On a Deflation Method for the Symmetric Generalized Eigenvalue Problem

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ABSTRACT

A generalization of the Fix-Heiberger reduction is used to deflate the infinite and the singular structure from a symmetric matrix pencil $A - \lambda B$. The finite eigenvalues can be determined from the remaining symmetric problem. With the aid of this deflation method it is shown that the Kronecker canonical form of $A - \lambda B$ is very special if $B$ is positive semidefinite.

1. INTRODUCTION

For the general matrix pencil

$$A - \lambda B,$$  

(1)

where $A$ and $B$ are both $m \times n$ matrices, there exist an $m \times m$ matrix $P$ and an $n \times n$ matrix $Q$ whose elements are independent of $\lambda$, such that $P(A - \lambda B)Q$ has the Kronecker canonical form [1]

$$P(A - \lambda B)Q = \begin{bmatrix}
A_N - \lambda B_N & A_\varepsilon - \lambda B_\varepsilon \\
& A_\eta - \lambda B_\eta
\end{bmatrix},$$  

(2)


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where:

(i) $A_N - \lambda B_N$ is a square matrix pencil, and is of the following form:

$$A_N - \lambda B_N = \begin{bmatrix} J - \lambda I & I - \lambda N \end{bmatrix},$$  

where the matrix $J$ is in the Jordan canonical form, and $N$ is a Jordan canonical matrix with zero diagonal elements.

(ii) $A_e - \lambda B_e$ is a block diagonal matrix pencil whose diagonal blocks consist of bidiagonal matrices, each of which is an $\epsilon_i \times (\epsilon_i + 1)$ matrix of the form

$$\begin{bmatrix} \epsilon_i & \epsilon_i + 1 \\ \epsilon_i & \epsilon_i + 1 \\ \epsilon_i & \epsilon_i + 1 \\ \vdots & \vdots \\ -\lambda & 1 \\ -\lambda & 1 \end{bmatrix},$$  

where $\epsilon_i$ is a nonnegative integer and is called the Kronecker column index.

(iii) $A_\eta - \lambda B_\eta$ is also a block diagonal matrix pencil whose diagonal blocks consist of bidiagonal matrices, each of which is an $(\eta_i + 1) \times \eta_i$ matrix of the form

$$\begin{bmatrix} \eta_i & \eta_i \\ \eta_i & \eta_i \\ \eta_i & \eta_i \\ \vdots & \vdots \\ -\lambda & 1 \\ -\lambda & 1 \end{bmatrix},$$  

where $\eta_i$ is a nonnegative integer and is called the Kronecker row index.

If $A$ and $B$ are both square matrices and $\det(A - \lambda B) \neq 0$, then (1) is called a regular matrix pencil. In this case, there are no blocks whose forms are (4) or (5), i.e., (3) is the canonical form of the regular matrix pencil; it is then also called the Weierstrass canonical form.

In this paper, we generalize the Fix-Heiberg reduction to deflate the infinite and the singular structure from a symmetric matrix pencil $A - \lambda B$ and determine the finite eigenvalues from the remaining symmetric problem.
With the aid of this deflation method we show that the Kronecker canonical form of $A - \lambda B$ is very special if $B$ is symmetric positive semidefinite.

2. GENERAL CASE

We start with generalizing the Fix-Heiberger reduction [6, 9] to deflate $A$ and $B$ simultaneously when $A - \lambda B$ is a general $n \times n$ symmetric matrix pencil.

1. Construct an orthogonal matrix $Q_1$ to diagonalize $B$:

$$B^{(0)} \equiv Q_1^T B Q_1 = \begin{bmatrix} D_{11} & 0 \\ 0 & 0 \end{bmatrix}_{n_1},$$

where $D_{11}$ is a diagonal matrix whose entries consist of the $n_1$ nonzero eigenvalues of $B$. Apply the same transformation to $A$, and partition the resulting matrix $A^{(0)}$ into the form

$$A^{(0)} \equiv Q_1^T A Q_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}_{n_1}.$$

2. Diagonalize $A_{22}$ with the orthogonal matrix $Q_{22}$:

$$Q_{22}^T A_{22} Q_{22} = \text{diag}(D_{33}, 0),$$

where $D_{33}$ is a diagonal matrix whose entries consist of the nonzero eigenvalues of $A_{22}$. Then apply the congruent transformation associated with $Q_2 = \text{diag}(I, Q_{22})$ to get

$$A^{(1)} \equiv Q_2^T A^{(0)} Q_2 = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12}^T & D_{33} & 0 \\ A_{13}^T & 0 & 0 \end{bmatrix}_{n_1},$$

$$B^{(1)} \equiv Q_2^T B^{(0)} Q_2 = \begin{bmatrix} D_{11} \\ 0 \\ 0 \end{bmatrix}.$$
From hereafter, we reuse some of the names of the blocks after processing them to avoid step superscripts.

3. Construct an \( n_1 \times n_1 \) orthogonal matrix \( Q_{11} \) and an \( n_4 \times n_4 \) orthogonal matrix \( Q_{33} \) such that

\[
Q_{11}^T A_{13} Q_{33} = \begin{bmatrix}
\tilde{n}_4 & 0 \\
0 & \tilde{n}_4 - \tilde{n}_4
\end{bmatrix}
\]

(8)

where \( A_{14} \) is an \( \tilde{n}_4 \times \tilde{n}_4 \) nonsingular matrix. Then apply the congruent transformation associated with

\[
Q_3 = \text{diag} \left( \frac{Q_{11}}{n_1}, I, \frac{Q_{33}}{n_4} \right)
\]

to get

\[
A^{(2)} \equiv Q_3^T A^{(1)} Q_3 = \begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} & 0 \\
A_{12}^T & A_{22} & A_{23} & 0 & 0 \\
A_{13}^T & A_{23}^T & D_{33} & 0 \\
A_{14}^T & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\equiv \begin{bmatrix}
\tilde{A}^{(2)} & 0 \\
0 & 0
\end{bmatrix}
\]

(9)

If \( A_{13} \) in (7) is of full column rank, then \( \tilde{n}_4 = 0 \), and so \( A^{(2)} = \tilde{A}^{(2)} \) and \( B^{(2)} = \tilde{B}^{(2)} \).

**Theorem 1.** If \( A - \lambda B \) is a regular matrix pencil [1], then the \( n_1 \times n_4 \) matrix \( A_{13} \) in (7) is of full column rank.
Proof. If, on the contrary, \( A_{13} \) is not of full column rank, then there exists an \( n_4 \)-dimensional vector \( y_3 \neq 0 \) such that \( A_{13}y_3 = 0 \). Hence for the \( n \)-dimensional nonzero vector \( y = [0, 0, y_3^T] \) we have

\[
(A^{(i)} - \lambda B^{(i)})y = 0.
\]

This holds for any \( \lambda \), so \( A - \lambda B \) is a singular matrix pencil.

For the generalized eigenvalue problem (1) we have separated out a subset of the Kronecker indices \( (n_4 - \tilde{n}_4 \) zero Kronecker row and column indices) [1]. Hence we only need to consider the generalized eigenvalue problem

\[
\tilde{A}^{(2)}x = \mu \tilde{B}^{(2)}x
\]

associated with the matrices in (9). Let \( x = [x_1^T, x_2^T, x_3^T, x_4^T]^T \), and substitute this \( x \) in (10). The problem is now separated into an \( (n_1 - \tilde{n}_4) \times (n_1 - \tilde{n}_4) \)

symmetric generalized eigenvalue problem

\[
(A_{22} - A_{23}D_{33}^{-1}A_{23}^T)x_2 = \mu B_{22}x_2,
\]

and the following equations:

\[
x_1 = 0, \quad x_3 = -D_{33}^{-1}A_{23}^Tx_2, \quad x_4 = -A_{14}^{-1}(A_{12}x_2 + A_{13}x_3 - \mu B_{12}x_2).
\]

The symmetric deflation procedure can be continued with respect to the matrix pencil \( (A_{22} - A_{23}D_{33}^{-1}A_{23}^T) - \lambda B_{22} \), which is in accordance with the matrices in (11). These procedures are continued \( l \) times until the resulting right-hand \( (B_{22})_l \) is nonsingular. In that case we get

\[
(A_{22} - A_{23}D_{33}^{-1}A_{23}^T)_l(x_2)_l = \mu (B_{22})_l(x_2)_l.
\]

We can apply QZ algorithm [2, 8] or HR algorithm [3, 4] to solve the problem (12). If \( n_1 - \tilde{n}_4 \) is zero, then there are no finite eigenvalues.

3. ONE OF THE MATRICES IS SYMMETRIC POSITIVE SEMIDEFINITE

If one of the matrices in the symmetric matrix pencil \( A - \lambda B \) is positive (or negative) semidefinite, then the canonical form of this matrix pencil has a very special form and the solution of the corresponding generalized eigenvalue problem will be simplified considerably.
THEOREM 2. Suppose the matrix $B$ in the symmetric matrix pencil $A - \lambda B$ is positive semidefinite. Then $A - \lambda B$ is a regular matrix pencil if and only if the $n_1 \times n_4$ matrix $A_{13}$ in (7) is of full column rank.

Proof. We have proved the "only if" part in Theorem 1. Now we prove the "if" part. Since $B$ is symmetric positive semidefinite, the diagonal entries of the diagonal matrix $D_{11}$ in (7) are all positive. Hence apply the congruent transformation associated with $D_2 = \text{diag}(D_{11}^{-1/2}, I, I)$ to get

\[
\hat{A}_1 \equiv D_2 A^{(1)} D_2 = \begin{bmatrix}
\hat{A}_{11} & \hat{A}_{12} & \hat{A}_{13} \\
\hat{A}_{12}^T & D_{33} & 0 \\
\hat{A}_{13}^T & 0 & D_{33}
\end{bmatrix}, \quad \hat{B}_1 \equiv D_2 B^{(1)} D_2 = \begin{bmatrix}
I \\
0 \\
0
\end{bmatrix},
\]

(13)

where $\hat{A}_{13}$ is still of full column rank. By applying Fix-Heiberger reduction we get the following strictly equivalent matrix pairs ($\hat{A}_2, \hat{B}_2$) and ($\hat{A}_3, \hat{B}_3$):

\[
\hat{A}_2 = \begin{bmatrix}
\hat{A}_{11} & \hat{A}_{12} & \hat{A}_{13} & \hat{A}_{14} \\
\hat{A}_{12}^T & \hat{A}_{22} & \hat{A}_{23} & 0 \\
\hat{A}_{13}^T & \hat{A}_{23}^T & D_{33} & 0 \\
\hat{A}_{14}^T & 0 & 0 & D_{33}
\end{bmatrix}, \quad \hat{B}_2 = \begin{bmatrix}
I \\
0 \\
0 \\
0
\end{bmatrix}; \quad (14)
\]

\[
\hat{A}_3 = \begin{bmatrix}
\hat{A}_{11} & \hat{A}_{12} & \hat{A}_{13} & \hat{A}_{14} \\
\hat{A}_{12}^T & \hat{A}_{22} & \hat{A}_{23} & 0 \\
\hat{A}_{13}^T & \hat{A}_{23}^T & D_{33} & 0 \\
\hat{A}_{14}^T & 0 & 0 & D_{33}
\end{bmatrix}, \quad \hat{B}_3 = \begin{bmatrix}
I \\
0 \\
0 \\
0
\end{bmatrix}. \quad (15)
\]

After row and column interchange we get

\[
\hat{A}_4 = \begin{bmatrix}
0 & \hat{A}_{14}^T \\
\hat{A}_{22} & \hat{A}_{12}^T \\
0 & \hat{A}_{23}^T \\
\hat{A}_{12} & \hat{A}_{11} & \hat{A}_{13} & \hat{A}_{14}
\end{bmatrix}, \quad \hat{B}_4 = \begin{bmatrix}
0 & 0 & 0 \\
I & 0 & 0 \\
0 & 0 & I
\end{bmatrix}. \quad (16)
\]
Then with

$$
\begin{bmatrix}
1 & I \\
-\hat{A}_{12}^T \hat{A}_{14}^{-T} & I \\
-\hat{A}_{12} & \hat{A}_{11} & \hat{A}_{13} & \hat{A}_{14}
\end{bmatrix},
$$

(17)

left multiply $\hat{A}_4$ and $\hat{B}_4$, and interchange rows 1 and 2 to get

$$
\hat{A}_5 =
\begin{bmatrix}
\hat{A}_{22} & & & \\
0 & \hat{A}_{14}^T & & \\
0 & 0 & D_{33} & \\
\hat{A}_{12} & \hat{A}_{11} & \hat{A}_{13} & \hat{A}_{14}
\end{bmatrix},
\quad
\hat{B}_5 =
\begin{bmatrix}
I & & & \\
0 & 0 & & \\
0 & 0 & 0 & \\
0 & I & 0 & 0
\end{bmatrix}.
$$

(18)

This is a staircase form [10, 11]. Since the diagonal blocks are all square and the

$$
\hat{A}_{14}^T \quad \text{and} \quad \begin{bmatrix}
D_{33} \\
\hat{A}_{13} & \hat{A}_{14}
\end{bmatrix}
$$

are nonsingular, the matrix pencil $\hat{A}_5 - \lambda \hat{B}_5$ is regular [10, 11]. The matrix pencil $A - \lambda B$ is also regular because it is strictly equivalent to $\hat{A}_5 - \lambda \hat{B}_5$. ■

For the canonical form of the singular matrix pencil we have

**Theorem 3.** Suppose $B$ in the symmetric matrix pencil $A - \lambda B$ is positive semidefinite. Then the canonical form of $A - \lambda B$ is as follows:

$$
A - \lambda B \sim
\begin{bmatrix}
D - \lambda I & I & \cdots & I \\
I_2 - \lambda I_2 & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
I_2 - \lambda I_2 & \cdots & \cdots & 0
\end{bmatrix}_{n_1 - n_4} \quad \begin{bmatrix}
n_1 - n_4 \\
n_3 \\
2n_4 \\
n_4 - n_4
\end{bmatrix},
$$

(19)

where $D$ is a real diagonal matrix, and

$$
J_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.
$$
Proof. Since $B$ is symmetric positive semidefinite, the matrix pair $(A^{(2)}, B^{(2)})$ in (9) is strictly equivalent to the following matrix pair:

$$\begin{bmatrix} \tilde{A}_5 \\ 0 \end{bmatrix}, \begin{bmatrix} \tilde{B}_5 \\ 0 \end{bmatrix} \begin{bmatrix} n_1 + n_3 + \bar{n}_4 \\ n_4 - \bar{n}_4 \end{bmatrix}, \quad (20)$$

where $\tilde{A}_5$ and $\tilde{B}_5$ have the same form as $\hat{A}_5$ and $\hat{B}_5$ in (18) respectively. From Theorem 2 we know $\hat{A}_5 - \lambda \hat{B}_5$ is strictly equivalent to the regular part of $A - \lambda B$. Hence the matrices in (20) are staircase forms of a singular matrix pair. From (18) and (20) we can immediately get the all conclusions of the theorem.

From Theorem 3 we learn that if matrix pencil $A - XB$, where $B$ is positive semidefinite, is singular, then $N(A) \cap N(B) \neq \{0\}$, i.e., $A$ and $B$ must have a common null vector. Conversely, if $N(A) \cap N(B) \neq \{0\}$, then $A - \lambda B$ is singular. Hence we have

**COROLLARY 1.** Suppose $B$ in the symmetric matrix pencil is positive semidefinite. Then $A - \lambda B$ is singular matrix pencil if and only if $N(A) \cap N(B) \neq \{0\}$.

For the $2 \times 2$ symmetric matrix pencil we have

**COROLLARY 2.** A $2 \times 2$ symmetric matrix pencil $A - \lambda B$ is singular if and only if $A$ and $B$ have a common null vector.

Applying Theorem 3, we can also prove the following theorem easily (cf. [5])

**THEOREM 4.** Symmetric matrices $A$ and $B$ with $B$ positive semidefinite can be diagonalized simultaneously by nonsingular congruent transformation if and only if the submatrices

$$\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}$$

in (6) have the same rank.

Proof. From Theorem 3 we know that $A$ and $B$ can be diagonalized simultaneously by equivalent transformation if and only if $\bar{n}_4 = 0$ in (19), i.e.,
A DEFLATION METHOD

\( \hat{A}_{13} = 0 \) in (13). Obviously,

\[
\begin{bmatrix}
\hat{A}_{12} & 0 \\
D_{33} & 0 \\
0 & 0
\end{bmatrix}
\]

\[
\text{rank}
\begin{bmatrix}
\hat{A}_{12} & 0 \\
D_{33} & 0 \\
0 & 0
\end{bmatrix}
\]

Therefore

\[
\text{rank}
\begin{bmatrix}
A_{12} \\
A_{22}
\end{bmatrix}
= \text{rank}(A_{22}).
\]

We now show that we can diagonalize \( A \) and \( B \) by congruent transformation. Since

\[
\text{rank}
\begin{bmatrix}
A_{12} \\
A_{22}
\end{bmatrix}
= \text{rank}(A_{22}),
\]

we have \( \hat{A}_{13} = 0 \) in (13). Hence

\[
\hat{A}_1 =
\begin{bmatrix}
\hat{A}_{11} & \hat{A}_{12} \\
\hat{A}_{12}^T & D_{33}
\end{bmatrix},
\hat{B}_1 =
\begin{bmatrix}
I \\
0 \\
0
\end{bmatrix}.
\]

Let

\[
H =
\begin{bmatrix}
I \\
- D_{33}^{-1}\hat{A}_{12}^T & I
\end{bmatrix}.
\]

We have

\[
H^T\hat{A}_1H =
\begin{bmatrix}
\hat{A}_{11} - \hat{A}_{12}D_{33}^{-1}\hat{A}_{12}^T & D_{33} \\
0 & 0
\end{bmatrix},
\]

\[
H^T\hat{B}_1H =
\begin{bmatrix}
I \\
0 \\
0
\end{bmatrix}.
\]

Since \( \hat{A}_{11} - \hat{A}_{12}D_{33}^{-1}\hat{A}_{12}^T \) is symmetric, the conclusion follows immediately.

When the matrix \( B \) is positive semidefinite, the solution of the corresponding generalized eigenvalue problem (1) can be reduced to the solution
of the \((n_1 - \tilde{n}_4) \times (n_1 - \tilde{n}_4)\) symmetric eigenvalue problem [cf. (11) and (14)]
\[
(\hat{A}_{22} - \hat{A}_{23} D_{33}^{-1} \hat{A}_{23}^T) x_2 = \mu x_2.
\]

Assuming that the matrix \(\hat{A}_{13}\) [cf. (13)] is of full column rank, Fix and Heiberger have already derived (21). From Theorem 2 we know that their hypothesis about \(\hat{A}_{13}\) is the regularity of the matrix pencil \(A - \lambda B\), and from Theorem 3 we know that the \(n_1 - \tilde{n}_4\) eigenvalues of (21) are the all finite eigenvalues of (1). The matrix (1) has \(n_3 + 2\tilde{n}_4\) infinite eigenvalues, and if the pencil is singular (i.e., \(n_4 > \tilde{n}_4\)), then there exist \(n_4 - \tilde{n}_4\) zero Kronecker indices.

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