathbf{u}_h - \mathbf{v}_h\|_{1,\Omega}^2 + \nu^2\|\mathbf{U}_h - \mathbf{V}_h\|_{1,\Omega}^2 + \|p_h - q_h\|_{1,\Omega}^2 \\
&\quad \leqslant C(\nu^2\|\mathbf{u} - \mathbf{v}_h\|_{1,\Omega}^2 + \nu^2\|\mathbf{U} - \mathbf{V}_h\|_{1,\Omega}^2 + \|p - q_h\|_{1,\Omega}^2)^{1/2} \\
&\qquad \times (\nu^2\|\mathbf{u}_h - \mathbf{v}_h\|_{1,\Omega}^2 + \nu^2\|\mathbf{U}_h - \mathbf{V}_h\|_{1,\Omega}^2 + \|p_h - q_h\|_{1,\Omega}^2)^{1/2}
\end{aligned}
$$

for all $(\mathbf{v}_h, \mathbf{V}_h, q_h) \in \mathscr{V}_h \times \mathscr{S}_h \times \mathscr{Q}_h$, and hence

$$
\begin{aligned}
&\nu\|\mathbf{u}_h - \mathbf{v}_h\|_{1,\Omega} + \nu\|\mathbf{U}_h - \mathbf{V}_h\|_{1,\Omega} + \|p_h - q_h\|_{1,\Omega} \\
&\quad \leqslant C(\nu\|\mathbf{u} - \mathbf{v}_h\|_{1,\Omega} + \nu\|\mathbf{U} - \mathbf{V}_h\|_{1,\Omega} + \|p - q_h\|_{1,\Omega})
\end{aligned}
$$

for all $(\mathbf{v}_h, \mathbf{V}_h, q_h) \in \mathscr{V}_h \times \mathscr{S}_h \times \mathscr{Q}_h$. Therefore, we get

$$\begin{aligned} &\nu\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \nu\|\mathbf{U} - \mathbf{U}_h\|_{1,\Omega} + \|p - p_h\|_{1,\Omega} \\ &\quad \leqslant \nu\|\mathbf{u} - \mathbf{v}_h\|_{1,\Omega} + \nu\|\mathbf{U} - \mathbf{V}_h\|_{1,\Omega} + \|p - q_h\|_{1,\Omega} \\ &\qquad + \nu\|\mathbf{u}_h - \mathbf{v}_h\|_{1,\Omega} + \nu\|\mathbf{U}_h - \mathbf{V}_h\|_{1,\Omega} + \|p_h - q_h\|_{1,\Omega} \\ &\quad \leqslant C(\nu\|\mathbf{u} - \mathbf{v}_h\|_{1,\Omega} + \nu\|\mathbf{U} - \mathbf{V}_h\|_{1,\Omega} + \|p - q_h\|_{1,\Omega}) \end{aligned}$$

for all $(\mathbf{v}_h, \mathbf{V}_h, q_h) \in \mathscr{V}_h \times \mathscr{S}_h \times \mathscr{Q}_h$. Now, choosing $\mathbf{v}_h \in \mathscr{V}_h$, $\mathbf{V}_h \in \mathscr{S}_h$, and $q_h \in \mathscr{Q}_h$ such that (3.4), (3.5), and (3.6) hold when $\mathbf{v}$, $\mathbf{V}$, and q are replaced by $\mathbf{u}$, $\mathbf{U}$, and p, respectively, we obtain estimate (4.23). This completes the proof. □

5. An H^1-equivalent error estimator

In this section we briefly introduce an $[H^1(\Omega)]^{d^2+d+1}$-equivalent error estimator. Let $(\mathbf{u}, \mathbf{U}, p) \in \mathscr{V} \times \mathscr{S} \times \mathscr{Q}$ and $(\mathbf{u}_h, \mathbf{U}_h, p_h) \in \mathscr{V}_h \times \mathscr{S}_h \times \mathscr{Q}_h$ be the solutions of problems (2.6) and (4.3), respectively. For each $h < h_0$, define

$$\begin{aligned} \mathscr{E}_h^2 &= \|\mathbf{f} - (-\nu(\nabla \cdot \mathbf{U}_h)^{\mathrm{T}} + \nabla p_h)\|_{0,\Omega}^2 + \nu^2\|\mathbf{0} - (\mathbf{U}_h - \nabla \mathbf{u}_h^{\mathrm{T}})\|_{0,\Omega}^2 \\ &\quad + \nu^2\|g - (\nabla \cdot \mathbf{u}_h + \delta p_h)\|_{0,\Omega}^2 + \nu^2\|\nabla g - (\nabla \mathrm{tr}\mathbf{U}_h + \delta \nabla p_h)\|_{0,\Omega}^2 \\ &\quad + \nu^2\|\mathbf{0} - (\nabla \times \mathbf{U}_h)\|_{0,\Omega}^2. \end{aligned}$$

Then $\mathscr{E}_h$ is a computable quantity, and

$$\begin{aligned} \mathscr{E}_h^2 &= \| - \nu(\nabla \cdot (\mathbf{U} - \mathbf{U}_h)^{\mathrm{T}}) + \nabla(p - p_h)\|_{0,\Omega}^2 + \nu^2\|(\mathbf{U} - \mathbf{U}_h) - \nabla(\mathbf{u} - \mathbf{u}_h)^{\mathrm{T}}\|_{0,\Omega}^2 \\ &\quad + \nu^2\|\nabla \cdot (\mathbf{u} - \mathbf{u}_h) + \delta(p - p_h)\|_{0,\Omega}^2 + \nu^2\|\nabla \mathrm{tr}(\mathbf{U} - \mathbf{U}_h) + \delta\nabla(p - p_h)\|_{0,\Omega}^2 \\ &\quad + \nu^2\|\nabla \times (\mathbf{U} - \mathbf{U}_h)\|_{0,\Omega}^2. \end{aligned}$$

Thus, from (2.11), we have

$$\begin{aligned} &C_1(\nu^2\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega}^2 + \nu^2\|\mathbf{U} - \mathbf{U}_h\|_{1,\Omega}^2 + \|p - p_h\|_{1,\Omega})^{1/2} \\ &\quad \leqslant \mathscr{E}_h \leqslant C_2(\nu^2\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega}^2 + \nu^2\|\mathbf{U} - \mathbf{U}_h\|_{1,\Omega}^2 + \|p - p_h\|_{1,\Omega})^{1/2}. \end{aligned}$$

Hence $\mathscr{E}_h$ is an $[H^1(\Omega)]^{d^2+d+1}$-equivalent a posteriori error estimator.

6. Concluding remarks

In this investigation, we study a finite volume scheme for approximating the solution of the generalized Stokes problem recast in the velocity–stress–pressure first-order system formulation by using continuous piecewise linear polynomials. This method is composed of a direct cell vertex finite volume discretization step and an algebraic least-squares step. An optimal rate of convergence in the H^1 norm for all the unknowns, velocity, stress, and pressure, is achieved.

The solution of the Navier–Stokes problem is sometimes approximated by a sequence of linear Stokes problems. Another often more adequate linearization of the Navier–Stokes equations is the Oseen problem [14] which offers an iterative process for the approximate solution. A promising continuous piecewise linear approach may be by considering the iterative process as follows. Consider the Navier–Stokes problem

$$\begin{aligned} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0 \quad & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} \quad & \text{on } \partial\Omega. \end{aligned} \tag{6.1}$$

Similar to system (2.6), formulating this nonlinear second-order problem as a first-order system in terms of $\mathbf{u}$, $\mathbf{U}$, and p, we have

$$\begin{aligned} -\nu(\nabla\cdot\mathbf{U})^{\mathrm{T}} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad & \text{in } \Omega, \\ \mathbf{U} - \nabla\mathbf{u}^{\mathrm{T}} = \mathbf{0} \quad & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0 \quad & \text{in } \Omega, \\ \nabla\mathrm{tr}\mathbf{U} = \mathbf{0} \quad & \text{in } \Omega, \\ \nabla\times\mathbf{U} = \mathbf{0} \quad & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} \quad & \text{on } \partial\Omega, \\ \mathbf{n}\times\mathbf{U} = \mathbf{0} \quad & \text{on } \partial\Omega. \end{aligned} \tag{6.2}$$

Next, let $(\mathbf{u}^0, \mathbf{U}^0, p^0)$ be the solution of the Stokes problem (2.6). For $n = 0, 1, 2, \ldots$, consider the following sequence of linear problems

$$\begin{aligned} -\nu(\nabla\cdot\mathbf{U}^{n+1})^{\mathrm{T}} + A^n\mathbf{u}^{n+1} + \nabla p^{n+1} = \mathbf{f} \quad & \text{in } \Omega, \\ \mathbf{U}^{n+1} - \nabla(\mathbf{u}^{n+1})^{\mathrm{T}} = \mathbf{0} \quad & \text{in } \Omega, \\ \nabla\cdot\mathbf{u}^{n+1} = 0 \quad & \text{in } \Omega, \\ \nabla\mathrm{tr}\mathbf{U}^{n+1} = \mathbf{0} \quad & \text{in } \Omega, \\ \nabla\times\mathbf{U}^{n+1} = \mathbf{0} \quad & \text{in } \Omega, \\ \mathbf{u}^{n+1} = \mathbf{0} \quad & \text{on } \partial\Omega, \\ \mathbf{n}\times\mathbf{U}^{n+1} = \mathbf{0} \quad & \text{on } \partial\Omega, \end{aligned} \tag{6.3}$$

where the coefficient matrix A^n is given by

$$A^n = \begin{bmatrix} \frac{\partial u_1^n}{\partial x_1} & \cdots & \frac{\partial u_1^n}{\partial x_d} \\ \vdots & & \vdots \\ \frac{\partial u_d^n}{\partial x_1} & \cdots & \frac{\partial u_d^n}{\partial x_d} \end{bmatrix}.$$

Now, for each stage n, $n \geqslant 0$, we solve the corresponding linear problem by using the finite volume scheme developed in Section 3. Since the velocity $\mathbf{u}$ is approximated by using piecewise linear polynomials with respect to the triangulation $\mathscr{T}_h$, the coefficient matrix A^n, $n \geqslant 0$, is a piecewise constant matrix at each stage. One might expect the sequence $\{(\mathbf{u}_h^n, \mathbf{U}_h^n, p_h^n)\}$ generated by the above iterative process converging to a solution of the velocity–stress–pressure Navier–Stokes problem (6.2) as $n \to \infty$ and $h \to 0$. This issue has become the subject of current research of the author.

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