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The local density of triangle-free graphs

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Abstract

How dense can every induced subgraph of $\lfloor \alpha n \rfloor$ vertices $(0<\alpha\leqslant 1)$ of a triangle-free graph of order n be? Tools will be developed to estimate the local density of graphs, based on the spectrum of the graph and on a fractional viewpoint. These tools are used to refute a conjecture of Erdős et al. about the local density of triangle-free graphs for a certain range of α, by estimating the local density of the Higman–Sims graph via its eigenvalues. Moreover, the local density will be related to a long-standing conjecture of Erdős, saying that every triangle-free graph can be made bipartite by the omission of at most $n^2/25$ edges. Finally, a conjecture about the spectrum of regular triangle-free graphs is raised, which can be seen as a common relaxation of the two previous questions.

Keywords: Local density; Triangle-free graph; Spectrum; Least eigenvalue; Making graphs bipartite

1. Introduction and main results

It is an easy exercise to verify that every graph on n vertices can be made bipartite by the omission of

$$\binom{\lceil n/2\rceil}{2}+\binom{\lfloor n/2\rfloor}{2}<n^2/4$$

edges. According to an old conjecture of Erdős [6] the deletion of $n^2/25$ edges suffices to make a triangle-free graph bipartite.

Conjecture 1.1 (Erdős [6]). *Every triangle-free graph on n vertices can be made bipartite by the omission of at most $n^2/25$ edges.*

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This would be best possible, since that many edges are needed to make the 5-cycle bipartite (the lexicographic product $C_5[\overline{K}_r]$ is an infinite family for which this bound is tight). The current record is that the deletion of $n^2/18$ edges suffices to make triangle-free graphs bipartite [7].

How dense can a triangle-free graph be locally? More precisely, our objective is to determine for real numbers α $(0 \leqslant \alpha \leqslant 1)$ the smallest real number $\beta(\alpha)$ with the following property:

Every triangle-free graph of order n has a subset of $\lfloor \alpha n \rfloor$ vertices which span at most $\beta(\alpha)n^2$ edges.

Generalizing an older conjecture of Erdős [6], saying that $\beta(1/2) = 1/50$, the following conjecture was raised.

Conjecture 1.2 (Erdős et al. [8]).

$$\beta(\alpha) = \begin{cases} (2\alpha - 1)/4 & \text{if } 17/30 \leqslant \alpha \leqslant 1, \\ (5\alpha - 2)/25 & \text{if } 53/120 \leqslant \alpha < 17/30. \end{cases}$$

Note that the case $\alpha = 1$ follows from Mantel's theorem [12]. Krivelevich [11] verified Conjecture 1.2 for $\alpha \geqslant 3/5$, extending a result of [8]. We will refute Conjecture 1.2 for $0.442 \simeq 53/120 \leqslant \alpha < 474/1000$ by showing

Theorem 1.3.

$$\beta(\alpha) \geqslant (1000\alpha - 326)/10\,000 \quad \text{if } \alpha \leqslant 1.$$

It is easily checked that $(1000\alpha - 326)/10\,000 > (5\alpha - 2)/25$ if $0 < \alpha < 474/1000$. Theorem 1.3 is proved by estimating the local density of the Higman–Sims graph via its eigenvalues. The Higman–Sims graph is a 22-regular triangle-free graph of order 100. It is famous for its automorphism group, which is a sporadic simple group [10]. Although the estimation is probably not best possible, the Higman–Sims graph definitely is not a counterexample to Conjecture 1.2 at $\alpha = 1/2$.

The case $\alpha = 1/2$ of Conjecture 1.2, which is still open, is of particular interest. This is a long-standing conjecture of the late Paul Erdős, who informed the author in December 1995, that he offers 100$ for a proof or disproof. Slightly generalizing an observation due to Krivelevich [11] we will show, that the truth of Conjecture 1.2 for $\alpha = 1/2$ verifies Conjecture 1.1, restricted to regular graphs. It is conceivable that extremal triangle-free graphs for both problems are regular or almost regular. If both conjectures were true this would indeed be the case.

As we will see in the sequel, the precise determination of $\beta(\alpha)$ might be very hard if α is significantly smaller than 3/5. Therefore we will investigate estimates obtained from graph spectra.

Let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ be the eigenvalues of the adjacency matrix of a graph G of order n (for more information, see e.g. the monograph of Cvetković et al. [5] on graph spectra).

The investigation of eigenvalues has recently attracted much attention for measuring expansion properties and pseudo-random behaviour of regular graphs (see [1, Section 9.2]). Especially, the relations between the largest, second largest and smallest eigenvalue are of importance in this context. If G is a regular graph then the largest eigenvalue λ_1 equals the degree of regularity. Moreover, $\lambda_1 > \lambda_2$ if and only if G is connected, and $\lambda_n \geqslant -\lambda_1$ where equality holds if and only if G has a bipartite component.

Fiedler [9] proved that the difference $\lambda_1 - \lambda_2$ is a lower bound for the vertex connectivity of a non-complete regular graph (in fact, the second smallest Laplacian eigenvalue is a lower bound for the connectivity of any graph). We will show that the difference between λ_n and $-\lambda_1$ is a measure, how far a regular graph is from being bipartite.

Theorem 1.4. *The minimum number of edges which need to be deleted to make a regular graph G of order n bipartite is at least $(\lambda_1 + \lambda_n)n/4$.*

In view of this result, and as we are investigating dense graphs, the following parameter appears to be an interesting object of study. Call the function

$$\Xi(G) = (\lambda_1 + \lambda_n)/n \tag{1}$$

the *spectral ratio* of the graph G. If the spectral ratio of a regular graph is large, this means that the graph is dense and its least eigenvalue is relatively large at the same time. For triangle-free graphs (or, more generally, for graphs with bounded clique number) these concepts seem to be opposed.

Conjecture 1.5. *If G is a regular triangle-free graph then*

$$\Xi(G) \leqslant 4/25.$$

A direct consequence of Theorem 1.4 is that Conjecture 1.1 holding for regular graphs would imply Conjecture 1.5. As the truth of Conjecture 1.2 for $\alpha = 1/2$ verifies Conjecture 1.1 restricted to regular graphs, Conjecture 1.5 can be viewed as a common relaxation of the two former conjectures. In contrast to the other conjectures, the bound of Conjecture 1.5 is perhaps not tight. The largest spectral ratio of a regular triangle-free graph known to the author is $7/50 = 0.14$, attained by the Higman–Sims graph. Anyway, it would be hazardous speculation to conjecture that this bound is optimal.

Generalizing this concept it might be worthwhile to determine the function

$$\xi(s) = \sup_G \Xi(G),$$

where the supremum extends over all regular graphs G which do not contain K_{s+1}.

Theorem 1.6. (a) $\xi(2)\geqslant 0.14$ *and* $\xi(s)\geqslant(s-2)/s$ *for* $s\geqslant 3$,
(b) $\xi(s)\leqslant 5-4(\sqrt{s^2-s}+1)/s$ *for* $s\geqslant 2$.

Note that the upper bound, which was contributed for $s\geqslant 3$ by one of the referees, satisfies $\xi(s)\leqslant 1-2/s+\mathcal{O}(1/s^2)$, hence it asymptotically approaches the lower bound. Note that for the case $s=2$, what we are mainly concerned with, the upper bound $\xi(2)\leqslant 3-2\sqrt{2}\simeq 0.1715$ does not deviate a lot from the lower bound.

2. Fractional subgraphs

For a finite, simple, and undirected graph G let $V(G)$ denote the vertex set and $E(G)$ the edge set. Their cardinalities, the order and size of G, are $|G|$ and $e(G)$, respectively. For a vertex $v\in V(G)$ and a subset $U\subseteq V(G)$ the neighbourhood $N_U(v)$ is the set of neighbours of v in U, and if $U=V(G)$ we simply write $N(v)$. By $\langle U\rangle$ we denote the subgraph of G induced by U and if a graph H is isomorphic to an induced subgraph of G we write $H\leqslant G$.

Our aim is to estimate the function

$$\Psi(G,k)=\min\{e(H)\colon\ |H|=k,\ H\leqslant G\}.$$

This function can be made continuous in a natural way by allowing subgraphs to have fractional parts of vertices. The density of these fractional subgraphs can be reinterpreted as limits of the (integral) subgraph density of lexicographic products $G[\overline{K}_r]$. It should be mentioned that several of these concepts are implicitly used by Krivelevich [11].

Let G be a graph and $w: V(G)\to[0,1]$ be a real-valued function, assigning a weight $w(x)=w_x$ to each vertex x. We call the tuple (G,w) a *fractional subgraph* of G, and we define the order of (G,w) by

$$|(G,w)|=\sum_{x\in V(G)} w_x,$$

and the size (number of edges) by

$$e(G,w)=\sum_{xy\in E(G)} w_x w_y.$$

Note that if all weights are integral, then the order and size of (G,w) correspond to the order and size of the subgraph induced by the vertices of weight 1. Let $\Psi(G,t)$ denote the smallest size of a fractional subgraph of order t of G.

Proposition 2.1. *Let G be a graph of order n and α be a real number* $(0\leqslant\alpha\leqslant 1)$. *Then G contains a fractional subgraph (G,w) of order αn and size $\Psi(G,\alpha n)$, where $n-1$ vertices have integral weights.*

Proof. Let (G,w) be a fractional subgraph of order αn and size $\Psi(G,\alpha n)$ where two vertices x and y have fractional weights. We will show that we can change the weights of x and y to make one of the weights integral, without changing the order and without increasing the size. This, indeed, implies that in a fractional subgraph of size $\Psi(G,\alpha n)$ with the largest number of integral vertices, the weights of all but at most one vertices are integral.

For a vertex v define the fractional degree by $d(v)=\sum_{u\in N(v)} w_u$. We may assume that $d(x)\leqslant d(y)$. Let $\varepsilon=\min(1-w_x,w_y)$. Certainly, (G,w') where $w'_x=w_x+\varepsilon$ and $w'_y=w_y-\varepsilon$ and $w'=w$ otherwise, is a fractional subgraph of the same order as (G,w), where x or y has integral weight. Moreover, $e(G,w')=e(G,w)-\varepsilon d(y)+\varepsilon d(x)-\delta_{xy}\varepsilon^2\leqslant e(G,w)$, where δ_{xy} is 1 if $xy\in E(G)$ and 0 otherwise, which completes the proof. □

So looking for fractional subgraphs with small density we can restrict our attention to (integral) induced subgraphs, where just one suitable vertex is chosen fractionally, and if αn is an integer then $\Psi(G,\alpha n)$ is the size of an induced subgraph of G.

Define the local density function of G by $\beta(G,\alpha)=\Psi(G,\alpha n)/n^2$. We can express $\beta(G,\alpha)$ in terms of the integral subgraph density of lexicographic products of G with $\overline{K}_r$. Recall that the lexicographic product $G[H]$ has vertex set $V(G)\times V(H)$ and (u,x) and (v,y) are adjacent, if and only if (1) $uv\in E(G)$ or (2) $u=v$ and $xy\in E(H)$.

Proposition 2.2. *If G is a graph of order n and $0\leqslant\alpha\leqslant 1$, then*

$$\beta(G,\alpha)=\sup_{r\to\infty}\Psi(G[\overline{K}_r],\lfloor\alpha rn\rfloor)/(rn)^2.$$

Proof. For integers k and r let H be an induced subgraph of $G[\overline{K}_r]$ of order k. For a vertex $u\in V(G)$ let s_u be the number of vertices (u,x), $x\in V(\overline{K}_r)$, contained in H. Then the density $e(H)/(rn)^2$ equals the density $e(G,w)/n^2$ of (G,w), where $w_u=s_u/r$. This follows from the fact that the contribution of an edge uv in $E(G)$ to the density of (G,w) equals the contribution of all the edges $(u,x)(v,y)\in E(H)$ to the density of H.

On the other hand, by Proposition 2.1 there is a fractional subgraph of G of order k/r and size $\Psi(G,k/r)$ with at most one fractional vertex, so $\Psi(G[\overline{K}_r],k)/(rn)^2=\beta(G,k/rn)$. As $\beta(G,\alpha)$ is continuous and $\lfloor\alpha rn\rfloor/rn\to\alpha$ for $r\to\infty$ we get

$$\beta(G,\alpha)=\sup_{r\to\infty}\Psi(G[\overline{K}_r],\lfloor\alpha rn\rfloor)/(rn)^2. \qquad \square$$

In particular, we have

$$\beta(\alpha)=\sup_{G\text{ triangle-free}}\beta(G,\alpha).$$

The original form of Conjecture 1.2 in [8] had the additional requirement of the order n being sufficiently large, and in [11] this requirement was given globally throughout the paper. Concerning the investigation of $\beta(\alpha)$ this restriction is not necessary. Any small order counterexample G would give rise to an infinite sequence of counterexamples $G[\overline{K}_r]$, $r \in \mathbb{N}$.

3. Eigenvalue estimates

It is difficult to determine the exact minimum density of subgraphs of larger order graphs, in fact, this is an $\mathscr{NP}$-hard problem. So we need tools that provide reasonable estimates. For regular graphs we get an estimate from the least eigenvalue of the graph.

Theorem 3.1 (Bussemaker et al. [3]). *Let G be an r-regular graph of order n with least eigenvalue λ_n and let k be an integer satisfying $0 \leqslant k \leqslant n$. Then*

$$\Psi(G,k) \geqslant k(kr + (n-k)\lambda_n)/2n.$$

Note that for $k = n/2$, we get $\Psi(G,k) \geqslant (n^2/8)\Xi(G)$. The following lemma will bound the size of a fractional subgraph from below based on values at consecutive integral points.

Lemma 3.2. *If k is an integer and t a real number satisfying $0 \leqslant k \leqslant t \leqslant k+1 \leqslant n$ then for every graph G of order $n > k$*

$$\Psi(G,t) \geqslant (t-k)\Psi(G,k+1) + (k+1-t)\Psi(G,k).$$

Proof. Let (G,w) be a fractional subgraph of order t and size $\Psi(G,t)$, where all but possibly one vertex v have integral weights. Such a fractional subgraph exists by Proposition 2.1. If $t = k$ or $t = k+1$ then the result is an immediate consequence. So assume $k < t < k+1$ whence $0 < w_v < 1$. Let $U \subset V(G)$ be the subset of vertices of weight >0. As $e(\langle U \setminus \{v\} \rangle) \geqslant \Psi(G,k)$ and $e(\langle U \rangle) \geqslant \Psi(G,k+1)$ and $e(G,w) = e(\langle U\setminus\{v\}\rangle) + w_v|N_U(v)| = (t-k)e(\langle U \rangle) + (k+1-t)e(\langle U \setminus \{v\}\rangle)$ the result immediately follows. $\square$

More precisely, the function is concave between consecutive integral points, since it is the minimum of linear functions on the interval $[k, k+1]$.

Next we estimate the local density of the Higman–Sims graph [10] (denoted H_{100}).

Proof of Theorem 1.3 It is well-known (see e.g. [4, Chapter 8]) that the Higman–Sims graph H_{100} is 22-regular on 100 vertices, and its least eigenvalue is -8. So we obtain by Theorem 3.1 that

$$\Psi(H_{100},k) \geqslant \lceil k(22k - 8(100-k))/200 \rceil = \lceil 3k^2/20 \rceil - 4k \tag{2}$$

for $0 \leqslant k \leqslant 100$. It is easy to check that $10k - 326 \leqslant \lceil 3k^2/20 \rceil - 4k$ for every integer k with $0 \leqslant k \leqslant 100$. Therefore, by Lemma 3.2

$$\beta(\alpha) \geqslant \beta(H_{100},\alpha) \geqslant (1000\alpha - 326)/10\,000. \quad \square$$

Let us now turn to the problem, how many edges need to be deleted to make a graph bipartite. Denote this number by $\Phi(G)$. If (U,W) is a maximum edge cut, i.e. (U,W) is a partition of $V(G)$, where the maximum number of edges are joining U to W, then $\Phi(G)$ is just the number of edges in U plus the number of edges in W. If G is a regular graph then $\Psi(G,k)$ is attained by a maximum edge cut (U,W) with $|U|=k$, so we get

$$\Phi(G) = \min_{0\leqslant k\leqslant n/2} \Psi(G,k) + \Psi(G,n-k),$$

for regular graphs G. Now the proof of Theorem 1.4 is a simple application of Theorem 3.1.

Proof of Theorem 1.4. Using the fact that the estimate in Theorem 3.1 is a convex function we get

$$\Psi(G,k) + \Psi(G,n-k) \geqslant \frac{k(k\lambda_1 + (n-k)\lambda_n) + (n-k)((n-k)\lambda_1 + k\lambda_n)}{2n}$$

$$\geqslant \frac{2(n/2)^2(\lambda_1+\lambda_n)}{2n} = (\lambda_1+\lambda_n)n/4.$$

So $\Phi(G)\geqslant(\lambda_1+\lambda_n)n/4$. $\square$

We conclude this section with the proof of Theorem 1.6. The upper bound for $s\geqslant 3$ is due to an anonymous referee, whose proof is repeated here.

Proof of Theorem 1.6. (a) The lower bounds are obtained from the Higman–Sims graph for $s=2$ and for $s\geqslant 3$ from K_s.

(b) We will proceed by induction on s. Observe that the function $5-4(\sqrt{s^2-s}+1)/s$ is increasing with s. First consider the case $s=2$. Let G be an r-regular triangle-free graph of order n. We have $0=\Psi(G,r)\geqslant r(r^2+(n-r)\lambda_n)/2n$ by Theorem 3.1, as the neighbourhood of every vertex forms an independent set. We get $\lambda_n\leqslant -r^2/(n-r)$ which implies $\Xi(G)=(r+\lambda_n)/n\leqslant r(1-\frac{r}{n-r})/n$. Replacing $r=\alpha n$ we get

$$\Xi(G)\leqslant\alpha\left(1-\frac{\alpha}{1-\alpha}\right)\leqslant 3-2\sqrt{2},$$

since the function has its maximum at $\alpha = 1-1/\sqrt{2}$.

Now suppose that $s\geqslant 3$. Let G be a K_{s+1}-free r-regular graph of order n and set $r=\alpha n$. Since $\lambda_n\leqslant 0$ we may suppose that $\alpha\geqslant(s-2)/s$. If G does not contain a copy of K_s, then, by induction the statement holds. So let $V_0\subseteq V(G)$ be the vertex set of a clique on s vertices. Denote $\overline{V_0}=V(G)\setminus V_0$, then $\sum_{v\in V_0}|N_{\overline{V_0}}(v)| = s\alpha n - 2e(\langle V_0\rangle) = s\alpha n - s(s-1)$. Recall that G is K_{s+1}-free, therefore every vertex of $\overline{V_0}$ is joined to at most $s-1$ vertices of V_0. A simple account gives that there are at least $s\alpha n-(s-2)n-s$ vertices of $\overline{V_0}$, each having at least $s-1$ neighbours in V_0. So $\overline{V_0}$ contains a set U of at least $\alpha n-(s-2)n/s-1$ vertices, all of which have the same $s-1$ neighbours in V_0. Let v be the non-neighbour of the vertices of U in V_0, then $U^+ = U \cup \{v\}$ is an independent set of cardinality at least $\alpha n - (s-2)n/s$. Now Theorem 3.1 implies that

$$\lambda_n \leqslant -\frac{|U^+|\alpha n}{n - |U^+|} \leqslant -\frac{(\alpha n - (s-2)n/s)\alpha n}{n - \alpha n + (s-2)n/s}$$

and hence

$$\frac{\lambda_1 + \lambda_n}{n} \leqslant \alpha \left(1 - \frac{\alpha s - s + 2}{2s - \alpha s - 2}\right) =: f(\alpha).$$

The function $f(\alpha)$ attains its maximum on the interval $[(s-2)/s, 1]$ at $\alpha^* = (2s - 2 - \sqrt{s^2 - s})/s$, where $f(\alpha^*) = 5 - 4(\sqrt{s^2 - s} + 1)/s$. □

4. Final remarks

Conjecture 1.2 is probably false for a larger range of α. By a (not verified) computer program we have discovered that the quartic residue Cayley graph H_{41} of $\mathbb{Z}_{41}$ (i.e. the generators are $\{\pm 1, \pm 4, \pm 10, \pm 16, \pm 18\}$ satisfies $\beta(H_{41}, \alpha) > \beta(C_5, \alpha)$ for $10/41 \leqslant \alpha < 2062/4305 \simeq 0.479$. Anyway, it is not unlikely, that H_{100} is still a counterexample for $\alpha = 0.48$, but it definitely is not a counterexample for $\alpha = 1/2$. This can be derived from the well-known fact that the vertices of the Higman–Sims graph can be partitioned into two sets each of which induces a Hoffman-Singleton graph [4, Chapter 8, Example 1]. This implies, together with the lower bound of (2), $\beta(H_{100}, 1/2) = 0.0175$. Using the computer program `mtf` [2] for generating and analysing maximal triangle-free graphs, Brandt et al. [2] verified the case $\alpha = 1/2$ of Conjecture 1.2 for triangle-free graphs up to order 24.

According to another conjecture of Erdős, every K_4-free graph can be made bipartite by the omission of at most $n^2/9$ edges, which is the number of edges needed to make the complete balanced tripartite graph bipartite. Note that in view of the upper bound $\xi(3) \leqslant 0.4006\ldots$ in Theorem 1.6 there is no hope to find counterexamples to this conjecture just by the eigenvalue estimate in Theorem 1.4, in contrast to Conjecture 1.1.

While the estimate for $\beta(G, \alpha)$ derived from Theorem 3.1 is a convex function, the function itself is not necessarily convex. This is, e.g., the case for the local density function of the Clebsch graph (cf. [4, Chapter 8]), which is not difficult to compute by hand.

A surprising consequence is that the local density function of the Clebsch graph exceeds the function of C_5 within two intervals of α, namely $0.313 \simeq 5/16 < \alpha < 103/220 \simeq 0.468$ and $0.588 \simeq 47/80 < \alpha < 29/48 \simeq 0.604$. The dependence of the local density function on the structure of the graph seems to indicate that a precise calculation of $\beta(\alpha)$ might be very hard when α is significantly smaller than $3/5$, because many graphs are potential candidates to determine $\beta(\alpha)$ for smaller α.

An appendix containing details about density functions and the spectral ratio of certain triangle-free graphs and the current records for $\beta(\alpha)$ is available and can be requested directly from the author.

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