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A Combinatorial Proof of the Log-Concavity of the Numbers of Permutations with k Runs

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We combinatorially prove that the number $R(n, k)$ of permutations of length n having k runs is a log-concave sequence in k, for all n. We also give a new combinatorial proof for the log-concavity of the Eulerian numbers. © 2000 Academic Press

1. INTRODUCTION

Let $p = p_1 p_2 \cdots p_n$ be a permutation of the set $\{1, 2, ..., n\}$ written in the one-line notation. We say that p get changes direction at position i, if either $p_{i-1} < p_i > p_{i+j}$, or $p_{i-1} > p_i > p_{i+1}$, in other words, when p_i is either a *peak* or a *valley*. We say that p has k runs if there are $k-1$ indices i so that p changes direction at these positions. So, for example, $p = 3561247$ has 3 runs as p changes direction when $i = 3$ and when $i = 4$. A geometric way to represent a permutation and its runs by a diagram is shown in Fig. 1. The runs are the line segments (or edges) between two consecutive entries where p changes direction. So a permutation has k runs if it can be represented by k line segments so that the segments go "up" and "down"

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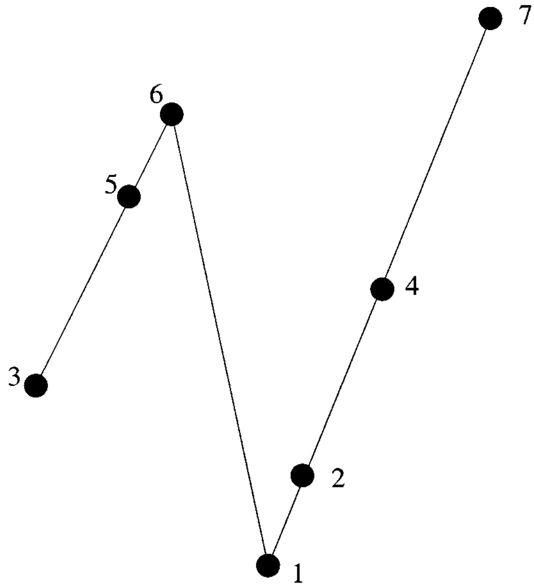

FIG. 1. The permutation 3561247 has three runs.

exactly when the entries of the permutation do. The theory of runs has been studied in [4, Sect. 5.1.3] in connection with sorting and searching.

In this paper, we study the numbers $R(n, k)$ of permutations of length n or, in what follows, n-permutations with k runs. We will show that for any fixed n, the sequence $R(n, k)$, $k = 0, 1, ..., n-1$ is log-concave, that is, $R(n, k-1) \cdot R(n, k+1) \leqslant R(n, k)^2$. In particular, this implies [1, 6] that this same sequence is unimodal, that is, there exists an m so that $R(n, 1) \leqslant R(n, 2) \leqslant \cdots \leqslant R(n, m) \geqslant R(n, m+1) \geqslant \cdots \geqslant R(n, n-1)$. We will also show that roughly half of the roots of the generating function $R_n(x) = \sum_{k=1}^{n-1} R(n, k)\, x^k$ are equal to -1, and give a combinatorial interpretation for the term which remains after one divides $R_n(x)$ by all the $(x+1)$ factors. While doing that, we will also give a new proof of the well-known fact [2, p. 292] that the Eulerian numbers are log-concave.

2. THE FACTORIZATION OF $R_n(x)$

Let $p = p_1 p_2 \cdots p_n$ be a permutation. We say that i is a *descent* of p if $p_i > p_{i+1}$, while we say that i is an *ascent* of p if $p_i < p_{i+1}$.

In our study of n-permutations with a given number of runs, we can clearly assume that i is an ascent of p. Taking the permutation $q = q_1 q_2 \cdots q_n$, where $q_i = n+1-p_i$, we get the *complement* of p, which has the same number of runs as p. This implies in particular that for any given i, there are as many n-permutations with k runs in which $p_i < p_{i+1}$ as there are such permutations in which $p_{i+1} > p_{i+1}$.

Let $R_n(x) = \sum_{k=1}^{n-1} R(n,k)\,x^k$ be the ordinary generating function of n-permutations by the number of runs. So we have $R_2(x) = 2x$, $R_3(x) = 2x + 4x^2$, and $R_4(x) = 10x^3 + 12x^2 + 2x$. One sees that all coefficients of $R_n(x)$ are even, which is explained by the symmetry described above.

It is easy to notice the phenomenon that $R_n(x)$ seems to be divisible by $(x+1)$ if $n \geqslant 4$. Further analysis of numerical data leads to the significantly stronger conjecture that $R_n(x)$ is divisible by $(x+1)^m$, where $m = \lfloor (n-2)/2 \rfloor$.

Our goal is to prove that conjecture, and also, to find a combinatorial interpretation for the polynomial obtained after dividing $R_n(x)$ by the highest possible power of $(x+1)$. For that purpose, we introduce the following definition.

DEFINITION 2.1. For $j \leqslant m = \lfloor (n-2)/2 \rfloor$, we say that p is a *j-half-ascending permutation* if, for all positive integers $i \leqslant j$, we have $p_{n+1-2i} < p_{n+2-2i}$. If $j = m$, then we will simply say that p is a *half-ascending permutation.*

So p is a 1-half-ascending permutation if $p_{n-1} < p_n$. In a j-half-ascending permutation, we have j relations, and they involve the rightmost j disjoint pairs of entries. The term half-ascending refers to the fact that at least half of the involved positions are ascents. There are $n! \cdot 2^{-j}$ j-half-ascending permutations.

Now we define a modified version of the polynomials $R_n(x)$ for j-half-ascending permutations.

DEFINITION 2.2. Let p be a $j+1$-half-ascending permutation. Let $r_j(p)$ be the number of runs of the substring $p_1, p_2, \ldots, p_{n-2j}$, and let $s_j(p)$ be the number of descents of the substring $p_{n-2j}, p_{n+1-2j}, \ldots, p_n$. Denote $t_j(p) = r_j(p) + s_j(p)$ and define

$$R_{n,j}(x) = \sum_{p \in S_n} x^{t_j(p)}.$$

In particular, we will denote $R_{n,m}(x)$ by $T_n(x)$, that is, $T_n(x)$ is the generating function for half-ascending permutations.

So in other words, we count the runs in the non-half-ascending part and the first two elements of the half-ascending part, and we count the descents in the rest of the half-ascending part (and on that part, as it will be discussed, the number of descents determines that of runs.) Let I be the involution interchanging p_{n-1} and p_n. Now we are in a position to state and prove the main result of this section.

LEMMA 2.3. *For all $n \geqslant 4$ and $1 \leqslant j \leqslant m$, we have*

$$\frac{R_n(x)}{2(x+1)^j} = R_{n,j}(x),$$

where $m = \lfloor (n-2)/2 \rfloor$.

Proof. By induction on *j*. For $j = 1$, the statement says that $R_n(x)/(x+1) = 2R_{n,1}(x)$.

Take all permutations in which $p_{n-3} < p_{n-2}$. The generating function of these by the number of runs is $R_n(x)/2$. Involution I makes pairs of permutations, and each pair contains two elements whose numbers of runs differ by 1. Dividing $R_n(x)/2$ by $(x+1)$ we obtain the run-generating function for the set of permutations which contains one element of each of these pairs, namely, the one having the smaller number of runs. Observe that for these permutations, the number of runs is equal to the value of $t_1(p)$ for the permutation in that, pair in which $p_{n_1} < p_n$ (by checking both possibilities $p_{n-2} < p_{n-1}$ and $p_{n-2} > p_{n-1}$; see the following example), so $R_n(x)/2(x+1) = R_{n,1}(x)$.

We point out that it is not true in general that in each pair made by I, the permutation having the smaller number of runs is the one with $p_{n-1} < p_n$. What is true is that we can *suppose* that $p_{n-1} < p_n$ if we count permutations by the defined parameter $t_1(p)$ instead of the number of runs. This latter could be viewed as the $t_0(p)$ parameter. Before turning to the induction step, the reader may want to study the following example.

EXAMPLE 2.4. If $n = 4$, then we have 6 permutations in which $p_3 < p_4$ and $p_1 < p_2$: 1234, 1324, 1423, 2314, 2413, 3412. We have $t_1(1234) = 1$ and $t_1(p) = 2$ for all the other five permutations, showing that indeed, that indeed, $R_{4,1}x = 2(5x^2 + x)$. The images of these six permutations by I are, respectively, 1243, 1342, 1432, 2341, 2431, 3421, and one verifies that in each of these pairs, the permutation with the smaller number of runs has a number of runs equal to the $t_1(p)$-value of the element of that pair in which $p_{n-1} < p_n$. This argument carries over for $n > 4$, too, for it is only the last four elements where the number of runs can be affected by I.

Now suppose we know the statement for $j-1$ and prove it for *j*. As above, apply I to the two rightmost entries of our permutations to get pairs as in the initial case, and apply the induction hypothesis to the leftmost $n-2$ elements. By the induction hypothesis, the string of the leftmost $n-2$ elements can be replaced by a *j*-half-ascending $n-2$-permutation, and the number of runs can be replaced by the t_{j-1}-parameter. In particular, $p_{n-3}<p_{n-2}$ will hold, and therefore we can verify that our statement holds in both cases ($p_{n-2}<p_{n-1}$ or $p_{n-2}>p_{n-1}$) exactly as we did in the proof of the initial case, and the previous example. ▮

Thus in particular, we have

$$\frac{R_n(x)}{2(x+1)^m}=T_n(x),$$

so we have proved that $m=\lfloor (n-2)/2\rfloor$ of the roots of $R_n(x)$ are equal to -1, and certainly, one other root is equal to 0 as all permutations have at least one run. It is possible to prove analytically [2, Sect. 7.1, Theorem B] that the other half of the roots of $R_n(x)$, that is, the roots of $T_n(x)$, are all real, negative, and distinct. That implies [6] that the coefficients of $R_n(x)$ and $T_n(x)$ are log-concave.

However, in the next section we will *combinatorially* prove that the coefficients of $T_n(x)$ form a log-concave sequence. Let $U(n,k)$ be the coefficient of x^k in $T_n(x)$. Let $\mathscr{U}(n,k)$ be the set of half-ascending permutations with k descents, so $|\mathscr{U}(n,k)|=U(n,k)$.

Now suppose for simplicity that n is even and assume that p is a half-ascending permutation, that is, $p_{2i-1}<p_{2i}$ for all i, $1\leqslant i\leqslant n/2$. The following proposition summarizes the different ways we can describe the same parameter of p.

PROPOSITION 2.5. *Let p be a half-ascending permutation. Then p has $2k+1$ runs if and only if p has k descents, or, in other words, when $t(p)=k+1$.*

If n is odd, then the rest of our argument is a little more tedious, though conceptually not more difficult. We do not want to break the course of our proof here, so we will go on with the assumption that n is even, then, in the second part of the proof of Theorem 4.2, we will indicate what modifications are necessary to include the case of odd n.

So in order to prove that the sequence $R(n,k)$ is log-concave in k, we need to prove that the sequence $U(n,k)$ enumerating half-ascending n-permutations with k descents is log-concave. That is sufficient as the convolution of two log-concave sequences is log-concave [6].

3. A LATTICE PATH INTERPRETATION

Following [3], we will set up a bijection from the set $\mathscr{A}(n,k)$ of n-permutations with k descents onto that of labeled northeastern lattice paths with n edges, exactly k of which are vertical. However, our lattice paths will be different from those in [3]; in particular, they will preserve the information if the position i is an ascent or descent.

Let $\mathscr{P}(n)$ be the set of labeled northeastern lattice paths with the n edges $a_1, a_2, ..., a_n$ and the corresponding positive integers as labels $e_1, e_2, ..., e_n$ so that the following hold:

(1) the edge a_1 is horizontal and $e_1 = 1$,

(2) if the edges a_i and a_{i+1} are both vertical, or both horizontal, then $e_i \geqslant e_{i+1}$,

(3) if a_i and a_{i+1} are perpendicular to each other, then $e_i + e_{i+1} \leqslant i+1$.

We will not distinguish between paths which can obtained from each other by translations. Let $\mathscr{P}(n, k)$ be the set of all such labeled lattice paths which have k vertical edges, and let $P(n, k) = |\mathscr{P}(n, k)|$.

PROPOSITION 3.1. *The following two properties of paths in $\mathscr{P}(n)$ are immediate from the definitions.*

- *For all $i \geqslant 2$, we have $e_i \leqslant i-1$.*
- *Fix the label e_i. Then if e_{i+1} can take value v, then it can take all positive integer values $w \leqslant v$.*

Also note that all restrictions on e_{i+1} are given by e_i, independently of preceding e_j, $j < i$. The following bijection is the main result in this section.

THEOREM 3.2. *The following description defines a bijection from $\mathscr{A}(n)$ onto $\mathscr{P}(n)$, where $\mathscr{A}(n)$ is the set of all n-permutations. Let $p \in \mathscr{A}(n)$. To obtain the edge a_i and the label e_i for $2 \leqslant i \leqslant n$, restrict the permutation p to the i first entries and relabel the entries to obtain the permutation $q = q_1 \cdots q_i$.*

- *If the position $i-1$ is a descent of the permutation p (equivalently, of the permutation q), let the edge a_i be vertical and the label e_i be equal to q_i.*
- *If the position $i-1$ is an ascent of the permutation p, let the edge a_i be horizontal and the label e_i be $i+1-q_i$.*

Moreover, this bijection restricts naturally to a bijection between $\mathscr{A}(n, k)$ and $\mathscr{P}(n, k)$ for $0 \leqslant k \leqslant n-1$.

Proof. It is straightforward to see that the map described is injective. Assume that $i-1$ and i are both descents of the permutation p. Let q, respectively r, be the permutation when restricted to the i, respectively

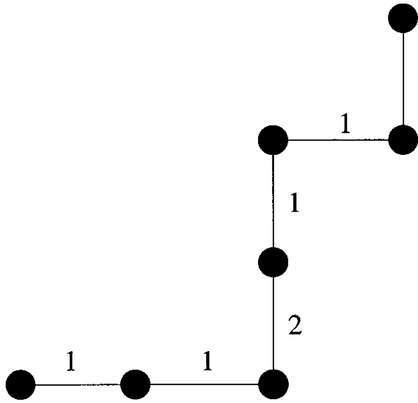

FIG. 2. The image of the permutation 243165.

$i+1$, first elements. Observe that q_i is either r_i or $r_i - 1$. Since $r_i > r_{i+1}$ we have $q_i \geq r_{i+1}$ and condition (2) is satisfied in this case. By similar reasoning the three remaining cases are shown, hence the map is into the set $\mathscr{P}(n)$.

To see that this is a bijection, we show that we can recover the permutation p from its image. It is sufficient to show that we can recover p_n, and then use induction on n for the rest of p. To recover p_n from its image, simply recall that p_n is equal to the label l of the last edge if that edge is vertical, and to $n+1-l$ if that edge is horizontal. Conditions (2) and (3) assure that this way we always get a number between 1 and n for p_n. ∎

The lattice path corresponding to the permutation 243165 is shown on Fig. 2.

The difference between our bijection and that of [3] is that in ours, the direction of a_i tells us whether p_{i-1} is a descent in p. This is why we can use this bijection to gain information on the class of half-ascending permutations.

COROLLARY 3.3. *The bijection in Theorem 3.2 restricts to a bijection from $\mathscr{U}(n, k)$ to lattice paths in $\mathscr{P}(n, k)$ where a_i is horizontal for all even indices i.*

4. THE LOG-CONCAVITY OF $U(n, k)$

In this section we give a new proof of the fact that the numbers $A(n, k) = |\mathscr{A}(n, k)|$ are unimodal in k, for any fixed n. This fact is already known and has an elegant proof [3]. However, our proof will also show the unimodality of the $U(n, k)$.

THEOREM 4.1. *For all positive integers* n *and all positive integers* $k \leqslant n-1$ *we have*

$$A(n, k-1) \cdot A(n, k+1) \leqslant A(n, k)^2$$

and also

$$U(n, k-1) \cdot U(n, k+1) \leqslant U(n, k)^2.$$

Proof. To prove the theorem combinatorially, we construct an *injection*

$$\Phi\colon \mathscr{P}(n, k-1) \times \mathscr{P}(n, k+1) \to \mathscr{P}(n, k) \times \mathscr{P}(n, k).$$

This injection Φ will be defined differently on different parts of the domain.

In particular, the restriction of Φ onto $\mathscr{V}(n, k-1) \times \mathscr{V}(n, k+1)$ will map into $\mathscr{V}(n, k) \times \mathscr{V}(n, k)$, where $\mathscr{V}(n, k)$ is the subset of $\mathscr{P}(n, k)$ consisting of lattice paths in which a_i is horizontal for all even i.

Let $(P, Q) \in \mathscr{P}(n, k-1) \times \mathscr{P}(n, k+1)$. Place the initial points of P and Q at $(0, 0)$ and $(1, -1)$, respectively. Then the endpoints of P and Q are $(n-k+1, k-1)$ and $(n-k, k)$, respectively, so P and Q intersect. Let X be their *first* intersection point (we order intersection points from southwest to northeast) and decompose $P = P_1 \cup P_2$ and $Q = Q_1 \cup Q_2$, where P_1 is a path from $(0, 0)$ to X, P_2 is a path from X to $(n-k, k)$, P_1 is a path from $(1, -1)$ to X, and Q_2 is a path from X to $(n-k+1, k-1)$. Let a, b, c, d be the labels of the four edges adjacent to X as shown in Fig. 4, the edges AX and XB originally belonging to P and the edges CX and XD originally belonging to Q. Then by condition (2) we have $a \geqslant b$ and $c \geqslant d$. (It is possible that these four edges are not all distinct; A and C are always distinct as X is the first intersection point, but it could be, that $B = D$ and so $BX = DX$; this singular case can be treated very similarly to the generic case we describe below and is hence omitted.) Let $P' = P_1 \cup Q_2$ and let $Q' = Q_1 \cup P_2$.

- If P' and Q' are valid paths, that is, if their labeling fulfills conditions (1)–(3), then we set $\Phi(P, Q) = (P', Q')$. See Fig. 3 for this construction. This way we have defined Φ for pairs $(P, Q) \in \mathscr{P}(n, k) \times \mathscr{P}(n, k)$ in which $a + d \leqslant i$ and $b + c \leqslant i$, where $i - 1$ is the sum of the two coordinates of X. We also point out that as we haven't changed any labels, in (P', Q') we still have $a \geqslant b$ and $c \geqslant d$ though it is no longer required as the edges in question are no longer parts of the same path.

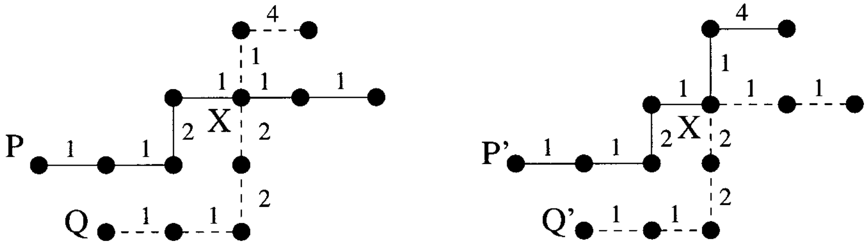

FIG. 3. Constructing the new pair of paths.

It is clear that $\Phi(P, Q) = (P', Q') \in \mathscr{P}(n, k) \times \mathscr{P}(n, k)$ (in particular, (P', Q') belongs to the subset of $\mathscr{P}(n, k) \times \mathscr{P}(n, k)$ consisting of *intersecting* pairs of paths), and that Φ is one-to-one.

• What remains to do is to define $\Phi(P, Q)$ for those $(P, Q) \in \mathscr{P}(n, k-1) \times \mathscr{P}(n, k+1)$ for which it cannot be defined this way, that is, when either $a + d > i$ or $b + c > i$ holds.

Change the label of the edge AX to $i - c$ and change the label of the edge CX to $i - a$ as seen in Fig. 5, then proceed as in the previous case to get $\Phi(P, Q) = (P', Q')$, where $P' = P_1 \cup Q_2$ and let $Q' = Q_1 \cup P_2$.

We claim that P' and Q' are valid paths. Indeed we had at least one of $a + d > i$ and $b + c > i$, so we must have $a + c > i$ as $a \geqslant b$ and $c \geqslant d$. Therefore, $i - a < c$ and $i - c < a$, so we have decreased the values of the labels of edges AX and CX, and that is always possible as showed in Proposition 3.1. Moreover, no constraints are violated in P' and Q' by the edges adjacent to X as $i - c + d \leqslant i$ and $i - a + b \leqslant i$. It is also clear that Φ is one-to-one on this part of the domain, too. Finally, we have to show that the image of this part of the domain is disjoint from that of the previous part. This is true because in this part of the domain we have at least one of $a + d > i$ and $b + c > i$, that is, at last one of $i - c < b$ and $i - a < d$, so in the image, at least one of the pairs of edges AX, XB and CX, XD does not

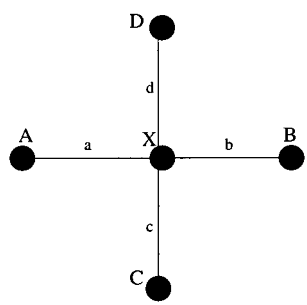

FIG. 4. Labels around the point X.

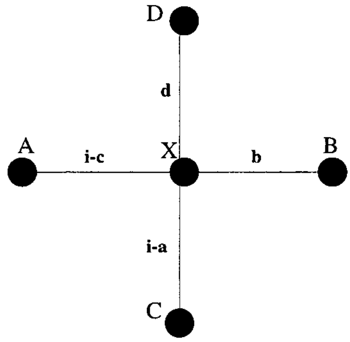

FIG. 5. Changed labels around the point X.

have the property that the label of the first edge is at least as large as that of the second one. And, as pointed out in the previous case, all elements of the image of the previous part of the domain do have that property.

To prove that the sequence $\{U(n, k)\}$ is log-concave, recall that half-ascending permutations in $\mathscr{U}(n, k)$ correspond to elements of $\mathscr{V}(n, k)$, that is, elements of $\mathscr{P}(n, k)$ in which all edges a_i are horizontal if i is even. We point out that this implies $B = D$. Then note that Φ does not change the indices of the edges, in other words, if $\Phi(P, Q) = (P', Q')$, and a given edge northeast from X was the ith edge of path P, then it will be the ith edge of path Q'. Therefore, Φ preserves the property that all even-indexed edges are horizontal, so the restriction of Φ into $\mathscr{V}(n, k-1) \times \mathscr{V}(n, k+1)$ maps into $\mathscr{V}(n, k) \times \mathscr{V}(n, k)$. As any restriction of Φ is certainly one-to-one, this proves that $U(n, k-1) \cdot U(n, k+1) \leqslant U(n, k)^2$. ∎

Now we are in a position to prove the main result of this paper.

THEOREM 4.2. *The polynomial $R_n(x)$ has log-concave coefficients, for all positive integers n.*

Proof. First suppose that n is even. For $n \leqslant 3$, the statement is true. If $n \geqslant 4$, then Lemma 2.3 shows that $R_n(x) = 2 \cdot (x+1)^m\, T_n(x)$. The coefficients of $(x+1)^m$ are just the binomial coefficients, which are clearly log-concave, while the coefficients of $T_n(x)$ are the $U(n, k)$, which are log-concave by Theorem 4.1. As the product of two polynomials with log-concave coefficients has log-concave coefficients [6], the proof is complete for n even.

If n is odd, then the equivalent of Proposition 2.5 is a bit more cumbersome. Again, we make use of symmetry by taking complements, but instead of assuming $p_1 < p_2$, let us assume that $p_2 < p_3$. Taking $R_{n,m}(x)$ then adds the restrictions $p_4 < p_5$, $p_6 < p_7$, ..., $p_{n-1} < p_n$. Then it is straightforward from the definition of $t_m(p)$ that $t_m(p) = d(p)$ where $d(p)$ is the number of descents of p, and we say, for simplicity, that the singleton p_1 has 0 runs.

So for odd n we have $T_n^{odd}(x) = \sum_{p \in S_n, p_2 < p_3} x^{t_m(p)} = \sum_{p \in S_n, p_2 < p_3} x^{d(p)}$, and then, in order to see that the coefficients of $T_n^{odd}(x)$ are log-concave, we can repeat the argument of Theorem 4.1. Indeed, the coefficient of x^k in $T_n^{odd}(x)$ equals the cardinality of $\mathscr{V}(n, k)$, the subset of $\mathscr{P}(n, k)$ in which the edges $a_3, a_5, ..., a_7$ are horizontal. And the fact that the $|\mathscr{V}(n, k)|$ are log-concave can be proved exactly as the corresponding statement for the $|\mathscr{V}(n, k)| = U(n, k)$, that is, by taking the relevant restriction of Φ.

This completes the proof of the theorem for all n. ∎

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