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On the Uniqueness of the Coefficient Ring in a Polynomial Ring

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If k is a field and X and Y are indeterminates then the statement "consider $R = k[X, Y]$ as a polynomial ring in one variable" is ambiguous, for there are infinitely many possible choices for the ring of coefficients (e.g., If $A_n = k[X + Y^n]$ then $A_n[Y] = A_m[Y] = R$ but $A_n \neq A_m$ if $m \neq n$). On the other hand, if Z denotes the integers then the polynomial ring $Z[X]$ has a unique subring over which it is a polynomial ring. This investigation began with our consideration of the first of these examples. In fact, Coleman had asked: If k is a field, then although $k[X, Y]$ can be written as a polynomial ring in many different ways, is it true that all of the possible coefficient rings are isomorphic? That is, if T is transcendental over A and $A[T] = k[X, Y]$, is A a polynomial ring over k? We found that this is indeed the case (see our (2.8)). We next proved the following: *If A is a one dimensional affine domain over a field and B is a ring such that $A[X] = B[Y]$ is an equality of polynomial rings, then either $A = B$ or there is a field k such that each of A and B is a polynomial ring in one variable over k.* This is a corollary of (3.3) in the present paper. Our (7.7) sketches a version of the original proof. In studying this argument, we found that there were implicit in it techniques for investigating the following general question: *Suppose A and B are commutative rings with identity and the polynomial rings $A[X_1, ..., X_n]$ and $B[Y_1, ..., Y_n]$ are isomorphic, how are A and B related? Are A and B isomorphic? In particular, when does the given isomorphism take A onto B?* This study is mainly centered on the latter portion of the question. We are concerned almost entirely with domains It is convenient to use the following terminology which is modeled after that introduced in [6]. We say that the ring A is ***invariant*** provided A satisfies the following condition: given a ring B and indeterminates $X_1,..., X_n, Y_1,..., Y_n$, if $A[X_1,..., X_n]$ is isomorphic to $B[Y_1,..., Y_n]$ then A is isomorphic to B. If for any ring B and any isomorphism $\phi : A[X_1,..., X_n] \to B[Y_1,..., Y_n]$, $\phi(A) = B$, then A is said to be ***strongly invariant***.

This article is divided into seven sections. The first of these is devoted to general considerations. In it we first observe that in attempting to decide if A is invariant (strongly invariant) it is always sufficient to assume the equality of polynomial rings $A[X_1,..., X_n] = B[Y_1,..., Y_n]$ and then try to show that A is isomorphic to (equal to) B. With this in mind we have adopted the convention that when we write $A[X_1,..., X_n]$ and $B[Y_1,..., Y_n]$ with no qualification, it is to be assumed that A and B are commutative rings with identity and the X's and Y's are indeterminates over A and B, respectively. With this convention our (1.1) reads: *If* $A[X_1,..., X_n] = B[Y_1,..., Y_n]$ *and* $A \subset B$. *then* $A = B$. One of the ways in which we implement (1.1) is through the idea of ***vertical ideals***. An ideal I of $A[X_1,..., X_n] = R$ is said to be ***vertical relative to*** A if there is an ideal $\mathfrak{A} \subset A$ such that $I = \mathfrak{A}R$. If $A[X_1,..., X_n] = B[Y_1,..., Y_n] = R$ then some ideals of R may be vertical relative to both A and B. We say that an ideal $\mathfrak{A} \subset A$ is ***vertical relative to*** B if $\mathfrak{A}R$ is vertical relative to B. We show that if each prime ideal of A is vertical relative to B then $A \subset B$. Thus $A = B$ by (1.1). In many cases it is sufficient to have only a few primes of R which are vertical relative to both A and B. In the case of domains our (1.15) guarantees that if the prime $p \subset R$ is minimal with respect to the property that R_p is not integrally closed, then it must be vertical relative to both A and B. We use this result in later sections to prove that a large class of nonnormal one-dimensional domains are strongly invariant [see (3.2) and (6.6)].

In the domain case, a very useful invariant is R_c, the algebraic closure in R of the subring generated by the units of R. In (1.9) we observe that if A is a domain such that $A = A_c$ then A is strongly invariant.

In Section 2 we restrict our attention to subrings of affine rings. An interesting result is (2.3) which says that if A is an affine domain over a field k, then any subfield of A is algebraic over k. Thus $\bar{k}$, the algebraic closure of k in A is the unique maximal subfield of A. An immediate corollary of this is the fact that if G is a finite group of automorphisms which acts on an affine domain over a field, then the fixed ring A_G is an affine ring over a field [see Remark (2.4)].

It is useful to have criteria which guarantee that a ring is a polynomial ring. One such result is our (2.11): *Suppose k is a field and k^* is a separable algebraic extension of k. If A is a one dimensional normal ring such that*

$$k \subseteq A \subseteq k^*[X_1,..., X_m]$$

then A is a polynomial ring over a field. In (2.12) we give an example to show that the separability assumption cannot be deleted, even in case k^* is finite algebraic over k. An immediate corollary of (2.11) is the fact that if k is a field then $k[X]$ is invariant (it is not strongly invariant since $k[X, Y] = k[X][Y] = k[Y][X]$). Another useful result in Section 2 is (2.13):

Let R be an integrally closed domain with quotient field K and let R^* be the integral closure of R in an algebraic field extension K^* of K. Let P be a prime ideal in R. If each prime ideal P^* in R^* lying over P is the radical of a principal ideal in R^*, then the P^* are finite in number and P is the radical of a principal ideal in R.

From this we get Corollary (2.14) which implies that if R is an integrally closed domain and R^* is an integral extension of R which is a prefactorial Krull ring, then R is a prefactorial Krull ring.

Section 3 is mainly devoted to proving (3.3):

Let A be an integral domain of transcendence degree one over a subfield k. Suppose $A[X_1, \ldots, X_n] = R = B[Y_1, \ldots, Y_n]$ and let k' denote the algebraic closure of k in A. If $A \not\approx B$, then A and B are both polynomial rings over the field k'. Consequently A is invariant and if A is not a polynomial ring, then A is strongly invariant.

In the affine case this theorem has a geometric interpretation which yields insight into the problem and motivates some of the terminology introduced in Section 1. Prior to any geometric discussion, however, we should say that while geometric considerations have influenced our approach and colored our terminology, all of our theorems and arguments are stated in purely ring theoretic terms. If A is a one-dimensional affine domain over a field k, then one can regard A as the coordinate ring of some irreducible affine curve Γ. Since $A[X] = A \otimes_k k[X]$, we can view $A[X]$ as the coordinate ring of a cylinder over Γ. A visual representation is presented in Fig. 1.

We think of the points P of Γ as the maximal ideals p of A, and the irreducible curves on the cylinder as the height one primes of $A[X]$. In this context the vertical line L of Fig. 1 which meets Γ at P_0 corresponds to the height one prime $p_0[X] \subset A[X]$. For this reason we refer to primes of $A[X]$ which are of the form $p[X]$ for some prime p in A as the *vertical* primes of $A[X]$.

The irreducible curves τ on the cylinder which are not vertical primes arise from height one primes of $A[X]$ which meet A only in the zero ideal.

If B is the coordinate ring of the curve Δ then the existence of an isomorphism $A[X] \to B[Y]$ corresponds to a reversible mapping ϕ from the cylinder over Γ onto the cylinder over Δ such that the coordinate mappings of ϕ and ϕ^{-1} are given by polynomials. (See Fig. 2).

To say that A is invariant is to say that the existence of ϕ implies the existence of a reversible mapping $\beta : \Gamma \to \Delta$ such that the coordinate mappings of β and β^{-1} are given by polynomials. To say that A is strongly invariant is

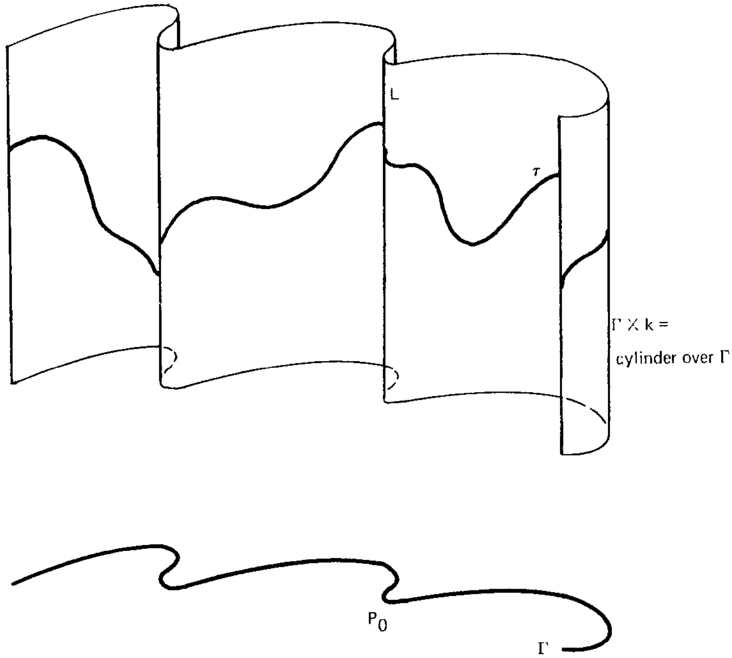

FIGURE 1

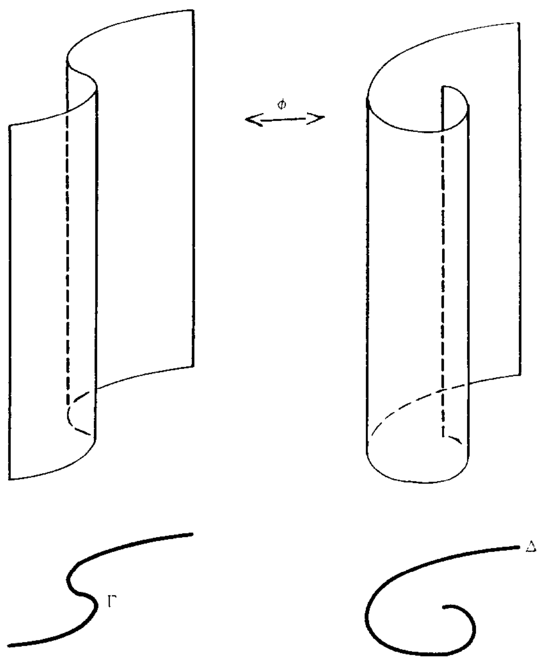

FIGURE 2

to say that the mapping ϕ necessarily takes "vertical lines onto vertical lines" and that ϕ followed by a "vertical shift and a rotation" will take Γ onto Δ.

Section 3 closes with an application of (3.4) to classifying certain derivations of $k[X, Y]$, in case k is a field of characteristic zero.

In Section 4 we attempt to generalize some of Section 3 by considering integral domains of transcendence degree one over a subring, rather than over a field. In (4.1) we show that if A is a unique factorization domain (UFD) and if D is a UFD such that (1) $A \subset D \subset A[X_1, ..., X_n]$ and (2) D is of transcendence degree one over A, then D is a polynomial ring over A. We provide an example then to show that the unique factorization hypothesis cannot be relaxed, even to the assumption that A is a Dedekind domain with finite class group.

In Section 4 we introduce the notions of *D-invariance* and *strong D-invariance.* If D is a ring and A a ring which is a D-algebra, then A is said to be D-invariant provided A satisfies the following condition: given a D-algebra B and indeterminates $X_1, ..., X_n$, $Y_1, ..., Y_n$, if $A[X_1, ..., X_n]$ is D-isomorphic to $B[Y_1, ..., Y_n]$ then A is D-isomorphic to B. If for any D-algebra B and any D-isomorphism $\phi : A[X_1, ..., X_n] \to B[Y_1, ..., Y_n]$, $\phi(A) = B$, then A is said to be strongly D-invariant. Our (4.1) referred to above implies that if D is a UFD then $D[Z]$ is D-invariant.

This notion of D-invariance has a nice local–global property which we state as (4.6). *Let D be an integral domain and let Z be an indeterminate over D. If for each prime ideal p of D, $D_p[Z]$ is D_p-invariant, then $D[Z]$ is D-invariant.* We close section 4 by showing that a theorem similar to (4.1) holds for HCF (highest common factor) rings (domains whose group of divisibility is a lattice-ordered group), and deriving corollaries similar to those of (4.1).

In Section 5 we study one-dimensional domains which are not strongly invariant. We show that such rings must be very closely related to polynomial rings, but we are not able to decide whether a one-dimensional domain which is not strongly invariant must be a polynomial ring. In (5.1) we find: *Let A be a one-dimensional integrally closed domain, If $A[X_1, ..., X_n] = B[Y_1, ..., Y_n]$ and $A \neq B$, then there exists an element s in A such that $A[1/s]$ is a prefactorial Dedekind domain containing a field over which A is of transcendence degree one. In fact there exist fields $k' \subset k$ with k algebraic over k' such that $k' \subset A[1/s] \subset k[T, 1/s]$ and such that $k[T, 1/s]$ is integral over $A[1/s]$.*

In case A is a locally finite intersection of valuation rings we are able to sharpen this result to conclude that there is a $u \in A \cap B$ such that $k = (A \cap B)[1/u]$ is a field and both $A[1/u]$ and $B[1/u]$ are polynomial rings over k. [This is stated as (5.4)].

From (5.4) we are led to consider domains A which contain an element u such that $A[1/u]$ is a polynomial ring over a field. In (5.7) we establish that if A is a Krull domain such that for some element $u \in A$, $A[1/u] = k[T]$, a polynomial ring over a field, then A is noetherian and of dimension one or two. In this case we are able to give necessary and sufficient conditions that A be one dimensional (i.e., that A be a Dedekind domain). These conditions are stated in terms of the essential valuations of the element u.

In Section 6 we prove that *any one-dimensional noetherian domain which contains a field of characteristic zero is either strongly invariant or a polynomial ring over a field.*

Section 7 is devoted mainly to articulating some of the questions which we haven't been able to answer. In particular, Question (7.1) asks: *If A and B are integral domains and $A[X_1, ..., X_n] = B[Y_1, ..., Y_n]$, does it follow that the quotient fields of A and B are isomorphic?* We observe that this is related to an unanswered question of Zariski. Our (7.4), (7.5) and (7.6) indicate an approach to Question (7.1) using techniques of Nagata and Abhyankar which have been applied to the Zariski problem.

1. Definitions, Conventions, and Some General Observations

A ring A is said to be *invariant* provided A satisfies the following condition: given a ring B and indeterminates $X_1, ..., X_n, Y_1, ..., Y_n$, if $A[X_1, ..., X_n]$ is isomorphic to $B[Y_1, ..., Y_n]$, then A is isomorphic to B. If for any ring B and any isomorphism $\phi : A[X_1, ..., X_n] \to B[Y_1, ..., Y_n]$, $\phi(A) = B$ then A is said to be *strongly invariant*. This is an extension to polynomial rings in n variables of the terminology introduced by Coleman and Enochs [6] for the case of polynomial rings in one variable. Here we are concerned only with the commutative case; thus in the sequel all rings are assumed to be commutative rings with identity. If R is a ring, M is an R module and Π is some property we say M has the property Π *locally* if $M_p = M \otimes R_p$ has the property Π for every prime p of R. To show that A is invariant (strongly invariant) it clearly suffices to show the following: if B is a subring of $A[X_1, ..., X_n]$ such that $A[X_1, ..., X_n]$ is a polynomial ring in n variables over B, then A is isomorphic to (equal to) B. In view of this fact, we will for conceptual simplicity in most of our presentation assume that $A[X_1, ..., X_n] = B[Y_1, ..., Y_n]$ rather than work through the given isomorphism.

We begin with some elementary observations. When we write $A[X_1, ..., X_n]$ and $B[Y_1, ..., Y_n]$ with no qualification it is to be assumed that A and B are commutative rings with identity and the X's and Y's are indeterminates over A and B, respectively,

(1.1) If $A[X_1, ..., X_n] = B[Y_1, ..., Y_n]$ and $A \subset B$, then $A = B$.

Proof. We have $B[Y_1, ..., Y_n] = B[X_1, ..., X_n]$ and the only thing to be seen is that $X_1, ..., X_n$ are algebraically independent over B. If B is noetherian, then this is clear, for the B-homomorphism of $B[Y_1, ..., Y_n]$ onto itself defined by mapping Y_i to X_i can certainly not, in this case, have a nonzero kernel (iteration of this homomorphism would yield a strictly ascending chain of ideals of $B[Y_1, ..., Y_n]$). In the general case if f is a polynomial in $B[Y_1, ..., Y_n]$ such that $f(X_1, ..., X_n) = 0$, then write the X's as polynomials in the Y's, the Y's as polynomials in the X's and adjoin the coefficients of these polynomials along with the coefficients of f to the prime subring of B. This yields a noetherian subring B' of B such that $f \in B'[Y_1, ..., Y_n]$ and $B'[Y_1, ..., Y_n] = B'[X_1, ..., X_n]$. By the noetherian case, $f(X_1, ..., X_n) = 0$ implies f is the zero polynomial. Hence $X_1, ..., X_n$ are algebraically independent over B and the proof of (1.1) is complete.

(1.2) Suppose $A[X_1, ..., X_n] = B[Y_1, ..., Y_n]$ and S is a multiplicative system of $A \cap B$ consisting of regular elements of A. If A_S is strongly invariant, then $A = B$. (Note that if $s \in A \cap B$ is not a zero divisor in A, then s is not a zero divisor in $A[X_1, ..., X_n]$ and hence not a zero divisor in B.)

An ideal I of $A[X_1, ..., X_n] = R$ is said to be *vertical relative to A* if there exists an ideal $\mathfrak{A}$ of A such that $\mathfrak{A}R = \mathfrak{A}[X_1, ..., X_n] = I$. Thus I is vertical relative to A precisely when I is the extension of an ideal of A and this, of course, is equivalent to $I = (I \cap A)R$. We note that vertical ideals behave well under infinite intersections—a property which is not true for extended ideals in an arbitrary ring extension.

(1.3) *If $R = A[X_1, ..., X_n]$ and $\{I_\alpha\}$ is a set of ideals of R which are vertical relative to A, then $\bigcap_\alpha I_\alpha$ is also vertical relative to A.*

Proof. Let $\mathfrak{A}_\alpha = I_\alpha \cap R$. Then

$$\bigcap_\alpha I_\alpha = \bigcap_\alpha \mathfrak{A}_\alpha[X_1, ..., X_n] = \left(\bigcap_\alpha \mathfrak{A}_\alpha\right)[X_1, ..., X_n] = \left(\bigcap_\alpha \mathfrak{A}_\alpha\right) R.$$

(1.4) If I is an ideal of $R = A[X_1, ..., X_n]$ which is vertical relative to A, then every prime ideal of R minimal with respect to the property of containing I (i.e., minimal prime divisor of I) is also vertical relative to A.

Proof. Since for any prime ideal P of R, $(P \cap A)R$ is also a prime ideal of R, we see that P is a minimal prime divisor of I if and only if $P \cap A$ is a minimal prime divisor of $I \cap A$.

(1.5) Let A be an integral domain with quotient field K and let A^* be the integral closure of A in an algebraic field extension L of K. Let $R = A[X_1, ..., X_n]$. Then we have the following:

(i) $A^*[X_1, ..., X_n] = R^*$ is the integral closure of R in $L(X_1, ..., X_n)$.

(ii) If P is a prime of R and P^* is a prime of R^* such that $P^* \cap R = P$, then P is vertical relative to A if and only if P^* is vertical relative to A^*.

Proof. The first assertion follows from the well-known fact that a polynomial ring extension of an integrally closed domain is again integrally closed. If P is a prime of R vertical relative to A, then PR^* is an ideal of R^* vertical relative to A^*. By (1.4) each minimal prime divisor of PR^* is vertical relative to A^*. Since R^* is integral over R, $P^* \cap R = P$ implies P^* is a minimal prime divisor of PR^*. Conversely, if P^* is a prime of R^* vertical relative to A^*, then $P^* \cap R = P$ must be a minimal prime divisor of $(P^* \cap A)R$. Since $(P^* \cap A)R$ is a prime ideal, we have $P = (P^* \cap A)R$.

(1.6) If A is an integral domain and I is an ideal of $A[X_1, ..., X_n]$ such that $I \cap A \neq 0$ and such that I is contained in the radical of a principal ideal (f) of $A[X_1, ..., X_n]$, then $f \in A$.

Proof. Let a be a nonzero element of $I \cap A$. Then f divides some power of a. Hence f must be of degree zero in each of the X_i.

As an obvious corollary to (1.6) we note the following:

(1.7) Suppose $A[X_1, ..., X_n] = R = B[Y_1, ..., Y_n]$. If R is an integral domain and $b \in B$ is such that $bR \cap A \neq 0$, then $b \in A$.

If R is an integral domain, we will use the notation R_u to denote the subring of R generated by the units of R, and R_c to denote the algebraic closure of R_u in R.

(1.8) If $A[X_1, ..., X_n] = R = B[Y_1, ..., Y_n]$ and R is an integral domain, then $A_u = B_u$, and $A_c = B_c \subset A \cap B$.

(1.9) If A is an integral domain, $A[X_1, ..., X_n] = B[Y_1, ..., Y_n]$ and $A = A_c$ then $A = B$. Consequently, if $A = A_c$, then A is strongly invariant.

Proof. By (1.8), we have $A \subset B$, so by (1.1), $A = B$.

Recall that the *Jacobson radical* of a ring R is defined to be the intersection of all maximal ideals of R.

(1.10) If A is an integral domain with nonzero Jacobson radical, then A is strongly invariant.

Proof. Let t be a nonzero element of the Jacobson radical of A. Then for any $a \in A$, $ta + 1$ is a unit of A, so $ta + 1 \in A_u$. Hence $ta \in A_u$, and $a \in A_c$. Thus $A = A_c$ and A is strongly invariant by (1.9).

(1.11) Suppose that $A[X_1, ..., X_n] = R = B[Y_1, ..., Y_n]$ and that R is an integral domain. If there exists a set $\{a_i\}$ of elements of A such that A is algebraic over the subring of A generated by the a_i and the units of A and such that $a_iR \cap B \neq 0$ for each i, then $A = B$.

Proof. By (1.7), each such $a_i \in B$. Since B is algebraically closed in $B[Y_1, ..., Y_n]$, we have $A \subset B$, so by (1.1), $A = B$.

If $A[X_1, ..., X_n] = R = B[Y_1, ..., Y_n]$, then certain ideals of R may be vertical relative to both A and B. If $\mathfrak{A}$ is an ideal of A such that $\mathfrak{A}R$ is vertical relative to B, then we will for convenience simply say that $\mathfrak{A}$ is *vertical relative to B*.

(1.12) Suppose that $A[X_1, ..., X_n] = R = B[Y_1, ..., Y_n]$ and that R is an integral domain. Then $A = B$ if and only if every prime ideal of A is vertical relative to B.

Proof. If every prime ideal of A is vertical relative to B, then $aR \cap B \neq 0$ for every nonzero $a \in A$. Hence by (1.11), $A = B$. The converse is obvious.

Recall that a ring A is called a *Hilbert ring* if every prime ideal of A is an intersection of maximal ideals. We note the following consequence of (1.12) and (1.3).

(1.13) If A is an integral domain which is a Hilbert ring and if $A[X_1, ..., X_n] = B[Y_1, ..., Y_n]$, then $A = B$ if and only if every maximal ideal of A is vertical relative to B.

If R is an integral domain, the *nonnormal locus* of R is the set of prime ideals p of R such that the localization R_p is not integrally closed.

(1.14) Let A be an integral domain and let $R = A[X_1, ..., X_n]$. If Q is a prime ideal of A, then Q is in the nonnormal locus of A if and only if $Q[X_1, ..., X_n] = P$ is in the nonnormal locus of R.

Proof. We have $R_P = A[X_1, ..., X_n]_{Q[X_1, ..., X_n]} = A_Q(X_1, ..., X_n)$, and R_P is integrally closed if and only if A_Q is integrally closed.

(1.15) Let A be an integral domain and let $R = A[X_1, ..., X_n]$. If P is a member of the nonnormal locus of R, then $P \cap A = Q$ is a member of the nonnormal locus of A. Consequently, any minimal member of the nonnormal locus of R is vertical relative to A.

Proof. R_p is a localization of $A_Q[X_1, ..., X_n]$; thus if R_p is not integrally closed, neither is A_Q. Hence Q is a member of the nonnormal locus of A. By (1.13), QR is in the nonnormal locus of R. Since $QR \subset P$, if P is a minimal member of the nonnormal locus of R, $QR = P$, and P is vertical relative to A.

We have the following corollary to (1.15).

(1.16) If A is an integral domain and $A[X_1, ..., X_n] = R = B[Y_1, ..., Y_n]$, then every minimal member of the nonnormal locus of R is vertical relative to both A and B.

(1.17) *Remark.* With reference to figure three, in case A is a one-dimensional affine ring, (1.16) guarantees that lines such as l_1 and l_2 which are fibers over singularities are distinguished inasmuch as any biregular map from the cylinder over the curve Γ to that over some curve Δ must take lines such as l_1 and l_2 onto corresponding lines over singularities of Δ. (See Fig. 2).

Our (3.2) will use this fact to show that under such circumstances the coordinate ring A is a strongly invariant ring.

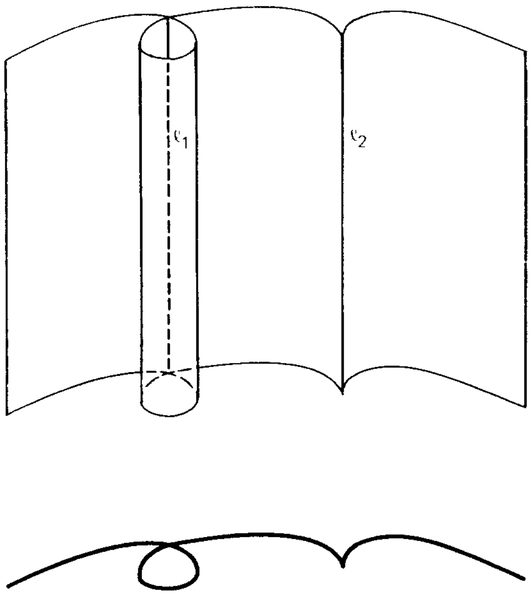

FIGURE 3

If $A \subset R$ are integral domains, then A is said to be *inertly imbedded* in R if every factor in R of an element in A is already in A [5]. Thus A is inertly imbedded in R if for any nonzero r, $t \in R$, $rt \in A$ implies r and $t \in A$. In [8], Evyatar and Zaks use the terminology *factorable subring* for what Cohn has called an inert imbedding.

(1.18) If A is an integral domain and $A[X_1, ..., X_n] = B[Y_1, ..., Y_n]$, then $A \cap B$ is an inert subring of both A and B.

(1.19) If $A[X_1, ..., X_n] = R = B[Y_1, ..., Y_n]$ and A is a unique factorization domain (UFD) then B and $A \cap B$ are also UFD's.

Proof. If A is a UFD, then so is $A[X_1, ..., X_n] = R$. It is easily seen that an inert subring of a UFD is again a UFD. Since $A \cap B$ is an inert subring of A and B is an inert subring of R, $A \cap B$ and B are also UFD's.

In concluding this introductory section, we note the following easy facts.

(1.20) If $A[X_1, ..., X_n] = R = B[Y_1, ..., Y_n]$. Then

(i) A is noetherian if and only if B is noetherian,

(ii) A is an integrally closed domain if and only if B is an integrally closed domain,

(iii) if D is a subring of $A \cap B$, then A is a finitely generated ring extension of D if and only if B is a finitely generated ring extension of D.

Proof. Statements (i) and (ii) are immediate, and (iii) follows from the fact that A and B, regarded as algebras over D, are homomorphic images of R.

2. Some Remarks on Subrings of Affine Rings

(2.1) If A is a d-dimensional affine domain over a field k and A contains d units which are algebraically independent over k, then A is strongly invariant.

Proof. In this case $A = A_c$, so (1.9) implies that A is strongly invariant.

(2.2) Corollary. *If A is a one-dimensional affine domain over a field k and A contains a nontrivial unit* (i.e., *a unit which is not algebraic over k*), *then A is strongly invariant.*

(2.3) If A is an affine domain over a field k and F is a subfield of A, then the elements of F are algebraic over k. Thus if $\bar{k}$ is the algebraic closure of k in A, then $F \subset \bar{k}$; and $\bar{k}$ is the unique maximal subfield of A.

Proof. By the normalization theorem [17, p. 45] there exist $X_1, ..., X_d \in A$ such that $X_1, ..., X_d$ are algebraically independent over k and A is an integral extension of $k[X_1, ..., X_d]$. Let V denote the valuation ring of $k(X_1, ..., X_d)$ obtained by giving value $-n$ to every polynomial in $k[X_1, ..., X_d]$ having total degree n. Let $V_1, ..., V_n$ denote the extensions of V to the quotient field of A. Note that $a \in A \cap V_1 \cap \cdots \cap V_n$ implies that a is algebraic over k, for the coefficients of the minimal polynomial for a over $k(X_1, ..., X_d)$ are in $k[X_1, ..., X_d] \cap V = k$. If F is a subfield of A, then $F \cap V_1 \cap \cdots \cap V_n$ is a finite intersection of valuation rings with quotient field F and hence is a domain with quotient field F [17, p. 38]. Thus we must have $F \subset \bar{k}$.

(2.4) *Remark.* Let A be an affine domain over a field, let G be a finite group of automorphisms of A, and let A^G denote the fixed ring of G acting on A. If k is the unique maximal subfield of A given by (2.3), then G must restrict to a finite group of automorphisms of k. Let k' denote the fixed field of G acting on k. Then A is an affine ring over k' and G is a finite group of k'-automorphisms of A. By a well-known theorem of Noether, it follows that A^G is also an affine ring over k' [16, p. 9].

(2.5) Let k be a field and let $X_1, ..., X_n$ be indeterminates over k. If A is a one-dimensional subring of $k[X_1, ..., X_n]$, then there is a k-homomorphism ϕ of $k[X_1, ..., X_n]$ onto a polynomial ring in one variable over k, say $k[X]$, such that ϕ restricted to A is an isomorphism. Consequently (within isomorphism) A is a subring of $k[X]$.

Proof. Let $(X_1, ..., X_n)$ denote the ideal of $k[X_1, ..., X_n] = R$ generated by $X_1, ..., X_n$. Consider $(X_1, ..., X_n) \cap A = P$. Since P is a prime ideal of A, either $P = (0)$ or P is a maximal ideal. If $P = (0)$, then the residue mapping of R onto $k[X_1, ..., X_n]/(X_1, ..., X_n)$ takes A isomorphically into k and we are done. If $P \neq (0)$ and $n > 1$, then consider the prime ideals $(X_1{}^m - X_n)$ of R, $m = 1, 2, ...$. If α is a nonzero element of P, then for some m, $X_1{}^m - X_n$ does not divide α in R. Thus $(X_1{}^m - X_n) \cap A$ is properly contained in P, so we must have $(X_1{}^m - X_n) \cap A = (0)$. Hence the residue mapping of R to $R/(X_1{}^m - X_n) \simeq k[X_1, ..., X_{n-1}]$ restricts to an isomorphism of A. Iteration of this process yields an isomorphism of A into $k[X]$.

(2.6) Let k be a field and let $X_1, ..., X_n$ be indeterminates over k. If A is a one-dimensional ring between k and $k[X_1, ..., X_n]$, then A is an affine ring and the integral closure of A is a polynomial ring. Thus if A is a one-dimensional integrally closed ring between k and $k[X_1, ..., X_n]$, then $A = k[t]$ for some $t \in k[X_1, ..., X_n]$.

Proof. By (2.5) we may assume that $k \subset A \subset k[X]$. Let f be an element of $A \backslash k$. Then $k[f] \subset A \subset k[X]$ and $k[X]$ is a finite $k[f]$-module. Thus A is a finite $k[f]$-module, so A is an affine ring over k. By the classical Luroth's theorem, the quotient field of A is a simple transcendental extension $k(Z)$ of k [20, p. 198]. Since $k(Z) \subset k(X)$ and $k[X]$ is integral over A, A is contained in all the valuation rings of $k(X)$ over k except the $1/X$-adic valuation ring. Thus if A is integrally closed, then A is the intersection of all the valuation rings of $k(Z)$ over k except the restriction to $k(Z)$ of the $1/X$-adic valuation rings of $k(X)$. Since the $1/X$-adic valuation of $k(X)$ is rational (i.e., has residue field k), its restriction to $k(Z)$ is also rational. Hence the only valuation ring of $k(Z)$ over k not containing A is either the $1/Z$-adic valuation ring or else given by a linear polynomial $p(Z)$ in $k(Z)$. If A is integrally closed, then, in the first case $A = k[Z]$ and in the second case $A = k[1/p(Z)]$.

(2.7) *Remark.* The fact that a normal ring A between a field k and the polynomial ring $k[X]$ has the form $A = k[t]$ is well known (see, for example [12, p. 256]) and as our argument shows, it is a consequence of the following fact. If A is a normal ring between a field k and a simple transcendental field extension $k(Z)$ of k, and if there is only one valuation of $k(Z)$ over k not containing A and this valuation is rational, then $A = k[t]$ for some $t \in k[Z]$.

From (2.6) we get a quick proof that a polynomial ring in one variable over a field is invariant.

(2.8) COROLLARY. *Let k be a field and let $A = k[X]$ be a polynomial ring in one variable over k. If $A[X_1, ..., X_n] = R = B[Y_1, ..., Y_n]$, then $B = k[t]$ for some t transcendental over k. Thus $A \simeq B$ and A is an invariant ring.*

Proof. We have $k \subset B \subset k[X, X_1, ..., X_n] = R$; and B is noetherian, for R is noetherian and B is a homomorphic image of R. Thus $\dim R = n + 1 = \dim B[Y_1, ..., Y_n] = \dim B + n$, and $\dim B = 1$. Also B is integrally closed, so by (2.6), $B = k[t]$ for some $t \in R \backslash k$.

(2.9) Let k^* be a finite separable algebraic field extension of a field k. If X is an indeterminate over k^* and A is a normal ring such that $k \subset A \subset k^*[X]$, then A has the form $k'[t]$, where k' is the algebraic closure of k in A.

Proof. We may assume that k is algebraically closed in A, i.e. $k = k'$. Let θ be a primitive element for k^* over k. By (2.6), we have $k^* \subset A[\theta] \subset k^*[Z]$ where $k^*(Z)$ is the quotient field of $A[\theta]$. The only valuation ring of $k^*(Z)$ over k^* which doesn't contain A is the $1/Z$-adic valuation ring. Hence this valuation ring is the only extension of its contraction V to the quotient field L of A. Also the $1/Z$-adic valuation ring is unramified over V, for the discriminant of θ is a unit of k. Thus if $[k^* : k] = n$, then k^* over the residue field of V is also a field extension of degree n. It follows that V has residue field k. Since $L(\theta) = k^*(Z)$ and the genus of a field does not go down under separable algebraic field extension [15, p. 621], we see that L/k is of genus zero with a "rational place" (one with residue field k). Hence L is a simple transcendental extension of k [4, p. 23], and by (2.7), $A = k[t]$.

(2.10) *Remark.* Let k^* be a separable algebraic field extension of a field k and let X be an indeterminate over k^*. If A is an integrally closed domain between $k[X]$ and $k^*[X]$, then A has the form $k'[X]$ for some field k' between k and k^*. This can be seen, for example, by passing to the normal closure of k^*/k and then using Galois theory to prove that the fields between k and k^* are in one-to-one correspondence with the fields between $k(X)$ and $k^*(X)$.

(2.11) Let k^* be a separable algebraic field extension of a field k and let $X_1, ..., X_n$ be indeterminates over k^*. If A is a one-dimensional normal ring such that $k \subset A \subset k^*[X_1, ..., X_n]$, then A has the form $k'[t]$ where k' is the algebraic closure of k in A.

Proof. We may assume that k is algebraically closed in A, so that $k = k'$. By the "cutting down lemma" (2.5), we have $k \subset A \subset k^*[X]$. Since k^* is algebraic over k and A is one-dimensional, we see that X is integral over A. Let A^* denote the finite integral extension of A generated by X, let L and L^* denote the quotient fields of A and A^*, and let k_1 be the algebraic closure of k in L^*. Since A is integrally closed and k is assumed to be algebraically closed in A, we see that k is algebraically closed in L. Thus k_1/k and L/k are linearly disjoint, so k_1 must be a finite algebraic extension of k. If A' denotes the integral closure of A in L^*, then $k_1[X] \subset A' \subset k^*[X]$ so by (2.10), $A' = k_1[X]$. We now have $k \subset A \subset k_1[X]$ with k_1 a finite separable algebraic extension of k. By (2.9), A has the form $k[t]$.

(2.12) If one simply deletes the separability assumption, (2.11) is no longer true. The proof would break down at the point where separability is invoked to argue that L/k is of genus zero. In [15] Lang and Tate provide examples to show that genus can decrease under an algebraic extension. In fact, from [15, p. 624] we have the following example:

Let k be a field of characteristic $p \neq 0$ such that there are elements α and β in an extension of k with α^p, $\beta^p \in k$ and $[k(\alpha, \beta) : k] = p^2$. Let X be transcendental over $k(\alpha, \beta)$ and set $R = k(\alpha + \beta X, X) \cap k(\alpha, \beta)[X]$. Then $k < R < k(\alpha, \beta)[X]$. Since R is the intersection of a polynomial ring and a field, R is normal. It is not hard to show that the largest field contained in R is k, yet no residue class ring of R is equal to k. Thus R cannot be a polynomial ring.

(2.13) Let R be an integrally closed domain with quotient field K and let R^* be the integral closure of R in an algebraic field extension K^* of K. Let P be a prime ideal in R. If each prime ideal P^* in R^* lying over P (i.e. such that $P^* \cap R = P$) is the radical of a principal ideal in R^*, then the P^* are finite in number and P is the radical of a principal ideal in R.

Proof. Let $\{P_i^* \mid i \in \text{an index set } I\}$ denote the set of primes of R^* lying over P in R, and let $a_i \in R^*$ be such that $P_i^* = \sqrt{a_i R^*}$. Let $K_i = K(a_i)$ and let R_i denote the integral closure of R in K_i. Note that P_i^* is the only prime of R^* lying over $P_i^* \cap R_i$. Now in R_i there are only a finite number of prime ideals lying over P in R. Hence we can choose $b_i \in R_i$ such that $b_i \notin P_i$, but b_i is in every other prime of R_i lying over P. It follows that $b_i \in P_j^*$ for any $j \in I$, $j \neq i$. The ideal in R^* generated by the b_i, $i \in I$ is not contained in $\bigcup \{P_i^* \mid i \in I\}$. Hence there exists $b_1, \ldots, b_n \in \{b_i \mid i \in I\}$ and $r_i \in R^*$ such that $y = r_1 b_1 + \cdots + r_n b_n \notin \bigcup \{P_i^* \mid i \in I\}$. It then follows that $P_1^*, \ldots, P_n^*$ are all the primes of R^* lying over P in R. Let $L = K(a_1, \ldots, a_n)$, and let R' denote the integral closure of R in L. For $z \in L$, let $N(z)$ denote the norm of z with respect to the field extension L/K. Note that if $a = a_1 \cdots a_n$, then $\sqrt{aR'} = \bigcap_{i=1}^n P_i^* \cap R'$, and $P \subset \sqrt{aR'}$. It follows that if $\alpha = N(a)$, then $P = \sqrt{\alpha R}$. For if $b \in P$, then $b^m \in aR'$, say $b^m = az$ with $z \in R'$. Hence $N(b^m) = N(az)$. If $[L : K] = s$, then $N(b^m) = b^{ms} = N(a)\, N(z) \in \alpha R$. This completes the proof of (2.13).

Recall that an integral domain R is said to be *prefactorial* if every height one prime ideal of R is the radical of a principal ideal [1, p. 1140]. We note the following consequence of (2.13).

(2.14) Corollary. *Suppose that R is an integrally closed domain with quotient field K and that L is an algebraic field extension. If the integral closure of R in L is a prefactorial domain, then R is prefactorial.*

3. Integral Domains of Transcendence Degree One Over a Subfield

In this section we prove our main result on the invariance of one-dimensional affine domains. We consider, in fact, a slightly more general class of rings, namely integral domains which contain a field over which they are of transcendence degree one. Note that such a domain A has dimension $\leqslant 1$, for any valuation ring between A and the quotient field of A has rank $\leqslant 1$.

(3.1) Lemma. *Let A be an integrally closed domain of transcendence degree one over a subfield k. If $A[X_1 ,..., X_n] = R = B[Y_1 ,..., Y_n]$ and $A \neq B$, then A is a prefactorial Dedekind domain.*

Proof. Note that $k \subset B$ and that B also has transcendence degree one over k. Since $A \neq B$, it is clear that A and B are not fields; so both A and B are one-dimensional domains. By (1.12) there is a maximal ideal M of B such that $MB[Y_1 ,..., Y_n] \cap A = (0)$. Let k^* denote the field B/M and let $\phi : B[Y_1 ,..., Y_n] \to k^*[Z_1 ,..., Z_n]$ denote the residue class mapping of R mod MR where Z_i denotes the residue class of Y_i. Note that k^* is algebraic over k, the Z_i are algebraically independent over k^*, and that (within isomorphism) we have $k \subset A \subset k^*[Z_1 ,..., Z_n]$. By the "cutting down lemma" (2.5), we conclude that $k \subset A \subset k^*[Z]$. Since k^* is algebraic over k, we see that $k^*[Z]$ is integral over A. Let L denote the quotient field of A. Since A is integrally closed, we have $A = L \cap k^*[Z]$. Thus A is a one-dimensional Krull ring and hence a Dedekind domain. That A is prefactorial follows from (2.14).

(3.2) Theorem. *Let A be an integral domain of transcendence degree one over a subfield k. Suppose that $A[X_1 ,..., X_n] = R = B[Y_1 ,..., Y_n]$. If A is not integrally closed, then $A = B$. Consequently, a nonnormal domain of transcendence one over a field is strongly invariant.*[1]

Proof. Let A^*, B^*, and R^* denote the integral closures of A, B, and R in their respective quotient fields. By (1.5), we have $A^*[X_1 ,..., X_n] = R^* = B^*[Y_1 ,..., Y_n]$. If $A^* = B^*$, then the elements of B are integral over A and it follows that $A = B$. If $A^* \neq B^*$, then by (3.1), A^* and B^* are prefactorial Dedekind domains. By (1.15), there is a nonzero prime ideal P of A such that P is vertical relative to B. Let P^* be a prime of A^* such that $P^* \cap A = P$ and let $\alpha \in A^*$ be such that $\sqrt{(\alpha)} = P^*$. Then (1.7) implies that $\alpha \in B$. But α is a nonzero nonunit of A^*, and hence must be transcendental over k. Thus A^* and B^* are algebraic over $k[\alpha] \subset A^* \cap B^*$, so by (1.9), we must have $A^* = B^*$. It follows that $A = B$.

[1] See Remark (1.17).

(3.3) THEOREM. *Let A be an integral domain of transcendence degree one over a subfield k. Suppose that $A[X_1, ..., X_n] = R = B[Y_1, ..., Y_n]$, and let k' denote the algebraic closure of k in A. If $A \neq B$, then A and B are both polynomial rings over the field k'. Consequently, A is invariant, and if A is not a polynomial ring, then A is strongly invariant.*[2]

Proof. We may assume that k is algebraically closed in A so that $k = k'$. If $A \neq B$, then (3.2) implies that A is integrally closed. Hence by (3.1), A and B are prefactorial Dedekind domains. If a nonzero prime ideal P of A is such that $PR \cap B \neq (0)$ and if $\alpha \in A$ is such that $\sqrt{(\alpha)} = P$, then (1.7) implies that $\alpha \in B$. As in the proof of (3.2), this implies that A and B are algebraic over $k[\alpha] \subset A \cap B$, and hence that $A = B$. Thus for each nonzero prime ideal P of A, we must have $PR \cap B = (0)$. Let k^* denote the field A/P. Then k^* is algebraic over k and the residue mapping of R to R/PR yields (within isomorphism) $k \subset B \subset k^*[Z_1, ..., Z_n]$, where the Z_i are algebraically independent over k^*. The "cutting down lemma" (2.5), yields $k \subset B \subset k^*[Z]$. If k^* is separable over k, then (2.11) implies that B is a polynomial ring over k and we are done. Hence all that remains is to prove that there is a nonzero prime ideal P of A such that A/P is separable over k. Let V^* denote the valuation ring on the quotient field of R obtained by giving value 0 to all elements of B and giving to a polynomial in $B[Y_1, ..., Y_n]$ of (total) degree m the value $-m$. Let L denote the quotient field of A and let $V = V^* \cap L$. If $A \neq B$, then $A \not\subset B$ by (1.1), so some element of A is a polynomial of positive degree in $B[Y_1, ..., Y_n]$. Thus V is a rank one valuation ring on L and must be the unique valuation ring of L over k that does not contain A. Now the residue field of V is algebraic over k and contained in the residue field of V^*. The residue field of V^* is a pure transcendental extension of the quotient field of B. By assumption, k is algebraically closed in A and this clearly implies that k is algebraically closed in B. Since B is integrally closed, we see that k is algebraically closed in the quotient field of B and hence in the residue field of V^*. Thus k is the residue field of V. Let α be a generator for the maximal ideal of V (i.e., α is a local uniformizing parameter for V). Consider the value of α in the other valuation rings of L over k. Now from $k \subset A \subset k^*[Z]$, we see that every other valuation ring of L over k contains A. Let $V_1, ..., V_m$ denote the (necessarily finite number of) valuation rings of L over k distinct from V such that α is a nonunit of V_i. We have $A \subset V_i$ and if P_i denotes the center of V_i on A, then $A/P_i = k_i$ is the residue field of V_i. To complete the proof of (3.3) we show that some k_i is separable algebraic over k. Assume that k has characteristic $p > 0$. If k_i is not separable over k,

[2] A generalization of (3.3) to noetherian rings of transcendence degree one over a field is proved in [7].

then k_i contains a subfield k_i' such that k_i' is finite algebraic over k and $[k_i' : k]$ is divisible by p. Let E be an extension field of k generated by α and by a finite number of elements of L which residually modulo P_i generate the field k_i' for each $i = 1,..., m$. Let $v_i'(\alpha)$ denote the value of α in the rank one valuation ring $V_i' = V_i \cap E$. Since V has residue field k, we see that $V \cap E$ has residue field k and that $\sum_{i=1}^{m} v_i'(\alpha)[k_i' : k] + 1 = 0$ [4, p. 18]. This contradicts the fact that each $[k_i' : k]$ is divisible by p. Hence some k_i must be separable over k and the proof of (3.3) is complete.

(3.4) COROLLARY. *If A is a one-dimensional affine domain over a field then A is invariant; and A is strongly invariant if A is not a polynomial ring.*

(3.5) *Remark.* As a corollary to the fact that $k[X]$ is an invariant ring we have a classification of certain derivations of $k[X, Y]$ in case k is a field of characteristic zero. Consider the derivation of $k(X, Y)$ given by $D_Y(f) = \partial F/\partial Y$. This k-derivation obviously satisfies the following axioms:

(i) $D : k[X, Y] \to k[X, Y]$,

(ii) There exists $f \in k[X, Y]$ such that $D(f) = 1$,

(iii) If $R = \{r \in k[X, Y] \mid D(r) = 0\}$ then there exists $g \in k[X, Y]$ such that $R[g] = k[X, Y]$.

Let us say that two k-derivations D_1 and D_2 are equivalent if there exists a k-automorphism w of $k[X, Y]$ such that $w^{-1}D_1w = D_2$. It is clear that any derivation equivalent to D_Y must satisfy (i)–(iii) above. We now show that in characteristic zero, derivations satisfying the above axioms are equivalent.

(3.6) If k is a field of characteristic zero then a necessary and sufficient condition that a derivation D of $k[X, Y]$ be equivalent to D_Y is that it satisfy (i)–(iii) above.

Proof. The necessity is obvious without regard to characteristic. For the sufficiency suppose D is a k-derivation of $k[X, Y]$ satisfying (i)–(iii) above. By (ii) there is an $f \in k[X, Y]$ such that $D(f) = 1$. By (iii) f has a representation of the form $f = \sum_{i=0}^{n} r_i g^i$ where $D(r_i) = 0$. Thus $1 = D(f) = (\sum_{i=1}^{n} i r_i g^{i-1}) D(g)$. Hence $D(g)$ is a unit in $k[X, Y]$. We can therefore assume $D(g) = 1$. Now we show that g is transcendental over R. If not there is an expression $0 = \sum_{i=0}^{m} r_i g^i$ where m is minimal and $r_m \neq 0$. Applying D we get $\sum_{i=1}^{m} i r_i g^{i-1} = 0$ which, in view of our characteristic zero assumption contradicts the minimality of m. Thus g is transcendental over R and $R[g] = k[X, Y]$. Since $k[X]$ is invariant, we have $R = k[h]$ for some $h \in k[X, Y]$. This allows us to define a k-homomorphism w of $k[X, Y]$ by

$h \to X$ and $g \to Y$. Then if $p(X, Y) = \sum_{i=0}^{n} r_i(h)\, g^i$ is any element of $k[X,Y]$ (the r_i are polynomials with coefficients in k) we have

$$\begin{aligned} w(D(p)) &= w\left(\sum_{i=1}^{n} i r_i(h)\, g^{i-1}\right) \\ &= \sum_{i=1}^{n} i r_i(X)\, Y^{i-1} \\ &= D_Y\left(\sum_{i=0}^{n} r_i(X)\, Y^i\right) \\ &= D_Y(w(p)). \end{aligned}$$

Thus $wD = D_Y w$ or $w^{-1} D_Y w = D$. Hence D is equivalent to D_Y.

4. Integral Domains of Transcendence Degree One Over a Subring

If $A \subset D$ are integral domains, then by the transcendence degree of D over A we of course mean the transcendence degree of the quotient field of D over the quotient field of A. Recall that when we say a ring has a property locally, we mean that R_p has the property for every prime p of R.

(4.1) Theorem. *Let A be a unique factorization domain* (UFD) *and let $X_1,\ldots,X_n$ be indeterminates over A. If D is a* UFD *such that $A \subset D \subset A[X_1,\ldots,X_n]$ and such that D has transcendence degree one over A, then D is a polynomial ring over A.*

Proof. Let S denote the multiplicative system of nonzero elements of A. We have $A_S \subset D_S \subset A_S[X_1,\ldots,X_n]$. Since D_S is a UFD, D_S is integrally closed, so by (2.6), D_S is a polynomial ring over A_S. Hence we can choose $\theta \in D$ so that $D_S = A_S[\theta]$. We may assume that θ as a polynomial in $A[X_1,\ldots,X_n]$ has zero constant term. Moreover, we can choose θ to be an irreducible element of D; for if $\theta = d_1 d_2$ with $d_1, d_2 \in D$, then from $D \subset A_S[\theta]$ we see that not both d_1 and d_2 can be polynomials of positive degree in $A[X_1,\ldots,X_n]$. Assuming now that θ is an irreducible element of D and that θ as a polynomial in $A[X_1,\ldots,X_n]$ has zero constant term, we show that $D = A[\theta]$. If $d \in D$, then from $D \subset A_S[\theta]$ we have $d = \alpha_0 + \alpha_1\theta + \cdots + a_m\theta^m$ with $\alpha_i \in A_S$. Moreover, α_0 is the constant term of d as a polynomial in $A[X_1,\ldots,X_n]$, so $\alpha_0 \in A$. Hence $d \in A[\theta]$ if and only if $d - \alpha_0 \in A[\theta]$. Now θ divides $d - \alpha_0$ in $D_S = A_S[\theta]$ and θ is a prime element of D which extends to a prime element in D_S. Hence θ divides $d - \alpha_0$ in D and $\alpha_1 + \alpha_2\theta + \cdots + \alpha_m\theta^{m-1} \in D$. Repeating the above argument yields $\alpha_1 \in D$ and then $\alpha_2 + \alpha_3\theta +$

$\cdots + \alpha_m \theta^{m-2} \in D$. We conclude by induction that all the $\alpha_i \in A$ and hence that $D = A[\theta]$.

The following example illustrates the necessity of the unique factorization hypothesis of (4.1).

(4.2) EXAMPLE. Let A be any Dedekind domain which is not a principal ideal domain and let $\mathfrak{A} = (a, b)$ be a prime ideal of A which is not principal. We choose an element t of the quotient field of A such that $A : \mathfrak{A} = \mathfrak{A}^{-1} = (1, t)$. Consider the polynomial ring $A[X, Y]$ and let $\theta = aX + bY$. Let $R = A[\theta, t\theta]$. Note that R is a subring of $A[X, Y]$. We show that R is an inert, integrally closed domain of transcendence degree one over A, and that R is not a polynomial ring over A.

(i) *R is an inert subring of $A[X, Y]$.* If P is a prime ideal of A and $S = A \setminus P$, then let R_P denote the localization $R_S = A_P[\theta, t\theta]$. Since either t or $(1/t)$ is in A_P, R_P is a polynomial ring in one variable over A_P. Moreover, for each prime ideal P of A there is an f_p in $A[X, Y]$ such that $R_P[f_p] = A_P[X, Y]$. This follows because if $P \neq (a, b)$, then either a or b is a unit in A_P and therefore either X or Y serves for f_p. In case $P = (a, b)$, then either ta or tb is a unit in A_P and again either X or Y will serve for f_p. Thus R_P is an inert subring of $A_P[X, Y]$. Now suppose $r \in R$ and $r = cd$ where $c, d \in A[X, Y]$. Thus each of c and d is in R_P for every prime ideal P of A. Since $R = \bigcap R_P = \bigcap \{A_P[\theta, t\theta] \mid P \text{ is a prime of } A\}$, we conclude that R is an inert subring of $A[X, Y]$.

(ii) *R is integrally closed.* Since $R = \bigcap \{R_P \mid P \text{ is a prime of } A\}$ and each R_P is a polynomial ring over A_P, R is the intersection of normal rings.

(iii) *R is not a polynomial ring over A.* Suppose there exists T such that $R = A[T]$. We may assume that T as a polynomial in $A[X, Y]$ has no constant term. Then T must be of the form $\lambda\theta$ for some $\lambda \in K$, the quotient field of A. But $A[\lambda\theta] = A[\theta, t\theta]$ implies equality of the fractional A-ideals $(\lambda) = (1, t)$; and this implies that $(1/\lambda) = (a, b) = \mathfrak{A}$, which contradicts the fact that $\mathfrak{A}$ is not principal.

As a corollary to (4.1) we prove the invariance of the polynomial ring in one variable over a certain class of unique factorization domains. Recall that for an integral domain D, we are using D_u to denote the subring of D generated by the units of D, and D_c to denote the algebraic closure of D_u in D.

(4.3) COROLLARY. *Let D be a unique factorization domain such that $D = D_c$ and let $A = D[Z]$ be a polynomial ring in one variable over D. If $A[X_1, \ldots, X_n] = B[Y_1, \ldots, Y_n]$, then A is isomorphic to B. Consequently, the polynomial ring $D[Z]$ is invariant.*

Proof. Since D is a UFD, $D[Z, X_1, \ldots, X_n] = B[Y_1, \ldots, Y_n]$ is also a UFD; and B is an inert subring of $B[Y_1, \ldots, Y_n]$, so B is also a UFD. The assumption that $D = D_c$ implies that $D \subset B$, and by counting transcendence degrees, we see that B has transcendence degree one over D. Thus $D \subset B \subset D[Z, X_1, \ldots, X_n]$ and (4.1) implies that B is a polynomial ring in one variable over D.

(4.4) *Remark.* Corollary 4.3 yields a large class of invariant rings which are not strongly invariant. For example, if D is any semilocal UFD, then $D[Z]$ is an invariant ring which is not strongly invariant. This corollary also leads us to consider the following. Let D be a ring and A a ring which is a D-algebra. A is said to be *D-invariant* provided A satisfies the following condition: given a D-algebra B and indeterminates $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$ if $A[X_1, \ldots, X_n]$ is D-isomorphic to $B[Y_1, \ldots, Y_n]$ then A is D-isomorphic to B. If for any D-algebra B and any D-isomorphism $\phi : A[X_1, \ldots, X_n] \to B[Y_1, \ldots, Y_n]$, $\phi(A) = B$ then A is said to be ***strongly** D-invariant*. With this terminology (4.1) implies:

(4.5) If D is a unique factorization domain then the polynomial ring $D[Z]$ is D-invariant.

(4.6) Theorem. *Let D be an integral domain and let Z be an indeterminate over D. If for each prime ideal P of D, $D_P[Z]$ is D_P-invariant, then $D[Z]$ is D-invariant.*

Proof. Let $D[Z] = A$ and suppose that $A[X_1, \ldots, X_n] = R = B[Y_1, \ldots, Y_n]$ with $D \subset B$. For a prime ideal P of D, let $S = D \backslash P$ and let B_P denote the localization B_S. By hypothesis, B_P is isomorphic to $D_P[Z]$. Hence $B_P = D_P[f_p]$ for some element f_p and we may assume that f_p is in B. In particular, if K is the quotient field of D, we have an $f_0 \in B$ such that $K[f_0]$ is the localization of B at the multiplicative system of nonzero elements of D. We may assume that the Y_i and the f_p regarded as polynomials in $D[Z, X_1, \ldots, X_n]$ all have constant term zero. Then for each prime ideal P of D, we have $f_p = \lambda_p f_0$ with $\lambda_p \in K$.

Let $\mathfrak{A}$ be the D-module generated by the λ_p. Then we claim:

(i) $B = D[\mathfrak{A} f_0]$

(ii) B is a polynomial ring if and only if $\mathfrak{A}$ is principal.

(iii) $\mathfrak{A}$ is an invertible fractionary ideal of D.

We obviously have $D \subset D[\mathfrak{A} f_0] \subset B$. Moreover, for every prime P of D, $(D[\mathfrak{A} f_0])_P = D_P[f_p] = B_P$. Thus, as D-modules $D[\mathfrak{A} f_0]$ and B are everywhere locally equal and hence are equal. This establishes (i). To see (ii) suppose B is a polynomial ring. We may assume then $B = D[T]$ where T has no constant term. Then $f_0 = \beta T$ for some $\beta \in K$ and $D[T] = D[\mathfrak{A}\beta T]$. Thus $\mathfrak{A}\beta = D$ and $\mathfrak{A} = D\beta^{-1}$. If $\mathfrak{A}$ is principal then clearly B is a polynomial ring. This establishes (ii). To see that $\mathfrak{A}$ is an invertible fractional ideal we first observe that if d is any nonzero coefficient of f_0 (as an element of $D[Z, X_1, ..., X_n]$) then $d\mathfrak{A} \subset D$. Thus C_{f_0}, the ideal of D generated by the coefficients of f_0 is contained in $\mathfrak{A}^{-1} = D : \mathfrak{A}$. Moreover, $\mathfrak{A}C_{f_0} = D$. For if not, there is a prime ideal P such that $\mathfrak{A}C_{f_0} \subset P$. But this would imply that $f_p = \lambda_p f_0 \in PB_P = PD_P[f_p]$, a contradiction. Thus $\mathfrak{A}C_{f_0} = D$ and $\mathfrak{A}$ is invertible.

We now show that $\mathfrak{A}$ is principal. Let J be the ideal in R generated by the monomials of degree two in $Z, X_1, ..., X_n$. Then

$$\frac{R}{J} = D[\mathfrak{A}\bar{f}_0, \bar{Y}_1, ..., \bar{Y}_n] = D[\bar{Z}, \bar{X}_1, ..., \bar{X}_n].$$

Thus as D-modules this takes the form

$$D \oplus M = D \oplus D\bar{Z} \oplus \cdots \oplus D\bar{X}_n \simeq D^{(n+2)},$$

where $M = \mathfrak{A}\bar{f}_0 \oplus D\bar{Y}_1 \oplus \cdots \oplus D\bar{Y}_n$ and by $D^{(j)}$ we mean a free D-module of rank j. Thus $M \simeq \mathfrak{A}\bar{f}_0 + D^{(n)}$ and since $\bar{f}_0 \neq 0$, $M \simeq \mathfrak{A} \oplus D^{(n)} \simeq D^{(n+1)}$. A simple argument now shows that $\mathfrak{A}$ is principal. Considering M as $\mathfrak{A} \oplus D^{(n)}$ let $\{\xi_i\}_{i=1}^{n+1}$ be a free basis for M. Then

$$\xi_i = (a_i, d_{i1}, ..., d_{in}) \qquad \text{with} \quad a_i \in \mathfrak{A}.$$

It follows that $\det(\xi_1, ..., \xi_{n+1}) = a \in \mathfrak{A}$ and in fact $aD = \mathfrak{A}$. For let P be any prime of D. We claim $(aD)_P = \mathfrak{A}D_P$. Since $\mathfrak{A}$ is invertible, $\mathfrak{A}D_P = a_pD_P$ for some $a_p \in \mathfrak{A}$. Thus $e_1, ..., e_{n+1}$ is a free basis for M_P where $e_1 = (a_p, 0, ..., 0)$ and for $i > 1$, $e_i = (0, ..., 1, ..., 0)$ where the 1 is in the i-th place. But $\det(e_1, ..., e_{n+1}) = a_p$ and a_p differs from a by at most a unit multiple in D_P. Consequently $(aD)_P = \mathfrak{A}_P$ for every prime P and hence $\mathfrak{A}$ is principal. This concludes the proof of the theorem.

Our Theorem (4.6) provides some more information concerning examples like (4.2). With reference to (4.2), in trying to construct an example of a non-invariant domain, one might attempt to find a $T \in D[X, Y]$ such that $R[T] = D[X, Y]$. Theorem (4.6) shows that this is impossible, for we have the immediate corollary:

(4.7) COROLLARY. *If D is a domain which is locally a unique factorization domain, then $D[Z]$ is D-invariant. In particular, if D is a Dedekind domain, then $D[Z]$ is D-invariant.*

Proof. By (4.1), for each prime P, $D_P[Z]$ is D_P-invariant; so (4.6) applies.

We can prove a more general version of (4.7). An integral domain R is called an HCF-ring (for highest common factor) if the partially ordered group of nonzero principal fractional ideals of R is lattice ordered [5, p. 253]. Thus R is an HCF-ring if and only if for any two principal ideals (a)and (b) of R, $(a) \cap (b)$ is also principal. Unique factorization domains are HCF-rings and other examples are valuation rings or more generally Bezout domains (i.e., integral domains in which finitely generated ideals are principal), and polynomial rings over valuation rings or Bezout domains.

The following proposition is similar to (4.1).

(4.8) PROPOSITION. *Let A be an* HCF-*ring and let $X_1, \ldots, X_n$ be indeterminates over A. Suppose that D is an integral domain of transcendence degree one over A and that $A \subset D \subset A[X_1, \ldots, X_n]$. If D is an inert subring of $A[X_1, \ldots, X_n]$, then D is a polynomial ring over A.*

Proof. Let $K \subset L$ denote the quotient fields of $A \subset D$. Since D is an inert subring of the integrally closed domain $A[X_1, \ldots, X_n]$, $D = A[X_1, \ldots, X_n] \cap L$ and D is integrally closed. We have $K \subset D[K] \subset K[X_1, \ldots, X_n]$, so (2.6) implies that there exists an element θ such that $D[K] = K[\theta]$. We may assume that $\theta \in D$ and that θ as a polynomial in $A[X_1, \ldots, X_n]$ has zero constant term. Since A is an HCF-ring and since D is an inert subring of $A[X_1, \ldots, X_n]$, we may also assume that if $a_1, \ldots, a_m$ are the coefficients of θ as a polynomial in $A[X_1, \ldots, X_n]$, then the greatest common divisor of $a_1, \ldots, a_m$ in A is 1. We now make use of the fact that any lattice-ordered group can be lattice embedded in a direct product of totally ordered groups and that for the HCF-ring A this implies the existence of a set $\{V_\alpha\}$ of valuation rings such that $A = \bigcap_\alpha V_\alpha$ and such that the associated valuation maps are lattice homomorphisms [22, p. 37]. It follows that $(a_1, \ldots, a_m)V_\alpha = V_\alpha$ for each α. We extend the valuation ring V_α to a valuation ring V_α' of $K(X_1, \ldots, X_n)$ by defining the value of a polynomial in $K[X_1, \ldots, X_n]$ to be the infimum of the values of its coefficients. The X_i are units in V_α' and the residues of the X_i modulo the maximal ideal of V_α' are algebraically independent over the residue field of V_α'. Since θ has zero constant term and since $(a_1, \ldots, a_m) V_\alpha = V_\alpha$, we see that the residue of θ modulo the maximal ideal of V_α' is transcendental over the residue field of V_α. It follows that the V_α'-value of any polynomial in $K[\theta]$ is the infimum of the values of its coefficients. Hence if $b_0 + b_1\theta + \cdots + b_s\theta^s$ is an element of $K[\theta] \cap A[X_1, \ldots, X_n]$, then all the $b_i \in V_\alpha$. Since $A = \bigcap_\alpha V_\alpha$, we have

$$A[\theta] = K[\theta] \cap A[X_1, \ldots, X_n] = D,$$

which completes the proof of (4.8).

We have the following corollary to (4.6) and (4.8).

(4.9) COROLLARY. *If D is an integral domain which is locally an* HCF-*ring, then the polynomial ring $D[Z]$ is D-invariant. Thus if $D = D_c$ (i.e., D is* *algebraic over the subring of D generated by the units of D) and if D is locally at each maximal ideal an* HCF*-ring, then the polynomial ring $D[Z]$ is invariant.*

The proof is similar to that of (4.3). Examples of integral domains which are locally HCF-rings are Prüfer domains and polynomial rings over Prüfer domains.

5. One-Dimensional Domains which are not Strongly Invariant

In this section we give some conditions on a one-dimensional integral domain A in order that A not be strongly invariant. These conditions show that such domains must be closely related to polynomial rings. However, we must admit that there is a gap here, for we have not determined whether a one-dimensional domain which is not strongly invariant must be a polynomial ring.

(5.1) Proposition. *Let A be a one-dimensional integrally closed domain. If $A[X_1, ..., X_n] = B[Y_1, ..., Y_n]$ and $A \neq B$, then there exists an element s of A such that $A[1/s]$ is a prefactorial Dedekind domain containing a field over which A is of transcendence degree one. In fact there exist fields $k' \subset k$ with k algebraic over k' and an element T transcendental over k such that*

$$k' \subset A[1/s] \subset k[T, 1/s],$$

and such that $k[T, 1/s]$ is integral over $A[1/s]$.

Proof. We first show that there must exist a maximal ideal M of B such that $M[Y_1, ..., Y_n] \cap A = (0)$. This would follow from (1.12) if we knew that B were also one-dimensional. Of course if we have some additional hypothesis such as that A is noetherian, then $A[X_1, ..., X_n]$ has dimension $n + 1$, so B must have dimension one. However, without some such simplifying hypothesis, $A[X_1, ..., X_n]$ can have dimension greater than $n + 1$ [19, p. 511], so we give a somewhat indirect argument. By (1.10), A must have Jacobson radical (0). Thus A and hence $A[X_1, ..., X_n] = B[Y_1, ..., Y_n]$ are Hilbert rings [13, p. 18]. Since B is a homomorphic image of $B[Y_1, ..., Y_n]$, B is also a Hilbert ring. If M is a maximal ideal of B and $M[Y_1, ..., Y_n] \cap A = P \neq (0)$, then $P[X_1, ..., X_n] \subset M[Y_1, ..., Y_n]$, and since these two prime ideals have the same depth, we must have $P[X_1, ..., X_n] = M[Y_1, ..., Y_n]$, which means that M is vertical relative to A. It follows from (1.13) that $M[Y_1, ..., Y_n] \cap A = (0)$ for some maximal ideal M of B. Considering residue class rings modulo $M[Y_1, ..., Y_n]$ and applying the "cutting down lemma" (2.5), we obtain $A[x_1, ..., x_n] = k[T]$, where k is the field B/M and T is an indeterminate over k. By the normalization theorem [17, p. 45], there exists $s \in A$ such that $A[1/s, x_1, ..., x_n] = k[1/s, T]$ is a finite integral extension of $A[1/s]$. Since A is integrally closed, $A[1/s]$ is integrally closed. Thus $A[1/s]$ is a Dedekind domain, and by (2.14), $A[1/s]$ is prefactorial.

It remains to show that $A[1/s]$ contains a field k' such that k is an algebraic extension of k'. Let L' denote the quotient field of $A[1/s]$, and let L be the normal closure of the algebraic field extension $k(T)$ of L'. If D is the integral closure of $A[1/s]$ in L, then D is also the integral closure of $k[T, 1/s]$ in L, so D is an affine ring over k. The automorphisms of L over L' restrict to automorphisms of D. Let D^G denote the fixed ring of D under the group G of automorphisms of L over L'. By (2.4), D^G contains a subfield k_1 of k such that k is a finite algebraic extension of k_1. Moreover, D^G is the integral closure of $A[1/s]$ in a finite purely inseparable field extension. Hence if p is the characteristic of k_1, then for some positive integer e, $k_1^{p^e} \subset A[1/s]$. This completes the proof of (5.1).

(5.2) *Remark.* It is clear that if A is an integral domain for which the integral closure is strongly invariant, then A is strongly invariant. It seems conceivable that if A is a one-dimensional integrally closed domain which is not strongly invariant, then A is a polynomial ring. If this is the case, then polynomial rings are the only one-dimensional integral domains which are not strongly invariant. For if A is a one-dimensional domain whose integral closure A^* is a polynomial ring, then $A \neq A^*$ implies A is strongly invariant. This can be seen as follows: If $A[X_1, ..., X_n] = B[Y_1, ..., Y_n]$, then $A^*[X_1, ..., X_n] = B^*[U_1, ..., Y_n]$, where B^* is the integral closure of B. Since $A \neq A^*$, (1.15) and (1.5) imply that some maximal ideal of A^* is vertical relative to B^*. Since A^* is a polynomial ring, this implies that $A^* = B^*$, and hence that $A = B$.

For a certain class of one-dimensional integrally closed domains we can prove a sharper version of (5.1).

(5.3) Definition. An integral domain D is said to be a *locally finite intersection of valuation rings* if $D = \bigcap_\alpha V_\alpha$ where $\{V_\alpha\}$ is a set of valuation rings having the property that each nonzero element d of D is a unit in all but a finite number of the V_α. Thus the integral domains which are locally finite intersections of valuation rings include, for example, all noetherian integrally closed domains [21, p. 82].

(5.4) Let A be a one-dimensional integral domain which is a locally finite intersection of valuation rings. Suppose that $A[X_1, ..., X_n] = B[Y_1, ..., Y_n]$, with $A \neq B$, and let k denote the quotient field of $A \cap B$, then

(i) if $A \cap B = k$, then A and B are both polynomial rings over k,

(ii) if $A \cap B < k$, then $A \cap B$ is a finite intersection of rank one valuation rings and hence is a one-dimensional Prüfer domain with only a finite number of prime ideals,

(iii) in case (ii), if u is any nonzero element of the Jacobson radical of $A \cap B$, then both $A[1/u]$ and $B[1/u]$ are polynomial rings over k. In particular, the quotient fields of A and B are simple transcendental extensions of the quotient field of $A \cap B$.

Proof. By (5.1), there exists an element s of A such that $A[1/s]$ contains a field k' over which A is of transcendence degree one. If $A = \bigcap_\alpha V_\alpha$ is a representation of A as a locally finite intersection of valuation rings, we may assume that all the V_α are contained in the quotient field of A. Since A is one-dimensional, it follows that each V_α is a localization of A and is of rank one. By the locally finite assumption, u is a nonunit in only a finite number of the V_α, say $V_1, \ldots, V_m$. We have $A = A[1/s] \cap V_1 \cap \cdots \cap V_m$, so $A \cap k' = V_1 \cap \cdots \cap V_m \cap k'$. Thus $A \cap k'$ has quotient field k' and has nonzero Jacobson radical [17, p. 38]. It follows that $A \cap k'$ is contained in A_c (where A_c is the algebraic closure in A of the subring of A generated by the units of A). Since $A_c \subset B$, we have $k' \cap A \subset A \cap B$. Thus A must be of transcendence degree one over $A \cap B$. If $A \cap B$ is a field, then (3.3) implies that A and B are polynomial rings over $A \cap B$. If $A \cap B < k$, then since $k' \subset k$, we have $k \cap V_1 \cap \cdots \cap V_m \subset A \cap B < k$. Thus $A \cap B$ is a finite intersection of rank one valuation rings which establishes (ii), and (iii) follows by again applying (3.3).

(5.5) Corollary. *Let A be a Dedekind domain. If $A[X_1, \ldots, X_n] = B[Y_1, \ldots, Y_n]$ and $A \neq B$, then A is of transcendence degree one over $A \cap B$. Moreover, if $A \cap B$ is a field, then A and B are polynomial rings over $A \cap B$; if $A \cap B$ is not a field, then it is a semilocal principal ideal domain, and if u is any nonzero element in the Jacobson radical of $A \cap B$, then $A[1/u]$ and $B[1/u]$ are polynomial rings over the quotient field of $A \cap B$.*

(5.6) *Remark.* With reference to (5.5) and its notation, we know of no example where $A \cap B$ is not a field. We show in section 6 that if A contains a field of characteristic zero, then $A \cap B$ must be a field.

We close this section with a look at the structure of Dedekind domains of the type mentioned in (5.5). An integral domain A is called a *Krull domain* if A is a locally finite intersection of rank one discrete valuation rings. If A is a Krull domain and $\{P_\alpha\}$ is the set of height one primes of A, then $A = \bigcap_\alpha A_{P_\alpha}$, and each A_{P_α} is a rank one discrete valuation ring. The A_{P_α} are called the *essential valuation rings* of A. If $A = \bigcap_\beta W_\beta$ is a representation of A as a locally finite intersection of rank one valuation rings of the quotient field of A then each A_{P_α} must be in the set $\{W_\beta\}$.

Let k be a field and let $k(T)$ be a simple transcendental field extension. We wish to examine Krull domains and especially Dedekind domains (which are precisely the one-dimensional Krull domains) A such that for some element

$u \in A$, $A[1/u] = k[T]$. We note that for many fields k such domains $A \neq k[T]$ exist. For if $V_1, \ldots, V_n$ are rank one discrete valuation rings with quotient field k and $V_1^*, \ldots, V_m^*$ are extensions of the V_i to rank one discrete valuations of $k(T)$, then $A = \bigcap_{i=1}^{m} V_i^* \cap k[T]$ is a Krull domain; and for any nonzero u in the Jacobson radical of the semilocal domain $\bigcap_{i=1}^{n} V_i = J$, $u^t T$ will be in every V_i^* for some positive integer t. Hence $J[u^t T] \subset A$, so A has quotient field $k(T)$ and $A[1/u] = k[T]$. The following proposition shows that such Krull domains are sometimes Dedekind and gives a precise condition in order that this be the case.

(5.7) PROPOSITION. *Let A be a Krull domain such that for some element $u \in A$, $A[1/u] = k[T]$, where k is a field and T is transcendental over k. Then A is noetherian and of dimension one or two. A necessary and sufficient condition that A be one-dimensional (i.e., that A be a Dedekind domain) is the following*:

(*) *If V^* is any essential valuation ring of A in which u is a nonunit and $V = V^* \cap k$, then the residue field of V^* is algebraic over the residue field of V.*

Proof. Let $V_1^*, \ldots, V_m^*$ denote the essential valuation rings of A in which u is a nonunit and let $J = k \cap V_1^* \cap \cdots \cap V_m^*$. Then J is a semilocal principal ideal domain ([17, p. 38] and [21, p. 278]) and by multiplying T by a suitable high power of an element in the Jacobson radical of J, we get an element in A. Thus we may assume that $T \in A$. It follows that A is a Krull domain between the two-dimensional noetherian domain $J[T]$ and its quotient field, and hence that A is noetherian [10] and of dimension at most two [9, pp. 348–9].

Let $V_i^* \cap k = V_i$ and let F_i^* and F_i denote the residue fields of V_i^* and V_i. Let P_i^* denote the center of V_i^* on A, and let P_i denote the center of V_i on J. We have $F_i = J/P_i \subset A/P_i^* \subset F_i^*$, and if F_i^* is algebraic over F_i, then $A/P_i^* = F_i^*$, and P_i^* is a height one prime of A which is also a maximal ideal. Hence if (*) holds, then u is contained only in maximal ideals of A, and $A[1/u] = k[T]$ implies that A is one-dimensional.

Now suppose that some F_i^* is not algebraic over F_i. Since V_i^* is a localization of A, we can choose $\alpha \in A$ such that the residue of α in A/P_i^* is transcendental over F_i. Moreover, we can choose α so that $k(T)$ is a separable algebraic extension of $k(\alpha)$. For if T will serve for α, then the assertion is obvious; and if the residue of T in A/P_i^* is algebraic over F_i, then $k(T)$ is separable algebraic over either $k(\alpha)$ or $k(\alpha + T)$. Let B denote the integral closure of $J[\alpha]$ in $k(T)$. Then $B \subset A$, and B is a finite $J[\alpha]$-module, so B is a finitely generated J-algebra. Moreover, V_i^* is centered on a nonmaximal ideal of B, which implies that V^*_i is a localization and hence an essential valuation ring for B. If y is an element in the Jacobson radical of J, then $B[1/y] =$

$k[T]$. Hence there exist at most a finite number $W_1, \ldots, W_n$ of essential valuation rings for B which are not essential valuation rings for A and we have $B = A \cap W_1 \cap \cdots \cap W_n$. We can choose $s \in B$ such that s is not in the center of V_i^* on B, but such that s is a nonunit in each W_j. It follows that $A \subset B[1/s] \subset V_i^*$. Since $B[1/s]$ is a finitely generated J-algebra, we see that V_i^* can not be centered on a maximal ideal of $B[1/s]$, for this would imply that F_i^* is a finitely generated ring extension of F_i contradicting the assumption that F_i^* is transcendental over F_i. But if A were one-dimensional, then $A[1/s] = B[1/s]$ would be one-dimensional. Hence if condition (*) is not satisfied, then A must be two-dimensional.

6. Invariance of Dedekind Domains which Contain a Field of Characteristic Zero

This section is devoted to proving that a one-dimensional noetherian domain A containing a field of characteristic zero is either strongly invariant or a polynomial ring. As observed in (5.2), it will suffice to prove this when A is integrally closed and hence a Dedekind domain.

(6.1) *Remark.* Let A be a Dedekind domain and suppose that

$$A[X_1, \ldots, X_n] = R = B[Y_1, \ldots, Y_n]$$

and $A \neq B$. By (5.5), if $A \cap B = C$ is a field, then A and B are polynomial rings over $A \cap B$; and if $A \cap B = C$ is not a field, then C is a semilocal principal ideal domain. Assume that C is not a field and let $\pi \in C$ generate a maximal ideal of C. If πA is a prime ideal in A, then the field extension $C/\pi C \subset A/\pi A$ can not be separable.

Proof. If πA is prime, then πR and πB are also prime ideals and taking residues in $R/\pi R$, we have $A/\pi A[X_1, \ldots, X_n] = R/\pi R = B/\pi B[Y_1, \ldots, Y_n]$. Hence $A/\pi A = B/\pi B$. By (5.7), $A/\pi A$ is an algebraic field extension of $C/\pi C$. Suppose that $A/\pi A$ and hence $B/\pi B$ were separable algebraic over $C/\pi C$. We can write any $a \in A$ in the form $a = b + m$, where $b \in B$ and $m \in B[Y_1, \ldots, Y_n]$ is a polynomial with zero constant term. Letting "‾" denote residue class in $R/\pi R$, we have $\bar{a} = \bar{b} + \bar{m}$, and since $A/\pi A = B/\pi B$, we must have $\bar{a} = \bar{b}$ and $\bar{m} = 0$. Now $A \neq B$ implies that there exists $a \in A$ such that $a = b + m$ and $m \neq 0$. Choose $a = b + m$ such that in the π-adic valuation of R, m has the smallest value possible (i.e., among all elements of the form $a = b + m$ choose an a such that m is divisible by the smallest power of π). With the assumption that $B/\pi B$ is separable over $C/\pi C$, we show that this leads to contradiction. Let $\bar{g}(Z)$ be the minimal polynomial for $\bar{b}$ over $C/\pi C$ and lift $\bar{g}(Z)$ to a polynomial $g(Z)$ in $C[Z]$ of the same degree. From Taylor's formula we have

$$g(Z) = g(b) + g'(b)(Z - b) + h(Z)(Z - b)^2,$$

where $g'(Z)$ is the derivative of $g(Z)$ and $h(Z)$ is a polynomial in $B[Z]$. Hence $g(a) = g(b) + g'(b)m + h(a)m^2$. Since $\bar{g}(Z)$ is a separable polynomial, the residue of $g'(b)$ is not zero. This implies that the π-adic value of $g'(b)m + h(a)m^2$ is equal to the π-adic value of m. But the residue of $g(b)$ in $R/\pi R$ is zero, which means that π divides $g(a)$. Hence

$$g(a)/\pi = g(b)/\pi + (g(b')m + h(a)m^2)/\pi = b' + m',$$

with $b' \in B$ and m' a polynomial in $B[Y_1, ..., Y_n]$ with zero constant term and with π-adic value strictly less than the π-adic value of m. This contradiction of the choice of $a = b + m$ completes the proof of (6.1).

(6.2) Corollary. *If A is a principal ideal domain* (PID) *containing a field of characteristic zero, then A is invariant, and if A is not a polynomial ring, then A is strongly invariant.*

Proof. Suppose $A[X_1, ..., X_n] = B[Y_1, ..., Y_n]$ and $A \neq B$. If $A \cap B$ is a field, then (5.5) implies that A is a polynomial ring over $A \cap B$. Suppose that $A \cap B = C$ is not a field. Let πC be a maximal ideal of C. The fact that C is an inert subring of A and that A is a PID imply that πA must be a prime ideal. By (5.7) $A/\pi A$ is algebraic over $C/\pi C$ and since A contains a field of characteristic zero $A/\pi A$ is separable algebraic over $C/\pi C$. But (6.1) implies that this cannot happen. Hence $A \neq B$ implies that $A \cap B$ is a field.

In order to apply (6.1) to a wider class of rings, we note the following obvious facts.

(6.3) Let A be an integral domain and suppose that $A[X_1, ..., X_n] = B[Y_1, ..., Y_n]$. Let Δ be a set of elements, each of which is algebraic over $A \cap B$. Then $A[\Delta][X_1, ..., X_n] = B[\Delta][Y_1, ..., Y_n]$ and $A[\Delta] = B[\Delta]$ if and only if $A = B$.

(6.4) Let A be an integral domain and suppose that $A[X_1, ..., X_n] = R = B[Y_1, ..., Y_n]$. If an integral domain R^* is algebraic over R and if there exist domains A^* and B^* algebraic over A and B, respectively, such that $A^*[X_1, ..., X_n] = R^* = B^*[Y_1, ..., Y_n]$, then $A = B$ if and only if $A^* = B^*$.

(6.5) Theorem. *If A is a Dedekind domain containing a field of characteristic zero, then A is invariant, and if A is not a polynomial ring, then A is strongly invariant.*

Proof. Suppose that $A[X_1, ..., X_n] = R = B[Y_1, ..., Y_n]$ and that $A \neq B$. By (5.5), $A \cap B = C$ is a semilocal PID and we may assume that C is not a field. Let πC be a maximal ideal of C and let p be a height one prime of R which contains π. Suppose that π generates the m-th power of the maximal ideal of the valuation ring R_p, say $\pi = \mu\gamma^m$ where $\gamma R_p = pR_p$ and μ is a unit in R_p. If we adjoin to C a root θ of the polynomial $Z^m - \pi$, then adjoining θ to the quotient field of R_p is equivalent to adjoining a root of $Z^m - \mu = 0$. Since the discriminant of this polynomial is a unit, the extension $R_p[\theta]$ of R_p is unramified and θ generates the maximal ideal of each localization of the integral closure of $R_p[\theta]$. [21, p. 303]. Thus in view of (6.3), we may assume in our original notation that π generates the maximal ideal of R_p. Let $p = p_1, p_2, ..., p_s$ be all the height one primes of R which contain π. Since $\pi \in A \cap B$, each p_i is vertical relative to both A and B. Thus each R_{p_i} is an essential valuation ring for R which is the extension of essential valuation rings from both A and B. Let R^* denote the Krull domain obtained by intersecting all the essential valuation rings for R except $R_{p_2}, ..., R_{p_s}$. In Nagata's terminology, R^* is the ideal transform of R with respect to the ideal $I = p_2 \cap \cdots \cap p_s$ [16, p. 41]. The importance of R^* in our considerations is that since $A[X_1, ..., X_n] = R = B[Y_1, ..., Y_n]$, and since $R_{p_2}, ..., R_{p_s}$ are extensions of essential valuation rings from both A and B, we have $R^* = A^*[X_1, ..., X_n] = B^*[Y_1, ..., Y_n]$ where A^* and B^* are the Krull domains obtained by intersecting all the essential valuation rings for A and B, respectively, except those which extend to $R_{p_2}, ..., R_{p_s}$. By (6.4), it will suffice to contradict the assumption that $A^* \neq B^*$. In R^*, $pR_p \cap R^*$ is the only height one prime containing π and π generates the maximal ideal of R_p. Thus πR^* is prime in R^*, so $\pi A^* = \pi R^* \cap A^*$ is also prime. Since we are assuming that A, and hence $C = A \cap B$, contains a field of characteristic zero, (6.1) implies that $A^* = B^*$. This completes the proof of (6.5).

(6.6) Corollary. *If A is a one-dimensional integral domain containing a field of characteristic zero and if the integral closure of A is a Dedekind domain, then A is invariant, and A is strongly invariant if A is not a polynomial ring. In particular, one-dimensional noetherian domains containing a field of characteristic zero are invariant.*

Proof. If the integral closure A' of A is not a polynomial ring, then in view of (6.5) and (6.4) the assertion is clear. If $A \neq A'$, and A' is a polynomial ring, then it is observed in (5.2) that A is strongly invariant.

7. Questions

We conclude with a few observations and questions related to the material considered in this article. There is an unanswered question of Zariski concerning simple transcendental field extensions that seems somewhat similar to the problem we have considered. Nagata [18, p. 89] states the Zariski problem as follows.

Let K and K' be finitely generated fields over a field k. Assume that simple transcendental extensions of K and K' are k-isomorphic to each other. Does it follow that K and K' are k-isomorphic to each other?

The following question might be more easily settled, at least for $n = 1$.

(7.1) *Question.* If A and B are integral domains and $A[X_1, ..., X_n] = B[Y_1, ..., Y_n]$, does it follow that the quotient fields of A and B are isomorphic?

Nagata pointed out to us the following fact about the one variable case.

(7.2) If A is an integral domain and $A[X] = R = B[Y]$, then there exist isomorphisms of A into B and B into A.

Proof. If $B \subset A$, then $B = A$ and the assertion follows. If $B \neq A$, then we may assume that $Y \notin A$, for if $Y \in A$, we may replace Y by $Y - b$ where $b \in B \backslash A$. Let d be any element of $A \cap B$ and consider the canonical homomorphism $\phi : B[Y] \to B[Y]/(Y - d) \cong B$. The restriction of ϕ to A must be an isomorphism. For if $a \in A$ is in the kernel of ϕ, then $Y - d$ divides a in $A[X] = R$. But $Y \notin A$ and $d \in A$ imply that $Y - d$ is a nonconstant polynomial in $A[X]$. Hence $a = 0$. Since the situation is symmetric, the proof is complete.

Note that the proof of (7.2) shows that A and B are in fact simple ring extensions of isomorphic copies of each other.

(7.3) *Question.* If $A[X_1, ..., X_n] = B[Y_1, ..., Y_n]$, do there exist isomorphisms of A into B and B into A? In particular, does there exist an isomorphism $\phi : A \to B$ such that B is a finitely generated ring extension of $\phi(A)$?

The Zariski problem is known to have an affirmative answer in certain special cases, and this of course yields some information about (7.1). For example, the answer to the Zariski question being yes for k algebraically closed of characteristic zero and K and K' of transcendence degree two over k [18, p. 90], implies the following. If A is a two-dimensional affine domain over an algebraically closed field of characteristic zero and if $A[X] = B[Y]$, then the quotient fields of A and B are isomorphic.

Nagata in [18] makes use of a lemma of Abhyankar concerning quadratic transformations along a valuation ring [2, p. 336], to study the Zariski problem, and the following proposition illustrates the use of this same lemma in connection with (7.1).

(7.4) PROPOSITION. *Let A be an integral domain and suppose that A is of finite transcendence degree over the subring A_u of A generated by the units of A. Suppose that $A[X] = B[Y]$, and let K and L denote the quotient fields of A and B, respectively. If $A \neq B$, then K and L are both ruled over the quotient* *field k of A_u*—i.e., *K and L are both simple transcendental extensions of fields containing k.*

Proof. We have $A_u \subset A \cap B$, so k is a subfield of $K \cap L$. Since $A \neq B$, some element of B is a polynomial of positive degree in $A[X]$. Hence if V denotes the $1/X$-adic valuation ring of $K(X)$, then $V \cap L = V^*$ is properly contained in L. Since the residue field of V is canonically isomorphic to K, by counting transcendence degrees over k, we see that the residue field of V is transcendental over the residue field of V^*. It follows from the above mentioned lemma concerning quadratic transformations of a regular local ring along a prime divisor [2, p. 336], that K, the residue field of V, is a simple transcendental extension of a finite algebraic extension of the residue field of V^*. Since the situation is symmetric, the proof is complete.

(7.5) COROLLARY. *Let A be an integral domain with quotient field K, and suppose that A is of transcendence degree two over the subring A_u of A generated by the units of A. Let k be the quotient field of A_u and assume that K is a pure transcendental extension of some algebraic extension k^* of k. If $A[X] = B[Y]$, then the quotient fields of A and B are isomorphic.*

Proof. Let L denote the quotient field of B. We have $A_u \subset B$, so $k^* \subset L$. Let Z_1 and Z_2 be such that $k^*(Z_1, Z_2) = K$. Thus $k^*(Z_1, Z_2, X) = L(Y)$. By (7.4), there exists a field F and a transcendental element T over F such that $k \subset F$ and $F(T) = L$. Since k^* is algebraic over k, we have

$$k^* \subset F \subset k^*(Z_1, Z_2, X).$$

Hence by Igusa's generalization of the classical Luroth theorem ([11, or 18, p. 87]), F must be a simple transcendental extension of k^*. Since $L = F(T)$, it follows that L is a pure transcendental extension of k^*, and hence that L is isomorphic to K.

(7.6) COROLLARY. *Let A be a two-dimensional affine domain over an algebraically closed field k. If $A[X] = B[Y]$, then the quotient fields of A and B are isomorphic.*

Proof. Let K and L denote the quotient fields of A and B. By (7.4), K and L are both ruled over k. Suppose $K = F_1(T_1)$ and $L = F_2(T_2)$, with $k \subset F_i$. Then F_1 and F_2 are function fields in one variable over k. If either F_1 or F_2 is of genus zero, then it is a simple transcendental extension of k and (7.5) implies that K and L are isomorphic. If F_1 is of positive genus, then since F_1 is separably generated over k, every finite algebraic extension of F_1 is also of positive genus and hence not ruled over k. Thus in Nagata's terminology, F_1 is antirational over k, and $F_1(T_1, X) = F_2(T_2, Y)$ implies that $F_1 = F_2$ [18, p. 88]. Hence $K = F_1(T_1)$ is isomorphic to $L = F_1(T_2)$.

(7.7) *Remark.* The ideas of (7.4) provide a different proof of the "one variable" case of (3.3). We state the result and sketch an argument.

Let A be an integral domain of transcendence degree one over a field k and suppose $A[X] = B[Y]$ then either $A = B$ or each of A and B is a polynomial ring over $A \cap B$.

Proof. One simplifies (3.1) and (3.2) by the "one variable assumption" and reduces to the case where A and B are prefactorial Dedekind domains, and k is algebraically closed in K. Then as in the proof of (7.4) let K and L be the respective quotient fields of A and B, and let V denote the $1/X$-adic valuation ring on $K(X)$. Let $V^* = V \cap L$. Since $A \neq B$, V^* is properly contained in L. As in the proof of (7.4) K, the residue field of V is a simple transcendental extension of a finite algebraic extension of k' the residue field of $V^*(k \subset k' \subset k'' \subset k''(t) = K)$. Since k is algebraically closed in K, $k = k''$, thus the residue field of V^* is k and $K = k(t)$. It then follows that A satisfies the following

(1) $k \subset A \subset k(t) =$ quotient field of A,

(2) A is a prefactorial Dedekind domain,

(3) There is a k-valuation of $k(t)$, V^*, such that $A \not\subset V^*$ and the residue field of V^* is k.

(4) $k =$ units of A.

But (1)–(4) imply $A = k[\theta]$ for some $\theta \in k(t)$. For A is the intersection of a family of rank one discrete valuation rings since A is Dedekind. These are all k-valuations of $k(t)$. From (1) and (4) together with the fact that A is prefactorial, it follows that A is contained in every k-valuation of $k(t)$ except one. By (3) the unique k-valuation of $k(t)$ not containing A has residue field k. Thus it is either the $1/t$-adic valuation, or it is the f-adic valuation where f is an irreducible element of $k[t]$ of the form $t - \lambda$ for $\lambda \in k$. In the first case $A = k[t]$, in the latter $A = k[1/f]$.

In considering invariance and strong invariance when $A[X_1, ..., X_n] = B[Y_1, ..., Y_n]$, it would perhaps have been more precise to use the terms n-invariant and n-strongly invariant.

(7.8) *Question.* Is it possible for an integral domain to be n-invariant (or n-strongly invariant), but not be m-invariant (m-strongly invariant) for different positive integers m and n?

(7.9) *Question.* If A is a strongly invariant integral domain and

$$A[X_1, ..., X_n] = B[Y_1, ..., Y_m],$$

must it follow that $A \subset B$?

We have seen that one-dimensional affine domains over a field are invariant. Perhaps the most natural question is the following:

(7.10) *Question.* If A is a two-dimensional affine domain over a field k, is A invariant? In particular, is the polynomial ring $k[X, Y]$ invariant?

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