On the Structure of Certain \( p \)-Solvable Linear Groups II

DAVID L. WINTER

Department of Mathematics, Michigan State University, East Lansing, Michigan 48824

Communicated by Walter Feit

Received October 25, 1973

In [18], [8], and [21] complex \( p \)-solvable finite linear groups not having a normal \( p \)-Sylow subgroup were studied. There the structure of such groups was determined when the group is irreducible of degree \( n \leq 2p \). The next case is studied here.

**Theorem.** Let \( p \) be an odd prime. Let \( G \) be a finite \( p \)-solvable group which does not contain a normal \( p \)-Sylow subgroup. Assume that \( G \) has a faithful, irreducible representation over the complex number field of degree \( n = 2p + 1 \). Then \( G \) is solvable and \( n \) is a prime or a power of 3.

There are four sections in the paper. In the first section notation, past results and some preparations on \( p \)-solvable linear groups are presented. In the second section the inductive proof of the Theorem is begun and we reduce down to the case where \( G \) has a \( p \)-Sylow subgroup \( P \) of order \( p \) and a normal \( p \)-complement \( N \), where \( N \) modulo the center is a nonabelian simple group. In Section 3, detailed information about some Sylow normalizers of \( N \) is obtained. In the last section the strong interplay between various Sylow normalizers of \( N \) and the subgroup \( \mathcal{C}_N(P) \) is used in completing the proof. A recent result of Smith and Tyrer ([12]) helps dispose of some difficult cases that arise in Sections 3 and 4. The proof also apparently requires some deep classification theorems on finite simple groups in contrast to the earlier papers.

Although we have not addressed ourselves to the problem of describing the structure of \( G \) when \( G \) is solvable, the following result can be extracted from the proof of the Theorem.

**Corollary.** In addition to the hypotheses of the Theorem assume that \( G = PN \), where \( P \) is a \( p \)-Sylow subgroup of order \( p \) and \( N \) is a normal \( p \)-complement. If \( X \) is primitive, the only possible prime divisors of \( | N : \mathcal{C}(G) | \) are 2 and 3 if \( n \) is a power of 3 or 2, 3 and \( q \) if \( n = q \) is a prime.

A result when \( X \) is imprimitive is given at the end of Section 1.
1. Preparatory Results

We shall use the letter $Z$ for the center $Z'(G)$ of the particular group $G$. All group representations mentioned are over the field of complex numbers unless it is stated otherwise. Let $p$ be a prime. $O_p(H)$ is the largest normal $p$-subgroup of $H$ and $O^p(H)$ is the smallest normal subgroup of $H$ with $H/O^p(H)$ a $p$-group. $H$ is said to be $p$-closed if $H$ has a normal $p$-Sylow subgroup. Let $X$ be a representation of the group $H$. $X$ will be called $p$-closed if the group $X(H) = \{X(h) : h \in H\}$ is $p$-closed. A character $\chi$ of a group $H$ is called $p$-closed if a representation of $H$ with character $\chi$ is $p$-closed. $F(H)$ denotes the Fitting subgroup of $H$. f.p.f. stands for fixed-point-free(ly).

1.1. Let $p$ be an odd prime and let $G$ be a finite solvable group which is not $p$-closed. Let $G$ have a faithful, irreducible representation of degree $n = 2p + 1$. Then either $n$ is a prime or a power of 3.

Proof. By [19], there is a positive integer $m$ such that $n = mp$ or there is a prime $q$ and an integer $s \geq 1$ such that $n = mqs$, where $q^s = \pm 1 \pmod p$. This forces $m = 1, 2p + 1 = q^s$. If $s > 1, 2p = (q - 1)(1 + q + \cdots + q^{s-1})$, which implies $2 = q - 1, q = 3, p = 1 + 3 + 3^2 + \cdots + 3^{s-1}$ as required.

Let $G$ be a finite $p$-solvable group which is not $p$-closed. Let $G$ have a faithful, irreducible representation of degree $n < 2p$. By [8], $n$ must then be $p - 1, p, p + 1, 2p - 2$ or $2p - 1$ and $n$ must be a prime power. It follows that if $G$ is a finite $p$-solvable group which has a faithful representation all of whose irreducible constituents have degree less than $p - 1$, then $G$ is $p$-closed. These facts are used below without reference.

In the remainder of this section it is assumed that $G = PN$, where $P$ is a $p$-Sylow subgroup of odd prime order $p$ and $N$ is a normal $p$-complement. Set $B = C_N(P)$ and assume $B < N$.

1.2 ([7], 10.2.1). If $B = 1$, $N$ is nilpotent.

1.3 ([5]) $N = B[N, P]$. If $H$ is a $P$-invariant subgroup of $N$ and if $P$ fixes a coset $Hx$ of $H$ in $N$, then $Hx \cap B \neq \emptyset$. If $H$ is a $P$-invariant normal subgroup of $N$ and if $P$ acts trivially on $N/H$, then $N = BH$.

1.4. If $N$ is abelian, $N = B \times [N, P]$.

Proof. This is proved in ([7], 5.2.3) when $N$ has prime power order but the same proof can be used for 1.4.

Let $X$ be a $p$-closed representation of $PN$. Then $X(N) = X(B[N, P]) = X(B)[X(N), X(P)] = X(B)$. That is, $[N, P] \leq \ker X$ and $X$ is essentially a representation of $P \times B$. This fact is used repeatedly.
1.5. Let $Q_0$ be a $q$-subgroup of $B$ for some prime $q$. Then there exists a $P$-invariant $q$-Sylow subgroup of $N$ containing $Q_0$.

Proof. Let $Q$ be a maximal $P$-invariant $q$-subgroup of $N$ containing $Q_0$. Then $P < \mathcal{N}(Q)$ and $\mathcal{N}_N(Q)$ contains a $P$-invariant $q$-Sylow subgroup $Q_1$. Since $Q \leq Q_1$, $Q = Q_1$ and so $Q$ is a $q$-Sylow subgroup of $N$.

1.6 ([20] (2.3)). Let $X$ be a faithful, irreducible representation of $PN$ such that $X | N$ is irreducible and $X | P$ is unimodular. Let $\chi$ be the character of $X$. Then $\chi | P \times B = p\psi \pm \lambda$, where $\psi$ is the character of the regular representation of $PB|B$ and $\psi$ and $\lambda$ are characters of $PB|P$ with $\lambda$ irreducible.

It turns out that detailed information about $p$-solvable linear groups of degree $p + 1$ is required.

1.7. Let $G = PN$ have a faithful, irreducible representation $X$ of degree $p + 1$ with character $\chi$. Then $p + 1$ is a power of 2 and $N$ contains an abelian $H$all $2'$-subgroup.

(i) Let $X$ be imprimitive. If $p > 3$, then $N$ contains an abelian subgroup $A \triangleleft G$ such that $N/A$ is elementary of order $p + 1$ and $P$ acts f.p.f. on $N/A$. $\chi | A$ is a sum of $p + 1$ distinct linear characters permuted faithfully and doubly-transitively by $G/A$. If $p = 3$, $n = 4$ and $G$ has a normal subgroup $A$ such that $N/Z(A)$ is a 2-group.

(ii) Let $X$ be primitive. Then $N$ contains a normal subgroup $U$ which is either a 2-group, a $(2, 3)$-group or a $(2, 5)$-group and $N/U$ is nilpotent.

Proof. By ([18], Theorem 3), $N$ is solvable and $p + 1$ is a power of 2. Let $H$ be a $P$-invariant Hall $2'$-subgroup of $N$. By use of 1.6, it may be verified that $\chi | B = p\psi + \lambda$, where $\psi$ and $\lambda$ are linear characters of $B$. Hence if $H \leq B$, $H$ is abelian. Suppose this is not the case so that $PH$ is not $p$-closed. This implies that $\chi | PH$ is a sum of an irreducible constituent of degree $p$ and one of degree 1. $\chi | H$ is therefore a sum of linear characters and $H$ is abelian.

Let $X$ be imprimitive. Let $V$ be a vector space affording $X$ and let $V = V_1 \oplus \cdots \oplus V_r$, where $V_1, ..., V_r$ is a system of imprimitivity. Let $A$ be the normal subgroup of $G$ consisting of all elements of $G$ which fix all $V_i$. Suppose $PA \triangleleft G$. Then $PA$ is not $p$-closed. Hence $X | PA$ is irreducible unless $p = 3$ and $n = 4$. In the latter case $N/A$ has order 2 and $X | PA$ is a sum of two irreducible constituents of degree 2. By ([16], Theorem 1), $A/Z(A)$ is a 2-group. Therefore we can take $p > 3$, $X | PA$ irreducible if $PA \triangleleft G$. But $X | A$ is then irreducible which is a contradiction. Therefore
$PA$ is not normal and $G/A$ is not $p$-closed. Since $G/A$ is a transitive permutation group on $V_1, \ldots, V_r$, $r \geq p$ and we must have $r = p + 1$.

$G/A$ is then a transitive permutation group on the $p + 1$ distinct linear constituents of $\chi | A$ and $P$ fixes one of these and is transitive on the others. Let $PK, A \leq K < N$, be such that $PK/A$ is the subgroup of $G/A$ fixing a character. Then $\chi | PK$ must have a linear constituent and an irreducible constituent of degree $p$. Therefore $\chi | K$ is a sum of linear characters, $K$ is abelian and so $K = A$. Thus the subgroup of $G/A$ fixing a letter has order $p$ and $G/A$ is a Frobenius group of order $n(n - 1)$ and degree $n = p + 1$. The result (i) follows from the known structure of such groups.

Now let $X$ be primitive. Since $\chi | B = p\psi + \lambda$, $\psi$ and $\lambda$ linear characters, $| B : Z(G)| < 6^{p - 1} = 6$ by a result of Blichfeldt ([1], p. 101). Let $q$ be an odd prime divisor of $| N |$, let $Q$ be a $q$-Sylow subgroup invariant under $P$ and assume $N$ does not have a normal $q$-complement. Then $\mathcal{N}_N(Q)$ is nonabelian and $\chi | \mathcal{N}(Q)$ must be irreducible. $Q \leq B$ is impossible. Otherwise $\chi | Q = pb + \lambda = (p + 1)\lambda$ by Clifford's theorem implying $Q \leq Z$. Hence $PQ$ and $\mathcal{N}(Q)$ are not $p$-closed. $\chi | PQ$ contains an irreducible constituent of degree $p$ so $\chi | Q$ is a sum of $p + 1$ distinct linear constituents. This implies first, by a result of Brauer ([2], 3F), that $\mathcal{C}(Q)/\mathcal{Z}$ is a $(2, q)$-group. Second, it implies that $\chi | \mathcal{N}(Q)$ is imprimitive. By the preceding paragraph $\mathcal{N}(Q)/\mathcal{C}(Q)$ is a Frobenius group of degree $p + 1$ and order $p(p + 1)$. Suppose $B \cap Q \leq Z$. Then $P$ acts f.p.f. on $\mathcal{N}(Q)/\mathcal{C}(Q)Z$ and so the latter group is nilpotent. This implies that $\mathcal{N}(Q)$ is 2-closed and so is nilpotent against Burnside's Theorem. Therefore $B \cap Q \leq Z$. Since $| B : Z | < 6$, $q = 3$ or 5 and (ii) follows.

1.8. Let $G = PN$ be a transitive permutation group of degree $n = 2p + 1$. Then $N = BQ$, where $| B | \leq 2$ and $Q$ is a group of prime power order with $n | | Q |$.

Proof. Assume first that $G$ is imprimitive. Suppose there are $e$ blocks each containing $f$ points. Since $ef = 2p + 1$, both $e$ and $f$ are less than $p$. This implies that $P$ fixes all blocks and so also all points. This is a contradiction and $G$ must be primitive.

Let $w$ be a generator of $P$. If $w$ is a $p$-cycle, then $G = S_n$ or $A_n$ ([16], p. 39), a contradiction. Hence $w$ is the product of two $p$-cycles. This implies $B \leq \langle \tau \rangle$ where $\tau$ is an involution, a product of $p$ transpositions which interchanges the two orbits of $w$. In particular, $\tau$ is an odd permutation. $N$ therefore contains a normal subgroup $N_1$ of index 1 or 2 on which $P$ acts without fixed points. Hence $N_1$ is nilpotent and we let $Q$ be the $q$-Sylow subgroup of $N_1$ where $q$ is some prime divisor of $| N_1 |$. Then $Q \triangleleft G$ and since $G$ is primitive, $Q$ is transitive. Therefore $n | | Q |$ and this implies all parts of 1.8.
1.9. Let $X$ be a faithful, irreducible, imprimitive representation of $G = PN$ of degree $n = 2p + 1$. Then $n$ is a prime or a power of 3 and $G$ contains an abelian normal subgroup $K < N$ such that $N = N/K$ has the form $BQ$ where $B = C_S(P)$ has order 1 or 2, $Q$ is elementary abelian of order $n$ and $P$ acts irreducibly on $Q$.

Proof. Let $V$ be the underlying vector space affording $X$ and suppose $V = V_1 \oplus \cdots \oplus V_r$, where $V_1, \ldots, V_r$, $r > 1$, is a system of imprimitivity. Let $K$ be the normal subgroup of $G$ stabilizing all the subspaces $V_i$. Then $G/K$ is a transitive permutation group of degree $r$. $X/K$ being reducible implies $K$ is $p$-closed. Since $G$ is not $p$-closed, $p \mid |G : K|$. Hence $r \geq p$ and this forces $r = 2p + 1$ and $\dim V_i = 1$. By 1.8, $N$ contains a normal subgroup $Q > K$ of index 1 or 2 such that $P$ acts f.p.f. on $Q/K = Q$. $Q$ is a $q$-group for some prime $q$ and $n \mid |Q|$. Let $\chi$ be the character of $X$. $Q < G$ and $G/Q$ is cyclic of order $p$ or $2p$. Hence $\chi \mid Q$ is irreducible ([4], 9.12).

By 1.8, $B' \subseteq K$ which is abelian implying the contradiction $\chi_1(1) = 1$. Therefore $\chi_1$ is not $p$-closed and so $\chi_1(1) \geq p - 1$. Since $Q \cap K$ is an abelian normal subgroup of $I \cap Q$, $\chi_1(1) = q^{s_1}$, where $0 < s_1 < s$. As in 1.1, $q = 3$ and we then get a contradiction from $3(p - 1) \leq 3\chi_1(1) = 3s_1 \leq 3p = 2p + 1$. This proves that all irreducible constituents of $\chi \mid I \cap Q$ are linear. Hence $I \cap Q = K$ and $|Q : K| = 2p + 1$. Since $P$ acts f.p.f. on $Q/K$, it cannot contain a proper $P$-invariant subgroup. All parts of 1.9 now follow.

2. Initial Reductions

We shall use induction on the order of the central factor group. Let $G$ be a counterexample to the Theorem such that $G/Z$ has order as small as possible. We may then assume that $G$ is unimodular ([1], p. 14). Since $Z$ is cyclic, $|Z| \mid n$. Let $X$ denote the given complex, faithful, unimodular, irreducible representation of $G$ of degree $n = 2p + 1$ and let $\chi$ be the character of $X$. 
Because of the classification of linear groups of degree seven ([14], [15]) and the fact that 3-solvable permutation groups of degree 7 are solvable, \( p = 3 \), \( n = 7 \) is impossible. Hence \( p \geq 5, n \geq 11 \). 1.1 yields immediately

2.1. If \( H \) is an irreducible subgroup of \( G \) which is not \( p \)-closed and if \( H \) is solvable, then \( n \) is a prime or a power of 3. In particular, \( G \) is non-solvable.

2.2. \( G = \mathcal{O}_{pp'}(G) \) and \( |G : \mathcal{O}_{pp'}(G)| = p \).

Proof. Since \( \mathcal{O}_{pp'}(G) \) is not \( p \)-closed, \( X \mid \mathcal{O}_{pp'}(G) \) is irreducible. Solvability of \( \mathcal{O}_{pp'}(G) \) implies solvability of \( G \) by ([21], Lemma 1). Hence \( G = \mathcal{O}_{pp'}(G) \). There exists \( H \triangleleft G \) with \( |H : \mathcal{O}_{pp'}(G)| = p \). Induction forces \( H = G \).

2.3. \( \mathcal{O}_p(G) = 1 \).

Proof. Assume \( \mathcal{O}_p(G) \neq 1 \). By Clifford's theorem \( \chi \mid \mathcal{O}_p(G) = e \sum_{i=1}^r \theta_i \), where \( e \) and \( r \) are positive integers and \( \theta_1, \ldots, \theta_r \) are distinct conjugate characters of \( \mathcal{O}_p(G) \). Since \( p \mid n, \theta_i(1) = 1 \) and \( \mathcal{O}_p(G) \) is abelian, \( r > 1 \) because otherwise \( \mathcal{O}_p(G) \leq Z \) contradicting unimodularity of \( X \). Let \( K \) be the normal subgroup of \( G \) fixing all \( \theta_i \). Then \( G/K \) is a transitive permutation group of degree \( r \). \( \chi \mid K \) is reducible and so \( K \) is \( p \)-closed and therefore \( p \mid |G : K| \). This implies \( r \geq p \) whence \( r = 2p + 1 \) and \( K \) is abelian. Let \( P \) be a \( p \)-Sylow subgroup of \( G \). If \( PK \triangleleft G \), then \( \chi \mid PK \) is reducible and \( PK \) is \( p \)-closed, a contradiction. Therefore \( G/K \) is not \( p \)-closed. By 1.8, \( G/K \) is solvable and so \( G \) is solvable, proving 2.3.

From now on let \( P \) denote a fixed \( p \)-Sylow subgroup of \( G \) and let \( N = \mathcal{O}_{p'}(G) \). Then \( |P| = p \) and \( G = PN \). Also set \( B = \mathcal{C}_N(P) \).

As a consequence of 1.9 we have the following.

2.4. \( X \) is primitive.

2.5. If \( Z < H < N \) with \( H \triangleleft G \), then \( H \) is irreducible and solvable. In particular, \( N' = N \).

Proof. Suppose \( \chi \mid H \) is reducible. By primitivity, \( \chi \mid H \) is homogeneous, say \( \chi \mid H = e\mu, e \) a positive integer, \( \mu \) an irreducible character of \( H \). Then \( \chi \mid PH = \sum_{i=1}^e \nu_i \), where each \( \nu_i \) is an extension of \( \mu \). Since \( e > 1 \) and \( e \mid 2p + 1, e > 3 \) and so \( \mu_i(1) \leq (2p + 1)/3 < p - 1 \). This implies \( P \triangleleft PH \) and \( H \leq B \). But we know by 1.6 that \( \chi \mid B = p\psi + \lambda \), where \( \lambda \) is an irreducible character of \( B \) and \( \psi \) is a character of \( B \). This forces \( e \geq p \) and so \( e = 2p + 1, \mu(1) = 1 \). In other words, \( H \leq Z \).
Suppose \( \chi \mid H \) is irreducible, then \( H \leq B \) is impossible since \( \chi \mid B \) is reducible. Therefore, \( PH \) is not \( p \)-closed. Induction implies that \( H \) is solvable.

2.6. \( \chi \mid P \times B = \rho \psi + \lambda \) where \( \rho \) is the character of the regular representation of \( PB/B \), \( \psi \) is a character of \( PB/P \) of degree 2 and \( \lambda \) is a linear character of \( PB/P \).

Proof. By 1.6, \( \chi \mid P \times B = \rho \psi + \lambda \), where \( \rho \) is as described above and \( \psi \) and \( \lambda \) are characters of \( PB/P \) with \( \lambda \) irreducible.

Suppose first that the minus sign occurs. Then if \( r = (\psi, \lambda)_B \), \( r > 1 \). Furthermore, \( \chi \mid P \times B = (r - 1) \lambda + r \theta \lambda + \rho \psi_1 \), where \( \theta \) is the sum of the \( p - 1 \) nonprincipal characters of \( PB/B \) and \( \psi_1 \) is 0 or a character of \( PB/P \). Thus \( 2p + 1 = (r - 1) \lambda(1) + r( p - 1) \lambda(1) + p \psi(1) \). Hence \( r(p - 1) \leq 2p + 1 \) and so \( r = 1 \) or 2. It is easily checked, that there are no integers \( r \), \( \lambda(1) \), \( \psi(1) \) satisfying these conditions.

Now let \( \chi \mid P \times B = \rho \psi + \lambda \). Then \( \rho \psi(1) + \lambda(1) = 2p + 1 \) and \( 0 \leq \lambda(1) - 1 = p(2 - \psi(1)) \). Hence \( \psi(1) = 1 \) or 2. Assume \( \psi(1) = 1 \) and \( \lambda(1) = p + 1 \). Let \( V \) be a complex vector space affording the representation \( X \). Then \( \dim \mathscr{C}_\rho(P) = p + 2 \). We claim that if \( P = P_1, P_2, \ldots, P_k \) are any distinct conjugates of \( P \), then \( \dim \mathscr{C}_\rho(\langle P_1, \ldots, P_k \rangle) \geq p \). Suppose it is known that \( \dim \mathscr{C}_\rho(\langle P_1, \ldots, P_r \rangle) \geq p \) for some \( r \). Since \( \dim \mathscr{C}_\rho(\langle P_{r+1} \rangle) = p + 2 \), \( \dim \mathscr{C}_\rho(\langle P_1, \ldots, P_r, P_{r+1} \rangle) > 0 \). Let \( H = \langle P_1, \ldots, P_r, P_{r+1} \rangle \). Then \( \chi \mid H \) is irreducible. Since each non-\( p \)-closed constituent of \( \chi \mid H \) is such that on \( P \) it has at least \( p - 1 \) distinct eigenvalues, there is exactly one non-\( p \)-closed irreducible constituent \( \chi_1 \). Since \( \chi \) is rational on \( P \), so is \( \chi_1 \) and \( \chi_1 \) has the \( p - 1 \) nonreal \( p \)th roots of unity each occurring once as characteristic value on a nonidentity element of \( P \). If \( \chi_2 \) is the sum of the remaining irreducible constituents of \( \chi \mid H \), then \( P \leq \ker \chi_2 \) and so \( H = \ker \chi_2 \) since \( H \) is generated by conjugates of \( P \). \( \chi_1(1) \leq 2p \) but \( \chi_1(1) = 2p \) would imply \( \chi_1 \mid P = 2p \) which is not the case. Hence \( \chi_1(1) < 2p \). By ([8], Lemmas 2.1 and 2.9), \( \chi_1(1) \leq p + 1 \) and \( \dim \mathscr{C}_\rho(H) \geq p \) as desired. It follows that \( X \mid P^G \) is reducible. But this yields the contradiction that \( P < P^G \). The only remaining possibility is \( \psi(1) = 2 \), \( \lambda(1) = 1 \). This proves 2.6.

2.7. \( \mathcal{F}(G) = \mathcal{F}(N) = Z \).

Proof. Suppose 2.7 is false. By 2.5, we may assume \( \mathcal{F}(G) = R > Z \) is an \( r \)-group for some prime \( r \), \( n = r^s \), \( r = 3 \) if \( s > 1 \) and \( X \mid R \) is irreducible.

Let \( q \neq r \) be another odd prime divisor of \( |N| \) and let \( Q \) be a \( P \)-invariant \( q \)-Sylow subgroup of \( N \). Assuming \( P \) acts f.p.f on \( Q \), we shall obtain a contradiction.
Suppose first that $X \mid PQR$ is imprimitive. By 1.9, $PQR$ contains an abelian normal subgroup $K$ such that $|PQR : K|$ divides $2p(2p + 1)$. This implies $Q \leq K$ and so $Q < PQR$. Hence $Q \leq \mathcal{C}(R) = Z$ since $X \mid R$ is irreducible.

Now suppose that $X \mid PQR$ is primitive. Let $R_0$ be such that $Z < R_0 \leq R$ and $R_0/Z$ is a minimal normal subgroup of $PQR/Z$. By primitivity, $R_0$ is nonabelian. By the argument of 2.5, $X \mid R_0$ is irreducible. Furthermore, $R_0/Z$ is elementary abelian and so $Z \leq B \cap R_0 < R_0$. Clifford's theorem along with 2.6 now force $B \cap R_0 = Z$. Hence $P$ is f.p.f. on $R_0/Z$ and consequently on $QR_0/Z$. Therefore $QR_0/Z$ and so $QR_0$ are nilpotent forcing the contradiction $Q \leq \mathcal{C}(R_0) = Z$.

Letting $Q$ as above, we may let $y \in B \cap Q$ have order $q$. The action of $y$ on $R$ is nontrivial since $\mathcal{C}(R) = Z$. Let $q \geq 5$. By 1.6, the $q - 1$ nonreal $q$th roots of unity occur with the same multiplicity in $X(y)$. But this is contrary to 2.6. We conclude that $1 \leq 1$ has the form $2^a3^b$ or $2^a3^b7^c$. Since $N$ is nonsolvable, the latter must occur, $r \neq 3$ and $2p + 1 = r$. On the other hand, by [13], $r = 5, 7, 13, 17$ so $r = 2p + 1$ is impossible.

2.8. $N/Z$ is a nonabelian simple group.

Proof. By 2.5 and 2.7, $N/Z$ is a chief factor of $G$. If it is not simple, $N$ is the central product of $p$ isomorphic groups each of whose central factor groups is isomorphic to a fixed nonabelian simple group. This implies that $X \mid N$ is a tensor product of $p$ irreducible representations ([7], p. 102) of degree $d$ of these factors. Since $d \mid n$, $d \geq 3$ and so $2p + 1 \geq 3p$, a contradiction proving 2.8.

3. SYLOW NORMALIZERS

Throughout this section $q$ will denote an odd prime divisor of $|N|$ such that $B$ contains no $q$-Sylow subgroup of $N$. By 1.5, we choose a $q$-Sylow subgroup $Q$ of $N$ normalized by $P$ such that $B \cap Q$ is a $q$-Sylow subgroup of $B$.

3.1. $X \mid \mathcal{N}(Q)$ has a non-$p$-closed irreducible constituent of degree at least $p + 1$.

Proof. Suppose false and let $X_1$ be a non-$p$-closed irreducible constituent of $X \mid \mathcal{N}(Q)$. Then $\deg X_1 = p - 1$ or $p$. In the first case $X_1(Q) \leq \mathcal{X}(X_1(\mathcal{N}(Q)))$ ([18], Theorem 1) and in the second case $X_1(\mathcal{N}(Q))$ is abelian.

Let $X_2$ be a $p$-closed constituent. Then $X_2(Q) = X_2(B \cap Q[Q, P]) = X_2(B \cap Q)$ is abelian since an irreducible constituent of $X \mid B$ has degree at most 2. Hence $Q \simeq X(Q)$ is abelian and $[Q, P] \leq \mathcal{X}(\mathcal{N}(Q))$. By a transfer
3.2. If \( X \mid \mathcal{N}(Q) \) has a non-\( p \)-closed irreducible constituent of degree \( p + 1 \), then \( p + 1 \) is a power of 2, \( q \mid |B : Z| \) and \( q \) is the only odd prime divisor of \( |B : Z| \).

**Proof.** By assumption we may write \( X \mid \mathcal{N}(Q) = X_1 \oplus X_2 \), where \( X_1 \) is a non-\( p \)-closed irreducible constituent of degree \( p + 1 \). By 1.7, \( p + 1 \) is a power of 2 and so \( X_1 \mid Q \) is a sum of linear constituents since \( q \) is odd. Suppose \( X_2 \) is not \( p \)-closed. Then either \( X_2 \) is irreducible of degree \( p \) or contains an irreducible constituent of degree \( p - 1 \) with \( p - 1 \) a power of 2. The latter implies \( p = 3 \) so is impossible. Hence \( X_2 \) is irreducible of degree \( p \) and so \( X_2 \mid L \), where \( L = \mathcal{N}(Q) \cap N \), is a sum of linear constituents. Hence \( X_2(Q) \) is abelian, \( i = 1, 2 \) and \( Q \) is abelian. If \( X_2 \) is \( p \)-closed, then \( X_2(Q) = X_2(B \cap Q[Q, P]) = X_2(B \cap Q) \) is abelian because by 2.6 irreducible constituents of \( \chi \mid B \) have degree at most 2 < \( q \). Therefore in any case \( Q \) is abelian.

We claim that \( X_2 \mid L \) is a sum of linear constituents and \( Q \leq \ker X_2 \). If \( X_2 \) is not \( p \)-closed, then \( X_2 \) is irreducible of degree \( p \) and \( X_2 \mid L \) is a sum of linear constituents. Suppose \( X_2 \) is \( p \)-closed and let \( \chi_i \) be the character of \( X_i \), \( i = 1, 2 \). It is easily seen, using 1.6, that \( \chi_1 \mid B \cap L = p\psi_1 + \lambda_1 \) for some linear characters \( \psi_1, \lambda_1 \) of \( B \cap L \). But by 2.6, \( \chi \mid B \cap L = p\psi + \lambda \), where \( \psi \) has degree 2. It follows that \( \psi \mid B \cap L \) is reducible. Hence \( \chi_2 \mid B \cap L \) is a sum of linear characters. Since \( X_2(L) = X_2(B \cap L) \), \( X_2 \mid L \) is a sum of linear constituents. If \( Q \leq \ker X_2 \), then \( \varnothing(Q) \leq L \). By a theorem of Grün on \( q \)-normal groups ([7], 7.5.2), this would imply \( \varnothing(N) < N \). Hence \( Q \leq \ker X_2 \) as claimed.

Since \( Q \leq \ker X_2 \), \( X_2(PQ) \) is \( p \)-closed. This implies \( X_1(PQ) \) is not \( p \)-closed because \( Q \leq B \). Since \( q \) is odd, \( X_1 \mid PQ \) has an irreducible constituent of degree \( p \). Hence \( X_1 \mid Q \) is a sum of \( p + 1 \) distinct conjugate linear constituents. This yields that \( X_1 \) is imprimitive. By 1.7, \( L \) contains a subgroup \( A \) of index \( p + 1 \) with \( X_1(A) \) abelian. Since \( X_2(A) \) is also abelian, \( A \) is abelian. It follows that \( A = \mathcal{C}(Q) \) since \( L \mid A \) permutes faithfully the irreducible constituents of \( X_1 \mid A \).

We claim that \( |\mathcal{C}(Q) : QZ(G)| \) is a power of 2. Let \( r \) be an odd prime divisor of \( |\mathcal{C}(Q)| \), \( r \neq q \). Let \( R \) be the Sylow \( r \)-subgroup of \( \mathcal{C}(Q) \). Then \( \mathcal{N}(Q) \leq \mathcal{N}(R) \) and \( R \) is a Sylow \( r \)-subgroup of \( \mathcal{N}(Q) \). Suppose \( X_2(PR) \) is not \( p \)-closed. Then \( X_2 \) is irreducible of degree \( p \) and \( X_2 \mid R \) is a sum of \( p \) distinct linear characters. Since \( X \mid \mathcal{N}(Q) \) is a sum of irreducible constituents of degrees \( p + 1 \) and \( p \), \( X \mid \mathcal{N}(R) \) is also or else \( X \mid \mathcal{N}(R) \) is irreducible. If \( X \mid \mathcal{N}(R) \) is irreducible, then \( X \mid R \) is a sum of \( 2p + 1 \) distinct linear characters on which \( \mathcal{N}(R) \) acts transitively. By 1.9, this implies that \( \mathcal{N}(R) \) contains an
abelian normal subgroup $K$ such that $N(R)/K$ has order $pn$ or $2pn$. But this contradicts $|L:C(L)| = p + 1$. Hence $X|N(R) = \bar{X}_1 \oplus \bar{X}_2$, where we can assume $\bar{X}_i$ is an extension of $X_i$, $i = 1, 2$. Suppose $\bar{X}_1$ is primitive.

By 1.7, $N(R)$ must have a normal $r$-complement since $N(Q)$ cannot have a normal $q$-complement by Burnside's transfer theorem. Let $\bar{R} \supset R$ be an $r$-Sylow subgroup of $N(R)$ which leaves $Q$ invariant. This implies $\bar{R} \leq N(R)$ and so $\bar{R} = R$ is an $r$-Sylow subgroup of $G$. But since $N(R)$ has a normal $r$-complement, Burnside's theorem again yields a contradiction. Now suppose $\bar{X}_1$ is imprimitive.

By 1.7, $N(R) \leq N(Q)$ and so $N(R) = N(Q)$. If $X_1(PR)$ is not $p$-closed, then $X_1|R$ is a sum of $p + 1$ distinct conjugate constituents. Hence $X|R$ is a sum of $2p + 1$ distinct conjugate characters. By the method of ([2], 3F), $Q \leq C_4(G)$, forcing $Q \leq Z$. Therefore $X_1(PR)$ is $p$-closed, $X_1(R) \leq C(X_1(P))$ and so $X_1|R = p\psi_1 + \lambda_1$, where $\psi_1$ and $\lambda_1$ are linear characters of $R$. But $R < N(Q)$ implies $\psi_1 = \lambda_1$ and so $X_1(R)$ consists of scalars. Thus $X_1(R) \leq \mathcal{Z}(X_1(N(R) \cap N)), i = 1, 2$ and so $R \leq \mathcal{Z}(N(R) \cap N)$. A transfer theorem now contradicts $N = N'$. We conclude that $X_2(PR)$ is $p$-closed.

Hence $[R, P] \leq \ker X_2$. Let $V$ be a complex vector space affording $X$. We now have $\mathcal{C}_v(Q) \leq \mathcal{C}_v([R, P])$. We may assume that $X$ lies in some algebraic number field $K$ and that $X$ lies in $S$ where $S$ is the ring of $p$-integral elements of $K$, $p$ a prime ideal divisor of $r$ in the ring of integers of $K$.

Since $Q$ is abelian and $(q, r) = 1$, $X(Q)$ can be diagonalized by [22] and the method of Schur ([11], Section 3 or [4], 1.1). Hence we may assume that $X(Q)$ may be written over $S$ as $X = \text{diag} \{\theta_1, ..., \theta_{p+1}, 1, ..., 1\}$, where $\theta_1, ..., \theta_{p+1}$ are distinct linear characters. As $[R, P] \leq C(Q)$ and $\mathcal{C}_v(Q) \leq \mathcal{C}_v([R, P])$, $X([R, P])$ is diagonal. Hence, if $x \in [R, P]$, $X(x) = I (\text{mod } pS)$. By ([2], 3A), $[R, P] \leq C_v(G) \leq Z$. Therefore $R \leq B$. As above $X_1(R)$ consists of scalars. This is true now of $X_2(R)$ by the form of $X|B$ and so $R$ acts as scalars on $\mathcal{C}_v(Q)$. Repeating the above argument for $R$ in place of $[R, P]$, we get $R \leq Z$. This proves $|\mathcal{C}(Q) : QZ| = p$. Since $Q$ is abelian, a transfer theorem forces $B \cap Q \cap Z \leq Q \cap Z = 1$. But $B \cap Q \neq 1$ because otherwise $P$ would act f.p.f. on $L/\mathcal{C}(A)$ forcing $\mathcal{C}(L) < L$ and, in turn by Grün's theorem $\mathcal{C}(N) < N$.

Since $Q \leq \ker X_2$ and $X|B = p\psi + \lambda$, $\psi|B \cap Q = \psi_1 + 1_{B \cap Q}$, and $\psi_1 \neq 1_{B \cap Q} \neq \lambda|B \cap Q$ by unimodularity. Also $\psi_1 \neq \lambda$ by unimodularity. Hence $X|\mathcal{C}(B \cap Q) = \alpha + \beta + \gamma$ extensions of $p\psi_1$, $p 1_{B \cap Q}$, $\lambda|B \cap Q$, respectively. Therefore each non-$p$-closed constituent of $X|\mathcal{C}(B \cap Q)$ has degree $p$ ($p - 1$ cannot occur because $p - 1$, $p + 1$ both powers of 2 imply $p = 3$). Since each irreducible constituent of $X|R \cap \mathcal{C}(R \cap Q)$ is linear, each $p$-closed irreducible constituent of $X|\mathcal{C}(B \cap Q)$ is linear. Hence for any irreducible constituent $Y$ of $X|\mathcal{C}(B \cap Q)$, $Y(\mathcal{C}(B \cap Q) \cap N)$ is abelian. Since $Q \leq \mathcal{C}(B \cap Q) \cap N$, the latter is a subgroup of $\mathcal{C}(Q)$.
Since $\psi | B \cap Q$ is a sum of two characters which cannot be conjugate, $\psi | \mathcal{N}(B \cap Q) \cap B$ is reducible. Hence $\mathcal{N}(B \cap Q) \cap B$ is abelian and $B$ has a normal $q$-complement. Let $s$ be any other odd divisor of $| B |$. Then there is a $B \cap Q$-invariant $s$-Sylow subgroup $U$ of $B$. Since $| (B \cap Q)U |$ is odd, $\psi | (B \cap Q)U$ is reducible and so $(B \cap Q)U$ is abelian, $U \leq \mathcal{N}(B \cap Q) \cap N \leq \mathcal{N}(Q)$. Hence $U \leq Z$. Therefore, $| B : (B \cap Q)Z |$ is a power of 2. This completes the proof of 3.2.

3.3. $X | \mathcal{N}(Q)$ has no irreducible constituent of degree $2p - 2$.

Proof. Suppose on the contrary that $X | \mathcal{N}(Q) = X_1 \oplus X_2$, where $X_1$ is irreducible of degree $2p - 2$ and $X_2$ has degree 2. Since $p \geq 5$, $3 < p - 1$ and so $X_2$ is $p$-closed. Since by ([21], Lemma 3) $X_t(Q) \leq X_t(B)$, $PQ$ is $p$-closed. Since by ([21], Lemma 3) $X_t(Q) \leq X_t(B)$, $PQ$ is $p$-closed, a contradiction.

3.4. $X | \mathcal{N}(Q)$ cannot have an irreducible constituent of degree $2p - 1$.

Proof. Suppose $X | \mathcal{N}(Q) = X_1 \oplus X_2$, where $X_1$ is irreducible of degree $2p - 1$ and $X_2$ has degree 2. Since $p \geq 5$, $2 < p - 1$ and so $X_2$ is $p$-closed. Therefore $X_1$ is not $p$-closed. By [8], $2p - 1 = t^e$ for some $s$ and some prime $t$. By Clifford's theorem, let $X_1 | Q = e \sum_{i=1}^{n} Y_i$, where this is a direct sum of distinct conjugate irreducible representations $Y_1, \ldots, Y_r$ and $e$ is a positive integer. Suppose $r > 1$. If the $Y_i$ are nonlinear, then $r < p$ and $P$ fixes all $Y_i$. Then it would follow that $X_1 | PQ = e \sum_{i=1}^{r} \tilde{Y}_i$, where the $\tilde{Y}_i$ are extensions of the $Y_i$. But then degree $Y_i < p - 1$ would imply all $\tilde{Y}_i$ are $p$-closed and $X_1(PQ)$ is $p$-closed, a contradiction. If the $Y_i$ are linear, then $P$ does not fix all of them and so $r \geq p$. This implies that $r = 2p - 1$, $e = 1$. Let $K$ be the subgroup of $\mathcal{N}(Q)$ fixing all $Y_i$. Then $X_1 | K$ is a sum of $2p - 1$ distinct linear characters and $\mathcal{N}(Q)/K$ is a transitive permutation group of degree $2p - 1$ against ([8], Lemma 2.6). This proves $r = 1$.

Since $X_1(PQ)$ is not $p$-closed and all irreducible constituents of $X_1 | PQ$ are extensions of irreducible constituents of $X_1 | Q$, we must have $X_1 | Q$ irreducible. It follows that $X_1(\mathcal{A}(Q)) \leq \mathcal{A}(X_1(\mathcal{N}(Q)))$ and that $\mathcal{A}(Q) \leq B$. Let $y \in \mathcal{N}(Q) \cap [Q, P]$, $y \neq 1$. Then $y \in \ker X_2$ and so $X(y)$ has two characteristic values: one not equal to 1 of multiplicity $2p - 1$ and 1 of multiplicity 2. On the other hand, by 2.6, $X | \langle y \rangle = p \phi | \lambda$ which is a contradiction.

3.5. Let $X | \mathcal{N}(Q)$ have an irreducible constituent of degree $2p$.

(i) If $P$ is f.p.f. on $Q$, then $n = q$.

(ii) If $P$ is not f.p.f. on $Q$, then $| B \cap Q | = 3$ or 5, $| B : (B \cap Q)Z | \leq 2$ and $| \mathcal{N}_B(B \cap Q) : QZ |$ is a power of 2.
Proof. By assumption we may write $X \mid \mathcal{N}(Q) = X_1 \oplus X_2$, where $X_1$, $X_2$ are irreducible of degrees $2p$ and 1, respectively. Let $X_i$ have character $\chi_i$, $i = 1, 2$. Let $L = \mathcal{N}(Q) \cap N$. Then $\chi_1 \mid L = \sum_{i=1}^{p} \alpha_i$, where $\alpha_1, \ldots, \alpha_p$ are distinct irreducible characters of $L$ of degree 2 permuted cyclically by $P$.

3.5.1. $\chi \mid Q$ is a sum of $2p + 1$ distinct constituents, $\mathcal{C}(Q) = QZ$, and $Q \cap Z = 1$. $P$ has two orbits of length $p$ on the irreducible constituents of $\chi_1 \mid Q$.

Proof. Write $\alpha_i \mid Q = \alpha_{11} + \alpha_{12}$, $i = 1, \ldots, p$. Here $\alpha_{11}, \alpha_{12}$ is a complete set of conjugate characters of $Q$ under the action of $L$ for each $i$. If for some $i$, $P$ fixes $\alpha_i \mid Q$, then $P$ fixes all of $\alpha_1 \mid Q$, $\ldots$, $\alpha_p \mid Q$ and hence all $\alpha_{ij}$. This would imply that $\chi_1 \mid PQ$ is a sum of linear characters and $P$ would centralize $Q$. Hence $P$ fixes no $\alpha_i \mid Q$.

Suppose $P$ fixes $\alpha_i$ for some $i, j$. Let $w$ be a generator of $P$. Then $\alpha_i \mid Q$ and $\alpha_i^w \mid Q$ have a common constituent. Since $\alpha_i \mid Q$ and $\alpha_i^w \mid Q$ are each sums of a complete set of conjugate characters of $L$, $\alpha_i \mid Q = \alpha_i^w \mid Q$, a contradiction.

Suppose $\alpha_{11} = \alpha_{12}$ for some $i$. Then this would be true for all $i$. It would then follow that $X_1(Q) \leq \mathcal{Z}(X_1(L))$, $k = 1, 2$. Hence $Q \leq \mathcal{Z}(L)$ and $N$ would have a normal $q$-complement by Burnside’s theorem.

Since $\chi \mid Q$ is a sum of linear characters, $Q$ is abelian. By transfer theorems $Q \cap Z = 1$ and $Q \leq \ker \chi_2$. Using the simplicity of $N/Z$ and the method of ([2], 3F), we obtain $\mathcal{C}(Q) = Q \times Z$.

3.5.2. $L = SQ \times Z$, where $S$ is an elementary abelian 2-group which is $P$-invariant.

Proof. We may write $X_1 \mid L = Y_1 \oplus \cdots \oplus Y_p$, where $Y_i$ is a representation of $L$ having character $\alpha_i$. Since $\alpha_i \mid Q$ is a sum of two distinct constituents, $L$ has a normal subgroup $H_i$ of index 2 such that $Y_i(H_i)$ is abelian. Let $H = \cap H_i$. Then $H$ is abelian and $\mathcal{C}(Q) = Q \times Z \leq H$. Hence $H = \mathcal{C}(Q)$. Because $L/H$ is isomorphic to a subgroup of $L/H_1 \times \cdots \times L/H_p$, it is elementary abelian. If $S$ is taken to be a $P$-invariant Sylow 2-subgroup of $L$, 3.5.2 now follows.

3.5.3. (i) holds.

Proof. Here we assume $B \cap Q = 1$. We may write $S = B \cap S \times [S, P]$. Then $P$ acts f.p.f. on $[S, P]Q$ and so $[S, P] \leq \mathcal{C}(Q) = QZ$. Therefore $[S, P] = 1$ and $L = (B \cap S)Q \times Z$. It now follows that $PQ < PL$ and that $X_1 \mid PQ$ is a sum of two distinct irreducible constituents of degree $p$. Hence $PL$ has a normal subgroup, say $PK$, $K < N$, of index 2 such that $X_1 \mid PK$ is a sum of two irreducible constituents of degree $p$. $K$ is therefore abelian and
so \( K = \mathbb{C}(Q) \) and we have \(|\mathcal{N}_N(Q) : \mathbb{C}_N(Q)| = 2\). By Smith–Tyrer [12], \( Q \) is cyclic. Since \( P \) acts nontrivially on \( Q \), \( q - 1 = tp \) for some \( t \) or \( q = tp + 1 \). Suppose \( t > 2 \). Since \( q \) is odd, \( t \geq 4 \) and \( q \geq 2n - 1 \). By [17], \( N/Z \cong PSL(2, q) \), a contradiction because the latter has no automorphism of order \( p \).

3.5.4. If \( B \cap Q \neq 1 \), then \( \chi | B \cap Q = p\psi_1 + p\psi_2 + 1_{B \cap Q} \) where \( \psi_1 \neq \psi_2 \) are nonprincipal characters. \( |B \cap Q| = 3 \) or 5.

Proof. \( \chi_1 | B \cap Q = pa_1 = p\psi_1 + p\psi_2 \) for some linear characters \( \psi_1, \psi_2 \). \( \chi_2 | B \cap Q = 1_{B \cap Q} \) because \( Q \leq \ker \chi_2 \). If \( \psi_1(y) = \psi_2(y) \), then \( y \in \mathfrak{Z}(L) \) so \( y = 1 \) by transfer. If one of \( \psi_1, \psi_2 \) has a nontrivial kernel, unimodularity of \( X \) is contradicted. Hence \( \psi_1 \neq \psi_2 \) and \( B \cap Q \) is cyclic of order, say \( q^t \). It follows that \( B \cap Q \) contains a generator all of whose eigenvalues are in the set \( \{e^{2\pi i/q^t}, e^{2\pi i/q^t}, 1\} \). By Blichfeldt ([1], p. 96), \( q^t < 6 \). Hence \( |B \cap Q| = 3 \) or 5.

3.5.5. If \( B \cap Q \neq 1 \), \( B < \mathcal{N}(Q) \) and \( \mathcal{N}(B \cap Q) \leq \mathcal{N}(Q) \) so \( |\mathcal{N}_N(B \cap Q) : \mathbb{C}_N| \) is a power of 2.

Proof. By 2.6, \( \chi | B \cap Q = p\psi + \lambda \). Hence \( \lambda | B \cap Q = 1_{B \cap Q} \) by 3.5.4. Let \( V \) be the underlying vector space affording \( X \). There is a unique one-dimensional subspace \( V_1 \) on which \( B \cap Q \) acts trivially. It follows that \( V_1 \) admits \( B, \mathcal{N}(B \cap Q) \) and \( \mathcal{N}(Q) \). Hence \( \chi | \langle \mathcal{N}(Q), \mathcal{N}(B \cap Q), B \rangle = \tilde{X}_1 + \tilde{X}_2 \), where \( \tilde{X}_i \) is an extension of \( X_i \), \( i = 1, 2 \). Let us also write \( T = \langle \mathcal{N}(Q), B, \mathcal{N}(B \cap Q) \rangle \cap N, X | PT = \tilde{X}_1 + \tilde{X}_2, \tilde{X}_1, \) and \( \tilde{X}_2 \) extensions of \( X_1 \) and \( X_2 \). \( \chi_1 | T = \sum_{i=1}^p \tilde{X}_i \), where \( \tilde{X}_i \) is an extension of \( \alpha_i \). Let \( Y_i \) be a representation of \( T \) affording \( \tilde{X}_i, i = 1, \ldots, p \). \( \tilde{X}_1 \) represents \( Y_i | Y_1 \oplus \cdots \oplus Y_p \) and \( Y_1, \ldots, Y_p \) are conjugate representations of \( T \) of degree 2. Suppose \( Y_i \) is primitive. Then ([1]) \( Y_i(T)/\mathfrak{Z}(Y_i(T)) \cong A_4, S_4 \) or \( A_5 \). Since \( \alpha_i | B \cap Q = \psi_1 \) \( R \cap Q = \psi_1 + \psi_2 \) and \( \psi_1(y) = \psi_2(y) \) only for \( y = 1 \), \( Y_i(B \cap Q) \cap \mathfrak{Z}(Y_i(T)) = 1 \). Now \( Q = B \cap Q \times [Q, P] \). But \( Y_i(Q)/Y_i(Q) \cap \mathfrak{Z}(Y_i(T)) \cong Y_i(Q) \mathfrak{Z}(Y_i(T))/\mathfrak{Z}(Y_i(T)) \) has order \( q = 3 \) or 5. Hence it follows that \( Y_i([Q, P]) \leq \mathfrak{Z}(Y_i(T)) \). Since this must hold then for each \( i = 1, \ldots, p \), \( [Q, P] \leq \mathfrak{Z}(T) \). Therefore, \( [Q, P] \leq \mathfrak{Z}(L) \), a contradiction by transfer. Hence each \( Y_i \) is imprimitive. Therefore \( T \) has an abelian normal subgroup of 2-power index. Therefore \( PT \leq \mathcal{N}(Q) \) as desired.

3.5.6. \( |B : (B \cap Q)Z| \leq 2 \) if \( B \cap Q \neq 1 \).

Proof. By 3.5.1, \( X_1 | PQ \) is a sum of two distinct irreducible characters of degree \( p \). Since \( PQ < PBQ, X_1 | PBQ \) is irreducible or a sum of two distinct characters of degree \( p \). In the first case, the inertia group of an irreducible
constituent of $\chi_1 \mid PQ$ is a group $PB_1Q$, where $B_1$ is a subgroup of $B$ of index 2.

In this case further, $\chi \mid PB_1Q$ is reducible, $B_1Q$ is abelian and $B_1 \leq \mathcal{C}(Q) = QZ$ and the result follows. In the second case, $BQ$ is abelian and $B = B \cap Q \times Z$.

3.6. If $X \mid \mathcal{N}(Q)$ is irreducible with $X \mid Q$ reducible, then $q \parallel B : Z$.

If $1 \neq Q_0 \leq B \cap Q$, then $|\mathcal{N}_0(Q_0) : QZ|$ is a power of 2 and is at most 2 unless $p + 1$ is a power of 2.

3.6.1. $\chi \mid Q$ is a sum of $2p + 1$ distinct linear characters.

Proof. By Clifford’s theorem $\chi \mid Q = e \sum_{i=1}^r \theta_i$, where $\theta_1, \ldots, \theta_r$ are distinct irreducible conjugate characters of $Q$ and $e \theta_i(1) = 2p + 1$. Suppose $\theta_i$ is nonlinear. Then $\theta_i(1) \geq 3$ and $P$ fixes all of $\theta_1, \ldots, \theta_r$ since $n < 3p$. Therefore $\chi \mid PQ$ is a sum of $e$ irreducible constituents each of degree $\theta_i(1)$. By [19], there exists a positive integer $s$ such that $q^s \mid \theta_i(1)$ and $q^s \equiv 1 \pmod{p}$.

Hence $q^s > p + 1 > n/2$ and $q^s \mid n$. This implies $n = q^s \leq \theta_i(1)$, contradicting our assumption that $\chi \mid Q$ is reducible. Therefore $\theta_i(1) = 1$ and $2p + 1 = er$. Since $PQ \cong X(PQ)$ is not $p$-closed, $P$ does not fix all $\theta_i$. Hence $r \geq p$. This forces $r = n$ as desired.

3.6.2. $\mathcal{C}(Q) = Q \times Z, B \cap Q \neq 1$.

Proof. By ([2], 3F), $\mathcal{C}(Q) = QZ$ and by a transfer theorem $Q \cap Z = 1$. Suppose $P$ acts f.p.f. on $Q$. Then $1_0$ is the only linear character of $Q$ fixed by $P$.

But since $p \nmid n$, $P$ fixes a linear constituent of $\chi \mid Q$. This is a contradiction because $\chi \mid Q$ is a sum of distinct conjugates and $1_0$ is conjugate only to itself. Therefore $B \cap Q \neq 1$.

3.6.3. $|\mathcal{N}_0(Q) : QZ| = n$ or $2n$.

Proof. This follows from 3.6.1, 3.6.2, and 1.9.

3.6.4. The second statement of 3.6 holds.

Proof. Let $1 \neq Q_0 \leq B \cap Q$ and let $L = \mathcal{N}(Q_0)$. Since $Q \leq L$, $PL$ is not $p$-closed. We consider in turn each of the possibilities for the decomposition of $X \mid PL$. Suppose first that all non-$p$-closed irreducible constituents have degree $\leq p$. Let $X_1$ be one such. If $X_1$ has degree $p - 1$, then $X_1(Q) \leq \mathcal{Z}(X_1(PL))$ by ([18], Theorem 1) contradicting 3.6.1. If $X_1$ has degree $p$, then $X_1(L)$ is abelian, $X_1(Q) \leq \mathcal{Z}(X_1(L))$ and $X_1(B \cap L)$ consists of scalars. Thus $\psi \mid B \cap L$ is reducible and all irreducible constituents of $\chi \mid B \cap L$ are linear. Hence a $p$-closed irreducible constituent of $X \mid PL$ has degree 1.

Therefore, in this case, each irreducible constituent $Y$ of $X \mid PL$ satisfies $Y(Q) \leq \mathcal{Z}(Y(L))$. Hence $Q \leq \mathcal{Z}(L)$ and $L = \mathcal{C}(Q) \to QZ$. 

Now suppose \( X \mid PL \) has an irreducible, non-\( p \)-closed constituent \( X_1 \) of degree \( p + 1 \). Then \( p + 1 \) is a power of 2 and again \( \psi \mid B \cap L \) is reducible and all \( p \)-closed, irreducible constituents of \( X \mid PL \) are linear. By 1.7, \( X_1(L) \) has an abelian \( 2' \)-Hall subgroup. It follows that \( L \) contains an abelian subgroup of 2-power index. Hence \( |L : \mathcal{C}(B)| = |L : QZ| \) is a power of 2.

Assume now that \( X \mid PL \) has an irreducible constituent \( X_1 \) of degree \( 2p - 2 \). Then it has a complementary constituent \( X_2 \) of degree \( 3 < p - 1 \). By (21), Lemma 3) \( X_i(P) \trianglelefteq X_i(PQ) \), \( i = 1,2 \) and therefore we get the contradiction \( Q \leq B \).

Suppose now that \( X \mid PL \) has an irreducible constituent of degree \( 2p \) and let \( \chi_1 \) be the character of this constituent. Then \( \chi_1 \mid L = \sum_{i=1}^{p} \alpha_i \), where \( \alpha_1, ..., \alpha_p \) are irreducible characters of \( L \) of degree 2 permuted cyclically by \( P \). Hence \( \chi_1 \mid Q_0 = p\alpha_1 \mid Q_0 = p(\psi_1 + \psi_2) \), where \( \psi_1 \) and \( \psi_2 \) are linear characters of \( Q_0 \). By the work of Mitchell [9], \( G \) cannot contain a homology, i.e., an element \( y \) such that \( X(y) \) has an eigenvalue of multiplicity \( n - 1 \). Therefore \( \psi_1 \neq \psi_2 \). Thus each \( \alpha_i \) is imprimitive and it follows that \( L \) has an abelian normal subgroup of 2-power index. Hence \( \mathcal{C}(Q) = QZ \) has 2-power index in \( L \). But \( L \leq \mathcal{N}(Q) \) so the result follows from 3.6.3.

Finally suppose that \( X \mid PL \) is irreducible. Then \( \chi \mid Q_0 = p\psi + \lambda \) would be impossible by Clifford's theorem. This completes the proof of 3.6.4 and of 3.6.

4. Conclusion

In this section the structure of \( B \) is pinned down. It is then possible to describe the structure of \( N \) so specifically that the final contradiction can be obtained.

4.1. If \( \tau \in B \) is an involution, then \( \chi \mid \langle \tau \rangle = (p + 1)v + p1_{\langle \tau \rangle} \), where \( v \) is the nonprincipal linear character of \( \langle \tau \rangle \).

Proof. By unimodularity of \( X \) and [9], the multiplicity of \( v \) or \( 1_{\langle \tau \rangle} \) in \( \chi \mid \langle \tau \rangle \) is at most \( 2p - 1 \). From 2.6, \( \chi \mid \langle \tau \rangle = p\psi_1 + p\psi_2 + \lambda \mid \langle \tau \rangle \), where \( \psi_1 \) and \( \psi_2 \) are linear characters of \( \langle \tau \rangle \). Unimodularity now implies the result.

4.2. If \( B \) is nonabelian, then \( B \) contains a normal abelian subgroup \( A \) of odd order and index 2.

Proof. By 2.6, \( \chi \mid B = p\psi + \lambda \), where \( \psi \) is irreducible of degree 2. By 4.1, \( \ker \chi \) cannot contain an involution. Hence \( \lambda \) is faithful on a 2-Sylow subgroup which must therefore be cyclic. Hence \( B \) contains a normal 2-complement \( H \). Since \( 2 \tau \mid H \), \( \psi \mid H \) is reducible. If \( \psi \mid H \) is homogeneous, \( H \leq \mathcal{Z}(B) \).
implying $B$ abelian. Therefore $\psi \mid H$ is a sum of two distinct conjugate constituents. $\psi$ is consequently imprimitive and $B$ contains a normal subgroup $A$ of index 2 such that $\psi \mid A$ is reducible so that $A$ is abelian. Let $T$ be the Sylow 2-subgroup of $A$. Then $\psi \mid T = \psi_1 + \psi_1^\ast$, where $\psi_1$ is a linear character of $T$ and $x$ is a 2-element of $B \setminus A$. Since a 2-Sylow subgroup of $B$ is cyclic, $\psi_1 = \psi_1^\ast$ on $T$. By 4.1, $T = 1$ and $A$ has odd order.

4.3. Let $q$ be an odd prime divisor of $|N|$ and let $Q$ be a $q$-Sylow subgroup. If $Q \leq B$, then $\lambda \mid Q = 1$ and $|Q| = 3$ or 5.

Proof. Let $\mathcal{N}(Q) \cap N = L$. Let $\mu$ be an irreducible constituent of $\chi \mid \mathcal{N}(Q)$ such that $\mu \mid B \cap L$ contains $\lambda \mid B \cap L$. By 2.6, we may write $\chi \mid Q = p\psi_1 + p\psi_2 + \lambda$, where $\psi_1$ and $\psi_2$ are linear characters of $Q$. By [9], $\psi_1 \neq \psi_2$. Assume $Q \not\subseteq \ker \lambda$.

Suppose first that $\mu$ is not $p$-closed, $\lambda \mid Q = \psi_1$ or $\psi_2$ for if not by Clifford's theorem, $\mu(1) \leq 3 < p - 1$ which is impossible. Hence we may assume $\psi_1 \neq \psi_2 - \lambda \mid Q$. Since $\lambda \mid Q$ is conjugate only to itself in $B \cap L$ the same is true of $\psi_2$. Since two elements of $B \cap L$ are conjugate in $PL$ if and only if they are already conjugate in $B \cap L$ ([5]), $\psi_1$ and $\psi_2 = \lambda \mid Q$ are conjugate only to themselves in $\mathcal{N}(Q)$. This implies that if $U$ is an irreducible constituent of $X \mid \mathcal{N}(Q)$, $U(Q)$ consists of scalars. Therefore $Q \leq \mathcal{Z}(\mathcal{N}(Q))$, a contradiction by Burnside's transfer theorem.

Therefore $\mu$ is $p$-closed. But then $\mu$ is essentially a character of $P \times B \cap L$ and so $\mu \mid Q = \lambda$. Since $\mathcal{O}(\mathcal{N}(Q)) < \mathcal{N}(Q)$ implies $\mathcal{O}(G) < G$ by Grün's theorem, $Q \leq \ker \lambda$ as claimed.

Since $X$ is unimodular and $\chi \mid Q = p\psi_1 + p\psi_2 + \lambda$, there is an element of $Q$ whose eigenvalues are $e^{2\pi i/q^s}$, $e^{-2\pi i/q^s}$, 1 where $q^s$ is the exponent of $Q$. By Blichfeldt's theorem $q^s < 6$ and so $q^s = 3$ or 5. By unimodularity, each $\psi_i$ is faithful and so $Q$ is cyclic. Therefore $|Q| = 3$ or 5.

4.4. $|B : Z|$ is divisible by at most one odd prime.

Proof. Suppose the statement is false. By 4.2, it may be stated that $B$ contains an abelian normal subgroup $A$ of index 1 or 2.

$A$ contains at most one Sylow subgroup of $G$ of odd order. For otherwise, by the preceding lemma, $A$ would contain an element of order 15 a suitable power of which would contradict Blichfeldt's theorem.

By 1.5, it follows that there is an odd prime $q$, $q \mid |B : Z|$ and a $P$-invariant $q$-Sylow subgroup $Q > B \cap Q$ with $B \cap Q$ a $q$-Sylow subgroup of $B$. Since $A \leq \mathcal{O}(B \cap Q)$, it follows from the previous section, by elimination of other possibilities, that $X \mid Q$ is irreducible. Hence $\mathcal{n} = q^s$, where $q = 3$ if $s > 1$.

Combining the last two paragraphs, we deduce that $|B| = 2^{\alpha}q^{2\ell}$, where
2, q, t are distinct primes, \( t = 3 \) or 5, \( B \) contains a \( t \)-Sylow subgroup of \( G \) and \( X \mid Q \) is irreducible, \( Q \) a \( q \)-Sylow subgroup.

By the last section there can be no other prime divisors of \(|N|\). Hence \(|N| = 2^aq^bt, t = 3 \) or 5, \( n = q^s\). Suppose \( s = 1 \). By \[13\], \( q = 3, 5, 7, 13, \) or 17 against \( q = 2p + 1 \geq 7 \). Hence \( s > 1, q = 3 \) and \(|N| = 2^a3^b5\). By \[3\], there is no such simple group \( N/Z \) having an automorphism of order \( p \).

This proves 4.4.

Since \( N \) is nonsolvable, \(|N|\) is divisible by at least three distinct primes. From the last step, we deduce that there exists an odd prime \( q \) dividing \(|N|\) such that \( q \mid B : Z \). Let \( Q \) be a \( P \)-invariant \( q \)-Sylow subgroup. From Section 3 we see that either \( X \mid \mathcal{N}(Q) \) has an irreducible constituent of degree \( 2p \) or \( X \mid Q \) is irreducible. In the first case by \[3.5(i)\], \( n = q \). This precludes the existence of any other odd prime divisor \( r \) of \(|N|\) such that \( r \mid B : Z \). Thus \(|N|\) is divisible by precisely 3 distinct primes. Since in the first case \( n = q, n = 13 \) or 17 by \[13\] contradicting \( n = 2p + 1 \). Similarly if \( X \mid Q \) is irreducible and \( n = q \), we get a contradiction. Hence \( X \mid Q \) is irreducible, \( q = 3 \) and \( n \) is a power of 3.

4.5. \(|N| = 2^aq^b, n = 3^s \) for some \( s, q \) is a prime distinct from 2 and 3 such that \( q \mid B : Z \). \( 3 \mid B : Z \) and \( X \mid R \) is irreducible where \( R \) is a \( P \)-invariant Sylow 3-subgroup of \( N \).

The remainder of the proof will be divided into two cases according as \( X \mid \mathcal{N}(R) \) is imprimitive or primitive. We shall make use of the Smith-Tyrer theorem \[12\] which states that if \(|\mathcal{N}_N(R) : R \mathcal{C}_N(R)| = 2 \) and \( R \) has class 2, then \( N' \neq N \). We require first the following result.

4.6. \( 4 \mid B \).

Proof. Let \( q \) be the unique odd prime divisor of \(|B : Z|\) and let \( Q \) be a \( P \)-invariant \( q \)-Sylow subgroup of \( N \). If \( Q \trianglelefteq B \), then \(|Q| = 5 \) by 4.3 and 4.5 and so \(|N| = 2^a3^b5\) contrary to \[3\]. Therefore \( Q \nsubseteq B \). By 4.5 and Section 3, it follows that \( X \mid \mathcal{N}(Q) \) has the form described in 3.2, 3.5(ii) or 3.6. If \( p + 1 = 2^e \) for some \( e, n = 3^s = 2p + 1 = 2^{e+1} - 1 \) which implies the contradiction \( s = 0 \) or 1. Hence 3.2 cannot occur and 3.5(ii) or 3.6 with 4.2 now yields 4.6.

In 4.7 through 4.11 it will always be assumed that \( X \mid \mathcal{N}(R) \) is imprimitive.

4.7. \( \mathcal{N}_N(R) = \langle \tau \rangle R, \) where \( \tau \in B \) is an involution. \( R \) contains an abelian subgroup \( R_1 \) of index \( n = 3^s \). \( \mathcal{N}(R_1) = \mathcal{N}(R) \) and \( P \) acts irreducibly on \( R/R_1 \). \( \mathcal{C}(R_1) = R_1 \).

Proof. By 2.6, \( R \trianglelefteq B \) is impossible and so \( \mathcal{N}(R) \) is not \( p \)-closed. The proof of 1.9 shows that \( \mathcal{N}(R) \) contains an abelian normal subgroup \( K \) such
that \( \chi | K \) is a sum of \( 2p + 1 \) distinct linear characters and \( \mathcal{N}(R)/K \) has the structure indicated there. Let \( q \) be a prime divisor of \( |K| \), \( q \neq 3 \), and let \( Q \) be the Sylow \( q \)-subgroup of \( K \). Then \( Q < \mathcal{N}(R) \) and so \( Q \leq \mathcal{C}(R) = Z < R \) since \( \chi | R \) is irreducible and \( X \) is unimodular. This is a contradiction and we may write \( K = R_1 < R \). By \ref{1.9}, \( \mathcal{N}_N(R) = R_1 \) or \( \langle \tau \rangle R, \tau \in B \) an involution, \( \langle R : R_1 \rangle = n \) and \( P \) acts irreducibly on \( R/R_1 \).

\( \mathcal{N}(R) \leq \mathcal{N}(R_1) \) and \( X | \mathcal{N}(R_1) \) is imprimitive since \( X | R_1 \) is a sum of \( 2p + 1 \) distinct constituents. By the method of \ref{1.9}, \( \mathcal{N}(R_1) \) contains an abelian normal subgroup containing \( R_1 \) whose corresponding factor group is \( 3 \)-closed. By \((\mathcal{C}, 3\mathcal{F})\), \( \mathcal{C}(R_1) = R_1 \) and it follows that \( \mathcal{N}(R_1) = \mathcal{N}(R) \).

Glauberman \((\mathcal{G}, p. 37)\) has defined a characteristic subgroup \( K_\infty \) of \( R \). By the definition of \( K_\infty \), \( R_1 \leq K_\infty \) and since \( P \) acts irreducibly on \( R/R_1 \), \( K_\infty = R_1 \) or \( R \). In either case \( \mathcal{N}(K_\infty) = \mathcal{N}(R) \). By \((\mathcal{G}, \text{Theorem 12.10, p.} 37)\) \( N = N' \) implies \( \mathcal{N}(K_\infty)/\mathcal{C}_N(K_\infty) \) cannot be a \( 3 \)-group. Hence \( \mathcal{N}_N(R) = \langle \tau \rangle R, \tau \in B \) an involution.

4.8. Let \( R_0 = \mathcal{C}_N(\tau) \). Then \( Z < R_0 < R_1 \).

Proof. As in 4.7, \( \mathcal{N}_N(R)/\mathcal{C}_N(R) \) contains no normal subgroup of index 3. Hence \( \tau \) acts f.p.f. on \( R/R_1 \) and therefore \( R_0 \leq R_1 \). \( R_0 < R_1 \) because \( \mathcal{C}(R_1) = R_1 \). \( Z < R_0 \) for otherwise \( \tau \) would act f.p.f. on \( R/Z \), so \( R/Z \) would be abelian and \( R \) would have class 2. We would then have a contradiction by \[12\].

4.9. \( \mathcal{C}(R_0) \) is not \( 3 \)-closed.

Proof. \( R_1 < \mathcal{C}(R_0) \) and there is a \( P \)-invariant \( 3 \)-Sylow subgroup of \( \mathcal{C}(R_0) \). Since \( P \) acts irreducibly on \( R/R_1 \), \( R_1 \) is a \( 3 \)-Sylow subgroup of \( \mathcal{C}(R_0) \) or \( \mathcal{C}(R_0) \) contains a \( 3 \)-Sylow subgroup of \( N \). The latter is impossible because \( X | \mathcal{C}(R_0) \) is reducible while \( X | R \) is irreducible. Hence if \( \mathcal{C}(R_0) \) is \( 3 \)-closed, \( \mathcal{C}(R_0) \leq \mathcal{N}(R_1) = \mathcal{N}(R) \). We claim \( \mathcal{C}_N(\tau) \) is abelian. From 4.1 we can conclude that a \( p \)-closed irreducible constituent of \( \chi | \mathcal{C}(\tau) \) is linear and a non-\( p \)-closed irreducible constituent has degree at most \( p + 1 \). \( \chi | \mathcal{C}(\tau) \) cannot have an irreducible constituent of degree \( p - 1 = 2^a \) or \( p + 1 = 2^a \) for some \( a \) because \( 3^s = 2^s + 1 = 2(p - 1) + 3 = 2^{a+1} + 3 \) is impossible and \( 3^s = 2p + 1 = 2(p + 1) - 1 = 2^{a+1} - 1 \) has no solution except when \( s = 0 \) or \( s = 1 \). Hence a non-\( p \)-closed irreducible constituent has degree \( p \). Therefore \( \mathcal{C}_N(\tau) \) is abelian. If \( \mathcal{C}(R_0) \) is \( 3 \)-closed, we would have \( \mathcal{C}_N(\tau) \leq \mathcal{C}(R_0) < \mathcal{N}(R) \). By 4.7, \( 4 \tau \in \mathcal{C}_N(\tau) \) which implies the contradiction that \( \langle \tau \rangle \) is a \( 2 \)-Sylow subgroup of \( N \).

4.10. \( X | P\mathcal{C}(R_0) = X_1 \oplus X_2 \), where \( X_1 \) is irreducible of degree \( 2p \) and \( X_2 \) is linear.
Proof. Clearly $P$ normalizes $R_0 = \mathcal{C}(\tau) \cap R$ and so $P$ normalizes $\mathcal{C}(R_0)$. Since $B \cap R = Z$, $R_0 \leq B$ and hence $P \mathcal{C}(R_0)$ is not $p$-closed. As in the preceding step $X | P \mathcal{C}(R_0)$ cannot have a non-$p$-closed irreducible constituent of degree $p - 1$ or $p + 1$. If all non-$p$-closed irreducible constituents had degree $p$, $\mathcal{C}(R_0)$ would be reducible and $\mathcal{C}(R_0)$ would be abelian yielding the contradiction $\tau \in \mathcal{C}(R_0) \subseteq \mathcal{C}(R) = R_0$. If $X | P \mathcal{C}(R_0)$ had a non-$p$-closed irreducible constituent of degree $2p - 2$, then $2p - 2$ is a power of 2 implying $p = 1$ is a power of 2 and a contradiction is obtained as above. If $X_1$ is a non-$p$-closed irreducible constituent of degree $2p - 1$, then $X_1 | PR_0$ would have an irreducible constituent of degree $p$ and $X_1 | R_0$ would be a sum of linear constituents each having multiplicity 1. A contradiction could then be obtained by ([8], Lemma 2.6). If $X | P \mathcal{C}(R_0)$ were irreducible, $X | \mathcal{C}(R_0)$ would be irreducible yielding the contradiction $R_0 \leq Z$. The statement of 4.10 is the only remaining possibility.

4.11. $\mathcal{C}(R_0)$ contains a subgroup $H$ of index 2, $\tau \notin H$ and $H \triangleleft P \mathcal{C}(R_0)$.

Proof. $P \mathcal{C}(R_0)/\ker X_2$ is cyclic and by 1.3, $\mathcal{C}(R_0) = B \cap \mathcal{C}(R_0)/\ker X_2$. Let $X_3$ be the character of $X_1 \cdot X_2 | \mathcal{C}(R_0) = \sum_{i=1}^{p} x_i$, where $x_i$ are permuted cyclically by $P$. Since $P$ fixes $x_1 \in B \cap \mathcal{C}(R_0)$, $X_1 \in B \cap \mathcal{C}(R_0) = p \alpha_1$. By 4.1 then, $\tau \notin \ker X_2$ and since $4 \nmid | R |$ by 4.6, $P \mathcal{C}(R_0)$ contains a normal subgroup $H_1$ of index 2 with $\tau \notin H_1$. If we let $H = H_1 \cap \mathcal{C}(R_0)$, 4.11 is proved.

We can now complete the case $X | \mathcal{N}(R)$ imprimitive. Suppose $q \nmid | B \cap \mathcal{C}(R_0)|$. Then by 4.5 and 4.6, $P$ acts f.p.f. on $H/Z$ implying $H$ nilpotent against 4.9. Hence $q \nmid | B \cap H |$. Let $Q_0$ be a $q$-Sylow subgroup of $B \cap H$. Then $R_0 \leq \mathcal{C}(Q_0)$ contrary to 3.5 or 3.6 if $B$ does not contain a $q$-Sylow subgroup of $N$. If $B$ does contain a $q$-Sylow subgroup $Q$ of $N$, then, by 4.3, $| Q | = 5$ and a contradiction is obtained from [3].

A final contradiction is now obtained under the assumption that $X | \mathcal{N}(R)$ is primitive. As in 2.7, we may conclude that $| \mathcal{N}(R) : R |$ is a power of 2. Let $U$ be a $P$-invariant Sylow 2-subgroup of $\mathcal{N}(R)$. By 4.6, $| B \cap U | = 1$ or 2.

We claim $B \cap [U, P] = 1$. This is true by 1.2 if $U$ is abelian. So we suppose $U$ is nonabelian. This implies easily that $\chi \in [U, P] = \chi_1 + \chi_2$, where $\chi_1$ and $\chi_2$ are irreducible characters of degrees $2p$ and 1, respectively (see for example most of the proof of 4.10). Then $X_1 | B \cap U = p \psi | B \cap U$ and $[U, P] \leq \ker \chi_2$. By 4.1, $B \cap [U, P] = 1$.


Suppose that $X | PR$ is imprimitive. As in 1.9, $R$ contains an abelian normal subgroup $R_1$ of index $2p + 1$ such that $R/R_1$ is abelian. Therefore $R'$ is
abelian. By primitivity of $X \mid \mathcal{N}(R)$, $R' \leq Z$ and $R$ has class 2. [12] forces $B \cap U = 1$ but then Burnside's theorem applied to $N/Z$ and $\mathcal{N}(R)/Z$ gives a contradiction to $N/Z$ being simple.

Therefore $X \mid PR$ is primitive. Let $R_1 \leq R$ be such that $R_1/Z$ is a minimal normal subgroup of $PR/Z$. Then $R$ acts trivially on $R_1/Z$ and $P$ acts irreducibly on $R_1/Z$. By primitivity, $R_1$ is nonabelian and $X \mid R_1$ is irreducible (see 2.5). For $x, y \in R_1$, $1 = [x^3, y] = [x, y]^3$ ([7], 5.3.9) and since $R_1' \leq Z$ is cyclic, $|R_1'| = 3$. By a transfer theorem ([7], 7.4.4), $R_1/R_1' = (R_1 \cap (PR_1)/R_1') \times (Z/R_1')$. Let $R_0 = (PR_1)'$. Then $R_0 \vartriangleleft R_1$, $R_0$ is extraspecial and $P$ acts trivially on $R_0'$ and irreducibly on $R_1/R_0'$. Since $X \mid R_1$ is irreducible, $R_1 \vartriangleleft PR_1$ and $R_1 = R_0Z$, $X \mid R_0$ is homogeneous and so irreducible (as in 2.5). Hence $|R_0| = 3^{2s+1}$ and $p \mid (3^s - 1)(3^s + 1)$. By the proof of ([7], 11.2.5), $p \mid 3^s + 1$ which conflicts with $n + 1 = 2p + 2$. The proof of the Theorem is now complete.

Proof of the Corollary. We may assume by ([1], p. 14) that the given primitive representation $X$ is unimodular. Hence $|Z| \mid n$ since $Z$ is cyclic. By the Theorem, $N$ is solvable and $n$ is a power of 3 or a prime. Let $R \leq N$ be such that $R/Z$ is a minimal normal subgroup of $PN/Z$. As in 2.5, $X \mid R$ is irreducible. $R$ is therefore a group of prime power order such that $n \mid |R|$. The proof of 2.7 may now be used to complete the proof of the Corollary.

REFERENCES

5. G. GLAUBERMAN, Fixed points in groups with operator groups, Math. Z. 84 (1964), 120–125.
9. H. H. MITCHELL, Determination of all primitive collineation groups in more than four variables which contain homologies, Amer. J. Math. 36 (1914), 1–12.