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# Explicit analysis of Kähler deformations in 4D $\mathcal{N}=1$ supersymmetric quiver theories

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**Abstract**

Starting from the $\mathcal{N}=2$ $SYM_4$ quiver theory living on wrapped $N_i$ D5-branes around $S_i^2$ spheres of deformed ADE fibered Calabi–Yau threefolds (CY3) and considering deformations using *massive* vector multiplets, we explicitly build a new class of $\mathcal{N}=1$ quiver gauge theories. In these models, the quiver gauge group $\prod_i U(N_i)$ is spontaneously broken down to $\prod_i SU(N_i)$ and Kähler deformations are shown to be given by the real part of the integral $(2,1)$ form of CY3. We also give the superfield correspondence between the $\mathcal{N}=1$ quiver gauge models derived here and those constructed by Cachazo et al. [hep-th/0108120] using complex deformations. Other aspects of these two dual $\mathcal{N}=1$ supersymmetric field theories are discussed.


*Keywords:* Engineering of supersymmetric quiver gauge theories; Kähler and complex deformations in Calabi–Yau threefolds; Mirror symmetry; Wrapped branes on ADE geometries

## 1. Introduction

Recently four-dimensional $\mathcal{N}=1$ supersymmetric quiver gauge theories have been subject to an intensive interest [1–3]. These theories, which are engineered in different but dual ways, appear as low energy effective field theory of compactification of M-theory on G2 manifolds and type II string compactification on threefolds preserving $1/8$ of original supersymmetries [4–7]. A remarkable set of such field theoretical systems corresponds to those 4D $\mathcal{N}=1$ supersymmetric quiver gauge theories with gauge group $\prod_i U(N_i)$ and which are obtained through deformations of 4D $\mathcal{N}=2$ $\prod_i U(N_i)$ supersymmetric quiver gauge theories living on D5-branes wrapped on ADE fibered Calabi–Yau threefolds (CY3) [8,9]; see also [10]. Two classes (with and without monodromies) of such $4D$ $\mathcal{N}=1$ supersymmetric quiver gauge theories, following from the complex deformation of 4D $\mathcal{N}=2$ supersymmetric quiver gauge theories, have been constructed in [11–13]. In this Letter, we want to derive their mirrors using Kähler deformations rather than complex ones. Note that from the geometric point of view, this

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kind of dual models follows naturally using algebraic geometry methods and mirror symmetry exchanging Kähler and complex deformations; but from the supersymmetric field theory view the situation is far from obvious and needs a careful treatment. We will show, amongst others, that Kähler deformations in supersymmetric quiver field theories require massive gauge prepotentials; that is a spontaneously broken gauge symmetry $\prod_i U(N_i)$ down to $\prod_i SU(N_i)$ with all the features that go with this behaviour and also in particular the implementation of a Higgs superpotential and so adding further fundamental matters.

The presentation of this Letter is as follows: in Section 2, we describe the 4D $\mathcal{N}=2$ $\prod_i U(N_i)$ supersymmetric quiver gauge theories living on D5-branes wrapped on ADE fibered Calabi–Yau threefolds (CY3). We focus our attention on the special example of $U(N)$ gauge theory engineered on a $A_1$ fibered CY3 and use the simplest path involving the minimal degrees of freedom. Extension to ADE geometries is straightforward and some of its aspects may be found in [14]. In Section 3, we develop the study of the 4D $\mathcal{N}=1$ $\prod_i U(N_i)$ supersymmetric quiver gauge theories following from complex deformations of the $\mathcal{N}=2$ $\mathrm{SYM}_4$ quiver models. In Section 4, we consider the mirror of the previous $\mathcal{N}=1$ supersymmetric quiver gauge by using Kähler deformations rather than complex ones. In Section 5, we give our conclusion. Note that we will work in $\mathcal{N}=1$ superspace and make use of both real superspace $(x,\theta,\bar\theta)$ techniques as well as chiral ones $(x\pm i\theta\sigma\bar\theta,\theta,\bar\theta)$. For technical details; see for instance [14,15].

## 2. 4D $\mathcal{N}=2$ $\mathrm{SYM}_4$ quiver theories: $A_1$ model

The $\mathcal{N}=2$ supersymmetric $A_1$ quiver theory in four dimensions involves the following $\mathcal{N}=1$ degrees of freedom: (i) A $U(N)$ gauge multiplet $V$ which we take in the WZ gauge as $V=-\theta\sigma^\mu\bar\theta A_\mu - i\bar\theta^2\theta\lambda + i\theta^2\bar\theta\bar\lambda + \frac{1}{2}\theta^2\bar\theta^2 D$. This superfield has the special features

$$V^3=0,\qquad V=\frac{1}{N}U\mathrm{Y}+\sum_{a=1}^{N^2-1}V_aT^a,\quad U=\mathrm{Tr}(V),\tag{2.1}$$

which will be needed later on. Here $\mathrm{Y}\sim \mathrm{I}_{id}$ is the Abelian $U(1)$ generator of $U(N)$ and $\{T^a\}$ refer to the $SU(N)$ traceless generators. (ii) A chiral multiplet $\Phi$ in the adjoint representation of the gauge group $U(N)$. We will refer to it as adjoint matter and has the two following decompositions

$$\Phi=\frac{1}{N}\Theta\mathrm{Y}+\sum_{a=1}^{N^2-1}\Phi_aT^a,\qquad \Phi=\phi+\theta\psi+\theta^2F,\quad \bar{\mathcal{D}}\Phi=0,\tag{2.2}$$

where $\mathcal{D}$ stands for the supersymmetric covariant derivative; $\{\mathcal{D},\bar{\mathcal{D}}\}\sim 2\partial_\mu$, and $\Theta=\mathrm{Tr}(\Phi)$ is the $U(1)$ part of the adjoint of $U(N)$. We have also $[\Theta\mathrm{Y},\Phi]=[\Theta\mathrm{Y},U\mathrm{Y}]=[\Theta\mathrm{Y},V]=0$. For later use, we will focus on the supersummetric vacuum with a preserved $SU(N)$ gauge symmetry; that is matrix superfields with vevs such as $\langle\Phi_a\rangle=0$, but $\langle\Theta\rangle\neq 0$. Note that the computation of $\mathrm{Tr}[\Phi^m]$ in terms of $\Theta$ and $\Phi_a$ involves $SU(N)$ Casimirs; however, due to $\langle\Phi_a\rangle=0$ the vev of $\mathrm{Tr}[\Phi^m]$ simplifies to $\mathrm{Tr}[(\frac{1}{N}\Theta\mathrm{Y})^m]=N^{m-1}\Theta^m$ and so superpotentials of type $W(\Phi)=\sum_1^{n+1}\delta_m\,\mathrm{Tr}\,\Phi^m$ reduce to a polynomial in the $U(1)$ superfield $\Theta$. (iii) Four chiral multiplets $Q_{(\pm,\pm)}$ with the following $U(1)\times SU(N)$ charges: $Q_{(+,+)}\equiv Q_+$ and $Q_{(-,+)}\equiv P_-$ are in the representation $(\pm1,N)$ and $Q_{(+,-)}\equiv P_+$ and $Q_{(-,-)}\equiv Q_-$ are in the representation $(\pm1,\bar N)$. The antichiral superfields are in the complex conjugate of these representations. For convenience, we will work with the normalization of the $U(1)$ charge as $[\mathrm{Y},Q_\pm]=2Q_\pm$ and $[\mathrm{Y},P_\pm]=-2P_\pm$. These matter superfields have, in the chiral basis, the following $\theta$-expansions

$$Q_\pm=q_\pm+\theta\psi_\pm+\theta^2f_\pm,\qquad P_\pm=p_\pm+\theta\eta_\pm+\theta^2l_\pm,\tag{2.3}$$

where $q_\pm$, $p_\pm$ and so on stand for component fields. Note that the chiral composites $Q_+Q_-$ and $P_+P_-$ are in the $U(N)$ adjoint representation and may be expanded as in Eqs. (2.1) and (2.2). The same is valid for the Hermitian composites $Q_\pm Q^*_\pm$ and $P_\pm P^*_\pm$. Note also that these four $Q_\pm$ and $P_\pm$ chiral multiplets form two $\mathcal{N}=2$ hypermultiplets; one of them encodes the transverse coordinates of D5-branes; it describes their positions in the ten-dimensional type IIB string space, and the other is the usual moduli associated with the Kähler deformation of the $A_1$ singularity [17].

### 2.1. Action

The superspace Lagrangian density $\mathcal{L}_{N=2}(A_1)$ describing the $\mathcal{N}=2$ dynamics of the previous superfields reads as

$$\begin{aligned}\mathcal{L}_{N=2}(A_1) = \mathcal{L}_g(V) - 2\zeta\int d^4\theta\, U + \mathcal{L}_{ad}(\Phi) - \int d^2\theta\,\big(\beta\Theta + \mathrm{Tr}\big[\Phi(Q_-Q_+ - P_-P_+)\big] + \text{h.c.}\big)\\ + \int d^4\theta\,\mathrm{Tr}\big[Q^*_- e^{-2V}Q_- + Q^*_+ e^{2V}Q_+ + P^*_- e^{-2V}P_- + P^*_+ e^{2V}P_+\big],\end{aligned} \tag{2.4}$$

where $\mathcal{L}_g(V)$ and $\mathcal{L}_{ad}(\Phi)$ are, respectively, the gauge covariant Lagrangian densities for the $U(N)$ vector multiplets and adjoint matter superfields. The coupling constants $\zeta$ and $\beta$ are, respectively, real and complex parameters. They have both a field theoretical and geometric meanings and will play a crucial role in the present study. The supersymmetric scalar potential reads in terms of the auxiliary fields as $\mathcal{V} = \frac{1}{2}\mathrm{Tr}(D^2) + \mathrm{Tr}(FF^*) + \mathrm{Tr}(f^*_\pm f_\pm) + \mathrm{Tr}(l^*_\pm l_\pm)$ and the moduli space of its vacuum configuration is given by the following equations

$$\zeta = r_1 - r_2, \qquad \beta = t_1 - t_2, \tag{2.5}$$

where we have set

$$\begin{aligned} r_1 &= \big\langle \mathrm{Tr}(Q_+Q^*_+ + P_+P^*_+)\big\rangle, \qquad t_1 = \big\langle \mathrm{Tr}(Q_-Q_+)\big\rangle,\\ r_2 &= \big\langle \mathrm{Tr}(Q_-Q^*_- + P_-P^*_-)\big\rangle, \qquad t_2 = \big\langle \mathrm{Tr}(P_-P_+)\big\rangle.\end{aligned} \tag{2.6}$$

These parameters have a geometric interpretation in terms of Kähler and complex moduli of the $A_1$ fiber of the CY3. The real parameter $\zeta$ is the volume of the blown up sphere and the complex constant is just the so-called holomorphic volume of the complex deformation of $A_1$. In algebraic geometry, this means

$$\zeta = \int_{S^2_r} \mathcal{J}^{(1,1)}, \qquad \beta = \int_{S^2_h} \omega^{(2,0)}, \qquad \beta^* = \int_{S^2_a} \omega^{(0,2)}, \tag{2.7}$$

where $\mathcal{J}^{(1,1)}$ and $\omega^{(2,0)}$ are, respectively, the Kähler and complex holomorphic forms on the $A_1$ surface. Note in passing that the algebraic geometry equation of the complex deformed of the $A_1$ fiber of the CY3 reads as

$$x^2 + y^2 + (z - \Delta t)(z + \Delta t) = 0, \tag{2.8}$$

where $\Delta t$ stands for the holomorphic volume $(t_1 - t_2)$ of the complex deformation which, by help of Eq. (2.5), is also equal to $\beta$ and so Eq. (2.8) may be rewritten as $x^2 + y^2 + z^2 = \beta^2$.

### 2.2. Mirror $N=2$ models

On the supersymmetric field theory side, the $\zeta$ and $|\beta|$ parameters are involved in the $\mathcal{N}=2$ SYM gauge coupling constant $g^{(\mathrm{SYM})}_{N=2} \equiv g_{N=2}$ which read, in terms of the type IIB string coupling $g_s$, as

$$g_{N=2} = \sqrt{\frac{g_s}{V}}, \quad V = \sqrt{\zeta^2 + \beta\bar\beta}. \tag{2.9}$$

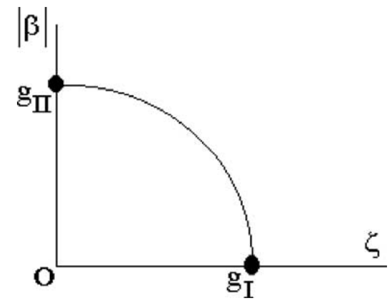


Fig. 1. In this figure we represent the projection of a flow $g_{N=2} = g_{N=2}(\vartheta)$ on the $(\zeta, |\beta|)$ plane. The black dot on the $\zeta$-axis represents $g^{(\mathrm{I})}_{N=2}$ and the one on the $|\beta|$-axis represents $g^{(\mathrm{II})}_{N=2}$.

Note that from the above relation, one sees that the $\mathcal{N}=2$ SYM coupling constant is a real two argument function; $g_{N=2} = g_{N=2}(\zeta, |\beta|)$, which we shall naively rewrite as $g_{N=2}(\zeta, \beta)$. Accordingly, one may think about this gauge coupling constant as describing a flow of $\mathcal{N}=2$ SYM models interpolating between two extreme models I and II with respective gauge coupling constants $g^{(\mathrm{I})}_{N=2}$ and $g^{(\mathrm{II})}_{N=2}$. The first is

$$g^{(\mathrm{I})}_{N=2} = \sqrt{\frac{g_s}{V_{\mathrm{I}}}}, \quad V_{\mathrm{I}} = \sqrt{\zeta^2}, \qquad \beta = 0, \tag{2.10}$$

with blown up volume $V_{\mathrm{I}}$ and the second involves pure holomorphic volume $V_{\mathrm{II}}$ type Weil–Peterson as

$$g^{(\mathrm{II})}_{N=2} = \sqrt{\frac{g_s}{V_{\mathrm{II}}}}, \quad V_{\mathrm{II}} = \sqrt{\beta\bar{\beta}}, \qquad \zeta = 0. \tag{2.11}$$

Setting $\zeta = \rho\cos\vartheta$ and $|\beta| = \rho\sin\vartheta$; with the spectral parameter $\vartheta$ bounded as $0 \leqslant \vartheta \leqslant \frac{\pi}{2}$, one gets an explicit relation for this $\mathcal{N}=2$ gauge coupling constant flow $g_{N=2} = g_{N=2}(\vartheta) = \sqrt{\frac{g_s}{V(\vartheta)}}$. In this view, the theories I and II with respective gauge couplings $g^{(\mathrm{I})}_{N=2}$ and $g^{(\mathrm{II})}_{N=2}$ correspond to $\vartheta = 0$ and $\frac{\pi}{2}$, they are mapped to each other under mirror symmetry acting as $\vartheta \to \frac{\pi}{4} - \vartheta$; see Fig. 1.

In $A_1$ geometric language, the $\mathcal{N}=2$ gauge models I correspond to the blowing up of $A_1$ surface in CY3; but zero holomorphic deformations, $\int_{S^2} \omega^{(2,0)} = 0$. The compact part of the $A_1$ singularity $x^2+y^2+z^2=0$ gets a non-zero volume as $(\mathrm{Re}\,x)^2 + (\mathrm{Re}\,y)^2 + (\mathrm{Re}\,z)^2 = \zeta$. This positive Kähler parameter $\zeta$ is same as in the superfield action Eq. (2.4). To fix the ideas $\zeta$ can be imagined of as corresponding to the derivative of a special Kähler deformation $K(h, \bar{h})$ where $h$ and $\bar{h}$ are Higgs fields to be specified later on; see Eq. (4.7). In other words $\zeta = \partial K_{\mathrm{FI}}/\partial C$ where $K_{\mathrm{FI}}$ is a *linear* Kähler deformation as $K_{\mathrm{FI}} \sim \zeta(C+U)$ and where $C = \frac{\upsilon^* H + \upsilon \bar{H}}{\upsilon\upsilon^*}$; see Eqs. (5.1)–(5.4). In the present Letter, $\zeta$ should be thought of as just the leading case of a *non-linear* Kähler superpotential $K(H, \bar{H})$ so that $(\mathrm{Re}\,x)^2 + (\mathrm{Re}\,y)^2 + (\mathrm{Re}\,z)^2 = \zeta$ gets replaced by $(\mathrm{Re}\,x)^2 + (\mathrm{Re}\,y)^2 + (\mathrm{Re}\,z)^2 = K'(h, \bar{h})$. Along with this Kähler analysis, one may also consider its mirror description using complex deformation of $A_1$ singularity. In this case the resulting $\mathcal{N}=2$ gauge model II corresponds exactly to the reverse of previous situation. Here $\int_{S^2} \mathcal{J}^{(1,1)} = 0$ but $\int_{S^2} \omega^{(2,0)} \neq 0$. As before the $A_1$ singularity $x^2+y^2+z^2=0$ gets now a holomorphic volume as $x^2+y^2+z^2=\beta^2$ where $\beta$ is as in the super action Eq. (2.4). Here also this $\beta$ appears as the derivative of *linear* complex deformation as $W_{\mathrm{FI}} \sim \beta\phi$ which in general should be thought of as just the leading case of a *non-linear* polynomial superpotential $W(\phi)$ so that $x^2+y^2+z^2=\beta^2$ extends to $x^2+y^2+z^2=W'^2(\phi)$ constituting a non-trivial fibered deformed $A_1$ in the CY3 we are interested in here. Note that $x^2+y^2+z^2=W'^2(\phi)$

describes a singular conifold in $\mathbb{C}^4$; no complex deformation of this conifold[1] is made here and so it should not be confused with geometric transition scenario of [11]. Note also that extension of $\zeta$ and $\beta$ to non-linear $K'$ and $W'$, respectively, break $\mathcal{N}=2$ supersymmetry down to $\mathcal{N}=1$. From the field theoretical point of view, these two models correspond to choosing the corresponding vevs Eqs. (2.5) and (2.6) such that $t_1(\phi)=t_2(\phi)$ and $r_1(c)\neq r_2(c)$ and inversely $t_1(\phi)\neq t_2(\phi)$ and $r_1(c)=r_2(c)$. The two symmetric situations indicate the existence of two mirror $\mathcal{N}=1$ supersymmetric $\mathrm{A}_1$ quiver gauge theories I and II with gauge couplings $\mathrm{g}_{N=1}^{(\mathrm{I})}$ and $\mathrm{g}_{N=1}^{(\mathrm{II})}$. Let us first study complex deformations of the previous $\mathcal{N}=2$ theory by introducing chiral superpotentials $W(\Phi)$. Later we consider how mirror Kähler superpotentials may be implemented.

## 3. $\mathcal{N}=1$ $\mathrm{A}_1$ quiver gauge theory I

This theory is obtained by performing complex deformations of the Lagrangian density $\mathcal{L}_{N=2}(A_1)$ Eq. (2.4). This is equivalent to introducing an extra chiral superpotential $W$ in the adjoint matter superfield and also in particular in the $U(1)$ factor $\Theta$ of adjoint matter $\Phi$ Eq. (2.2). In doing so, the Lagrangian density $\mathcal{L}_{N=2}(A_1)$ becomes

$$\mathcal{L}_{N=1}^{(\mathrm{I})}(A_1)=\mathcal{L}_{N=2}(A_1)+\left(\int d^2\theta\, W(\Theta)+\text{h.c.}\right). \tag{3.1}$$

In this relation, $\mathcal{N}=2$ supersymmetry is explicitly broken down to $\mathcal{N}=1$ due the presence of the nonlinear superpotential $W(\Theta)$; but $U(N)$ gauge invariance is still preserved. The superpotential $W(\Theta)$ generating complex deformations has two basic features which, in fact, are inter-related and play an important role at the quantum level: (i) the holomorphic property $\frac{\partial W(\Theta)}{\partial\Theta^*}=0$, which permits to benefit from the power of algebraic geometry and (ii) chirality

$$\int d^4\theta\, W(\Theta)=0 \tag{3.2}$$

allowing miraculous simplifications. Comparing the above Lagrangian density (3.1) with Eq. (2.4), one learns that complex deformation by the superpotential $W(\Theta)$ corresponds to promoting the previous complex FI type linear term with complex coupling constant $\beta$, namely $\beta\int d^2\theta\,(\Theta)$, to a more general chiral superfunction $\int d^2\theta\, W(\Theta)$. As a consequence $W'(\Theta)$ is no longer constant as in general it is $\Theta$ dependent. It follows then that the constant $\beta$ of Section 2 is now promoted to a $U(N)$ gauge invariant function $\mathcal{P}(\phi)$ as

$$\beta\to\mathcal{P}'(\phi)=\beta-W'(\phi), \tag{3.3}$$

where $W'(\phi)=\langle W'(\Theta)\rangle$. Moreover, putting the relation (3.3) back into the expression of the SYM gauge coupling g, one gets the following running $\mathcal{N}=1$ gauge coupling constant

$$\mathrm{g}_{N=1}(\phi)=\mathrm{g}_{N=2}(\zeta,\beta;\phi),\qquad \mathrm{g}_{N=1}^{(\mathrm{I})}(\phi)=\mathrm{g}_{N=2}(\zeta,\beta=0;\phi),\qquad \mathrm{g}_{N=1}^{(\mathrm{II})}(\phi)=\mathrm{g}_{N=2}(\zeta=0,\beta;\phi). \tag{3.4}$$

Note that $\mathcal{N}=2$ supersymmetry is recovered at the critical point $\phi_0$ of the superpotential; $W'(\phi_0)=0$, and so by expanding around this critical point, one may compute the deviations of the $\mathcal{N}=1$ gauge coupling from the $\mathcal{N}=2$ value.

$$\mathrm{g}_{N=1}(\phi)=\mathrm{g}_{N=2}(\beta)-\left(\phi W''\frac{\partial \mathrm{g}_{N=2}(\beta)}{\partial\beta}+\text{h.c.}\right)+O\big(\phi^2\big), \tag{3.5}$$

[1] Complex deformation of conifold singularity involves desingularisation moduli $\mu$ as $x^2+y^2+z^2=W'^2(\phi)+\mu$ which is required in geometric transition at large $N$ [18]. Here we have only $W'^2(\phi)$ (deformed $\mathrm{A}_1$ moduli); but no $\mu=0$.

where we have set $\phi_0 = 0$. The leading term of the $\phi$ expansion of $g_{N=1}(\phi)$ is $g_{N=2}(\beta)$ and the next one depends on $W''$. For free massless adjoint matter $g_{N=1}(\phi) = g_{N=2}(\beta)$ up to second order of $\phi$ expansion. Setting $\beta = 0$, one gets the variation of the coupling constant $g^{(\mathrm{I})}_{N=1}$ around the value of the $\mathcal{N} = 2$ one. Moreover, as the real coupling constant $\zeta$ has been untouched by the extension Eq. (3.1), it follows then that the defining equations of the moduli space of this $\mathcal{N} = 1$ supersymmetric quiver gauge theory reads as

$$\zeta = r_1 - r_2, \qquad \mathcal{P}'(\phi) = \beta - W'(\phi) = t_1(\phi) - t_2(\phi). \tag{3.6}$$

One of the special features of this expression is that under complex deformation Eq. (2.8) becomes

$$x^2 + y^2 + z^2 = \left(\mathcal{P}'(\phi)\right)^2, \tag{3.7}$$

showing that the CY3 is indeed a complex deformed $A_1$ surface fibered on the plane parameterized by the complex variable $\phi$. Furthermore, using the relation (3.6) and comparing with Eq. (2.7), it is not difficult to see that the superpotential of the adjoint matter considered above is in fact linked to CY3 complex moduli space as follows

$$W(\phi) = \beta\phi - \int_{S^2 \times J} \Omega, \tag{3.8}$$

where $\Omega = \omega \wedge d\tau$ is a (3, 0)-form on CY3 realized by an $A_1$ fiber on the complex plane and where one recognizes the FI terms $\beta\phi$. Such analysis extends straightforwardly to all ADE fibered CY3; with both finite and affine ADE geometries. This aspect and other features will be exposed in [14].

## 4. $\mathcal{N} = 1$ $A_1$ quiver gauge theory II

Applying mirror symmetry ideas to the above $\mathcal{N} = 1$ $A_1$ quiver gauge theory I, one expects to be able to build its superfield theoretical dual by starting from the Lagrangian density $\mathcal{L}_{N=2}(A_1)$ (Eq. (2.4)) and use Kähler deformations as

$$\mathcal{L}^{(\mathrm{II})}_{N=1}(A_1) = \mathcal{L}_{N=2}(A_1) + \delta_{\text{Kähler}}\mathcal{L}. \tag{4.1}$$

In superspace, $\delta_{\text{Kähler}}\mathcal{L}$ involves integration over the full superspace measure and reads as

$$\delta_{\text{Kähler}}\mathcal{L} = \int d^4\theta\, \mathcal{K}, \tag{4.2}$$

where $\mathcal{K}$ is a Kähler superpotential; that is some real superfunction we still have to specify. In what follows, we show that $\mathcal{K}$ has much to do with massive gauge superfields.

### *4.1. Massive gauge prepotential*

Although natural from geometric point of view due to mirror symmetry exchanging Kähler and complex deformations of CY3 [16], the superfield theoretical formulation of the dual theory II is far from obvious. The point is that in the derivation of $\mathcal{N} = 1$ quiver gauge theories I, the promotion of $\beta$ to chiral superpotentials $W(\phi)$ uses the scalar moduli of adjoint matter $\Theta$. However, for the Kähler deformations we are interested in here, one cannot use $\Theta$ by deforming the kinetic energy density $\int d^4\theta\, (\Theta^*\Theta)$ to

$$\int d^4\theta\, \mathcal{K}_{\text{adj}}\left(\Theta^*\Theta\right), \tag{4.3}$$

where $\mathcal{K}_{\text{adj}}(\Theta^*\Theta)$ is a Kähler superpotential for adjoint matter. A field theoretical reason for this is that $\Theta$ does not couple to the Abelian $U(1)$ gauge prepotential of the $U(N)$ gauge symmetry. The introduction of Kähler deformations for the $Q_\pm$ and $P_\pm$ fundamental matters

$$\int d^4\theta\, \mathcal{K}_{\text{fund}}\big(Q_\pm^* e^{\pm 2V} Q_\pm + P_\pm^* e^{\pm 2V} P_\pm\big) \tag{4.4}$$

does not solve the problem any more since this leads essentially to quite similar relations to Eqs. (2.5)–(2.7). The adjunction of superpotentials for fundamental matters does not work as well because it breaks $SU(N)$ gauge symmetry down to subgroups and this is ruled out by the $\mathrm{A}_1$ fibered CY3 we are considering here. However, there is still an issue since a careful analysis for the Kähler analogue of the chiral superpotential of complex deformations of theory I reveals that the difficulty we encounter in theory II is not a technical one. It is linked to the fact that in 4D $\mathcal{N}=1$ supersymmetric gauge theory II, the $\mathcal{N}=1$ massless gauge multiplet $(\frac{1}{2}, 1)$ has no scalar moduli that could play the role of the coordinate of the complex one dimension base of CY3. This is then a serious problem; but fortunately not a basic one since it may be overcome by considering massive $\mathcal{N}=1$ gauge multiplets $U^{(\text{mass})}$,

$$U^{(\text{mass})} \sim \left(0, \frac{1}{2}^2, 1\right)_M, \tag{4.5}$$

which have scalars contrary to massless gauge prepotentials. But how may this issue be implemented in the original $\mathcal{N}=2$ supersymmetric quiver gauge theory we started with? The answer is by spontaneously breaking the Abelian gauge sub-invariance as $U(N) \to SU(N)$. For general ADE geometries, the spontaneous breaking of the quiver gauge symmetry should be as $\prod_i U(N_i) \to \prod_i SU(N_i)$. Using this result, one still has to overcome the two following apparent difficulties.

### 4.2. *Two more things*

(1) From geometric point of view, we know that the variable $\tau$ parameterizing the complex one dimension base (plane) of the CY3 is associated with the complex scalar modulus of the adjoint matter multiplet $\Phi$ as shown on $(0^2, \frac{1}{2})$,

$$\tau \quad \leftrightarrow \quad \langle \operatorname{Tr} \Phi \rangle = \langle \Theta \rangle = \phi. \tag{4.6}$$

In the case of $\mathcal{N}=1$ massive gauge multiplets $U^{(\text{mass})}$, one has only one scalar modulus and it is legitimate to ask from where does come the lacking scalar? This is a crucial question since one needs one more scalar to be able to parameterize the two-dimensional base of CY3. The answer to this question is natural in massive $\mathrm{QFT}_4$; the missing scalar degree is, in fact, hidden in the $\mathcal{N}=1$ on shell massive gauge representation; it is just the longitudinal degree of freedom of the massive spin one particle $A_\mu$. This a good point in the right direction; but we still need to know how to extract this hidden scalar. The right answer to this technical difficulty follows from a remarkable feature of $\mathcal{N}=1$ supersymmetric theory which requires complex manifolds [15]. In the language of supersymmetric field theoretical representations, the real scalar $c$ appearing in $(0, \frac{1}{2}^2, 1)_M$ should, in fact, be thought of as the real part of a complex field $h$ as $c \sim h + h^*$ where now $h$ is the scalar component of chiral (Higgs) superfield,

$$H = h + \theta\psi + \theta^2 F, \tag{4.7}$$

which one suspects justly to be the right modulus for parameterizing the base of CY3.

(2) The second thing concerns the way to implement the massive vector multiplet into a $\mathcal{N}=2$ supersymmetric quiver gauge theory we started with. The answer is to think about the $\mathcal{N}=1$ massive gauge multiplet $(0, \frac{1}{2}^2, 1)_M$ as itself following from the decomposition of a $\mathcal{N}=2$ massive gauge multiplet $(0^5, \frac{1}{2}^4, 1)_M$ as shown on the following decomposition

$$\left(0^5, \frac{1}{2}^4, 1\right)_M \to \left(0, \frac{1}{2}^2, 1\right)_M \oplus \left(0^2, \frac{1}{2}\right) \oplus \left(0^2, \frac{1}{2}\right), \tag{4.8}$$

where $(0^2, \frac{1}{2}) \oplus (0^2, \frac{1}{2})$ are two chiral multiplets. Moreover, as the $\mathcal{N}=1$ massive gauge multiplet $(0, \frac{1}{2}^2, 1)_M$ may also be decomposed as the sum of an $\mathcal{N}=1$ massless gauge multiplet and an $\mathcal{N}=1$ chiral superfield, one then ends up with the following spectrum: (a) a massless Abelian gauge prepotential $U$ and (b) three chiral multiplets $H_{0,\pm}$ as shown here below

$$\left(0^5, \frac{1}{2}^4, 1\right)_M \to \left(\frac{1}{2}, 1\right) \oplus \left(0^2, \frac{1}{2}\right)_+ \oplus \left(0^2, \frac{1}{2}\right)_0 \oplus \left(0^2, \frac{1}{2}\right)_-, \tag{4.9}$$

where the charges $0, \pm 1$ appearing at the bottom of the matter multiplets refer to charges under the Abelian gauge factor of the $U(N)$ gauge symmetry.

### 4.3. The $N=1$ quiver gauge model II

This supersymmetric model involves the following $\mathcal{N}=1$ degrees of freedom: (a) A $U(N)$ gauge multiplet $V$ which has an Abelian part $U$ as in Eq. (2.1) and an $SU(N)$ part $V_a = \mathrm{Tr}(T_a V)$. (b) A chiral multiplet $\Phi$ in the adjoint representation of the gauge group $U(N)$. The Abelian part $\Theta$ of this adjoint matter is identified with the neutral superfield appearing in the decomposition (4.9). The non-Abelian term is given by the set $\Phi_a = \mathrm{Tr}(T_a \Phi)$. (c) Four chiral matter superfields $Q_\pm$ and $P_\pm$ transforming in the fundamental representations of the $U(1) \times SU(N)$ gauge symmetry as in Eq. (2.3). All these superfields exist already in the original $\mathcal{N}=2$ model we have described in Section 2. (d) Two more chiral multiplets $H_\pm$ carrying $\pm 2$ charges under the Abelian symmetry of the gauge group and transform as scalars with respect to $SU(N)$. The $H_\pm$ superfields are associated with the multiplets $(0^2, \frac{1}{2})_\pm$ appearing in the decomposition Eq. (4.9). In summary, we have the following $\mathcal{N}=1$ superfield spectrum: (i) the quartet

$$U, \quad H_0 \equiv \Theta, \quad H_+, \quad H_-, \tag{4.10}$$

which describe the degrees of freedom Abelian massive $\mathcal{N}=2$ multiplet $(0^5, \frac{1}{2}^4, 1)_M$ Eq. (4.9). The chiral multiplets should be thought of as Higgs superfields and whose Kähler superpotential

$$\int d^4\theta \, \mathcal{K}_{\mathrm{Higgs}}\big(H_+^* e^{2U} H_+ + H_-^* e^{-2U} H_-\big) \tag{4.11}$$

is exactly what we need; (ii) the $SU(N)$ massless $\mathcal{N}=2$ vector multiplet which in terms of the $\mathcal{N}=1$ superfield language we are using here reads as $V_a$ and $\Phi_a$; and (iii) finally the two $\mathcal{N}=2$ hypermultiplets $Q_\pm$ and $P_\pm$ describing fundamental matters. From this supersymmetric representation analysis, one learns that dynamics of massive $\mathcal{N}=2$ vector multiplet may be formulated in $\mathcal{N}=1$ superspace by starting with a massless vector multiplet $U$ and three chiral ones $H_{0,\pm}$ as introduced before. To get a massive gauge superfield, one gives non-trivial vevs to $H_\pm$; a fact which is achieved by introducing a superpotential $\mathcal{W}_{\mathrm{ext}}(H_+, H_0, H_-)$ describing couplings between chiral superfields. Since we are interested by the engineering of $\mathcal{N}=1$ quiver gauge theory using Kähler deformations, we will not insist on having $\mathcal{N}=2$ supersymmetric couplings for Higgs superfields. So we restrict the extra superpotential to $\mathcal{W}_{\mathrm{ext}} = \mathcal{W}_{\mathrm{ext}}(H_+, H_-)$ with the two following requirements: ($\alpha$) the full scalar potential $\mathcal{V}$ of the supersymmetric gauge Abelian model namely $\mathcal{V} = \frac{1}{2} D_U^2 + F_+ F_+^* + F_0 F_0^* + F_- F_-^*$ vanishes in the vacuum ($D_U = F_{0,\pm} = 0$) and ($\beta$) at least one of the chiral superfields $H_\pm$ acquires a vev when minimising $\mathcal{V}$ ($\frac{\partial \mathcal{V}}{\partial h_\pm} = 0$). Let us take these vevs as

$$\langle H_+ \rangle = \upsilon, \qquad \langle H_- \rangle = 0, \tag{4.12}$$

where $\upsilon$ is a complex parameter. A simple candidate for gauge invariant Higgs superpotential fulfilling features ($\alpha$) and ($\beta$) is $\mathcal{W}_{\mathrm{ext}} = m H_+ H_-$ with mass $m$ linked to $\zeta$ and $\upsilon$; i.e., $m = m(\zeta, \upsilon)$. With this in mind one can go ahead to work out the Kähler deformation program. In what follows, we describe the main lines and omit details.

### 4.4. *The action for $N=1$ quiver theory II*

From the above discussion, it follows that the Lagrangian density $\mathcal{L}_{N=1}^{(\mathrm{II})}(A_1)=\mathcal{L}_{N=2}(A_1)+\delta_{\text{Kähler}}\mathcal{L}$ (Eq. (4.1)) of the $\mathcal{N}=1$ supersymmetric quiver model II is given by the following superfunctional

$$\mathcal{L}_{N=1}^{(\mathrm{II})}(A_1)=\mathcal{L}_{N=2}(A_1)-\left(\int d^2\theta\, \mathcal{W}_{\text{ext}}(H_+,H_-)+\text{h.c.}\right)+\int d^4\theta\left[\mathcal{K}\big(H_+^* e^U H_+\big)+H_-^* e^{-U} H_-\right]. \tag{4.13}$$

In this relation we have endowed the matter superfield $H_+$ with a Kähler potential $\mathcal{K}(H_+^* e^U H_+)$ and left $H_-$ with a flat geometry. The introduction of a Kähler potential for $H_-$ does add nothing new since it is $H_+$ that is eaten by the gauge prepotential after symmetry breaking. $\mathcal{K}(H_+^* e^U H_+)$ is then crucial in the derivation of $\mathcal{N}=1$ quiver theories II; it is the mirror of $W(\Phi)$ of $\mathcal{N}=1$ quiver gauge theories I.

## 5. More results

In the Lagrangian density $\mathcal{L}_{N=2}(A_1)$ (Eq. (2.4)) of the $\mathcal{N}=2$ quiver theory, Kähler deformations are encoded in $2\zeta\int d^4\theta\,(U)$. This term should appear as a particular Kähler deformation in the $\mathcal{N}=1$ supersymmetric quiver theory II encoded in the term $\int d^4\theta\, \mathcal{K}(H_+^* e^U H_+)$. Choosing $\mathcal{K}$ as follows

$$\mathcal{K}_{\text{FI}}\big(H_+^* e^U H_+\big)=2\zeta\ln\big(H_+^* e^U H_+\big), \tag{5.1}$$

one recovers FI deformation; thanks to chirality $\int d^4\theta\,(H_+)=0$. Therefore Kähler deformations $\mathcal{R}$ that are mirror to the chiral potential $\mathcal{P}(\Theta)=\beta\Phi-W(\Theta)$ we have used in Eq. (3.3) read in general as

$$\mathcal{R}(\mathrm{Y})=2\zeta\ln(\mathrm{Y})-\mathcal{K}(\mathrm{Y}), \tag{5.2}$$

where $\mathrm{Y}=H_+^* e^U H_+$. In this result similarity between Kähler and complex deformation is perfect. It is a consequence of mirror symmetry in this super QFT and may also be rederived from the analysis of the Lagrangian density (4.13). The appearance of this composite Hermitian superfield Y is not fortuitous; it is just a manifestation of the massive gauge prepotential we have discussed before. Indeed parameterizing $H_+$ as

$$H_+=\upsilon\exp\left(\frac{H}{\upsilon}\right), \tag{5.3}$$

where now $H$ describes quantum fluctuation, we have for Y

$$\mathrm{Y}=\upsilon\upsilon^*\exp\left(\frac{\upsilon^* H+\upsilon H^*}{\upsilon\upsilon^*}+U\right). \tag{5.4}$$

But the term $\frac{\upsilon^* H+\upsilon H^*}{\upsilon\upsilon^*}+U$ in the exponential is nothing but the massive gauge prepotential $U^{(\text{mass})}$ of Eq. (4.5). Eqs. (5.4) and (5.2) give actually the relation between massive gauge multiplet and Kähler deformations. Moreover, the defining equations for the moduli space of the supersymmetric vacua of Kähler deformations in $\mathcal{N}=1$ quiver theories II following from (4.13) reading as

$$\mathcal{R}'(c)=(r_1-r_2), \tag{5.5}$$

where $c=\frac{\upsilon^* h+\upsilon h^*}{\upsilon\upsilon^*}$ and where $\mathcal{R}(c)=\mathcal{R}[\mathrm{y}(c)]$ and $\mathrm{y}(c)=\upsilon\upsilon^*\exp(c)$ are as follows

$$\mathcal{R}(\mathrm{y})=\big[\zeta\ln\mathrm{y}-\mathcal{K}(\mathrm{y})\big],\qquad \mathrm{y}(c)=\left[\upsilon\exp\left(\frac{h}{\upsilon}\right)\right]\left[\upsilon^*\exp\left(\frac{h^*}{\upsilon^*}\right)\right]\equiv w\bar{w}. \tag{5.6}$$

Eq. (5.5) shows that the blown sphere depends on the coordinate of the base of CY3. Like before, $\mathcal{N}=2$ supersymmetry is explicitly broken down to $\mathcal{N}=1$ except at the critical point $c_0$ of $\mathcal{R}(c)$ where it is recovered; but

$U(N)$ gauge invariance is spontaneously broken down to $SU(N)$. In terms of the quantum fluctuation superfields $H$ and $H^*$ Eq. (5.3), the critical point $\mathcal{R}'(c_0)=0$ is translated to

$$\left(\bar{\upsilon}\frac{\partial\mathcal{R}(H_0,H_0^*)}{\partial H_0}+\upsilon\frac{\partial\mathcal{R}(H_0,H_0^*)}{\partial H_0^*}\right)=0. \tag{5.7}$$

This relation should be thought of as the analogue of $\frac{\partial W}{\partial\Phi^*}=0$ in complex deformations. One can also compute the variation of the $\mathcal{N}=1$ running gauge coupling $\mathrm{g}_{N=1}(c)=\mathrm{g}_{N=1}(\zeta,\beta;c)$ around the value of the $\mathcal{N}=2$ one $\mathrm{g}_{N=2}(c_0)$ living at the critical point $\mathcal{K}'(c_0)=0$. One finds, for a generic point on the $\mathcal{N}=2$ supersymmetric flow $\mathrm{g}=\mathrm{g}(\vartheta)$, the following dual formula to Eq. (3.5)

$$\mathrm{g}_{N=1}[c]=\mathrm{g}_{N=2}(\zeta,\beta)-(c-c_0)\mathcal{K}''(\mathrm{c}_0)\frac{\partial\mathrm{g}_{N=1}(\zeta)}{\partial c_0}+O\big[(c-c_0)^2\big]. \tag{5.8}$$

Note by the way that one may also work out the mirror of Eqs. (3.7) and (3.8). Splitting $x$, $y$ and $z$ as $x=x_1+ix_2$ and so on, one may decompose the complex surface $x^2+y^2+z^2=0$ into a compact part $x_1^2+y_1^2+z_1^2=0$ and a non-compact one. Deformations of compact part as $x_1^2+y_1^2+(z_1-\Delta r)(z_1+\Delta r)=0$ and substituting $\Delta r$ as in Eq. (5.5), one gets the real analogue of Eq. (3.8), namely

$$x^2+y^2+z^2=\big(\mathcal{R}'(c)\big)^2. \tag{5.9}$$

Geometrically, this means that $\mathcal{R}(c)$ generates Kähler deformations of the CY3 and one can check that $\mathcal{R}(c)$ is given by the following

$$\mathcal{R}(c)=\zeta c+\int_{S^2\times J}\mathcal{K}^{(2,1)}+\int_{S^2\times\bar{J}}\mathcal{K}^{(1,2)}, \tag{5.10}$$

where $\mathcal{K}^{(2,1)}$ and $\mathcal{K}^{(1,2)}$ are, respectively, $(2,1)$ and $(1,2)$ forms on CY3 and where one recognizes the usual FI term $\zeta c$ of the $\mathcal{N}=1$ Abelian gauge theories. The correspondence between the two theories is then perfect.

## 6. Conclusion

In this Letter, we have developed the field theoretic analysis of deformations of 4D $\mathcal{N}=2$ quiver gauge theories living in D5-branes wrapped on $\mathrm{A}_1$ fibered CY3. Though it looks natural by using algebraic geometry methods and mirror symmetry exchanging complex and Kähler moduli, such study is far from obvious on the field theoretical side. After noting that the gauge coupling constant $\mathrm{g}_{N=2}$ of such a theory is given by a spectral flow

$$\mathrm{g}_{N=2}=\mathrm{g}_{N=2}(\vartheta),\qquad \tan\vartheta=\frac{|\beta|}{\zeta},\quad 0\leqslant\vartheta\leqslant\frac{\pi}{2}, \tag{6.1}$$

with $\mathrm{g}_{N=2}(0)$ and $\mathrm{g}_{N=2}(\frac{\pi}{2})$, respectively, associated with pure Kähler and pure complex deformations in the $\mathrm{A}_1$ fiber, we have considered deformations in the full moduli space of CY3. For complex deformations, geometry implies that we have the following: (a) if deformations are restricted to the ADE fibers, then $\mathcal{N}=2$ supersymmetry is preserved, up to a global shift of energy and (b) if they cover the full CY3, then $\mathcal{N}=2$ supersymmetry is broken down to $\mathcal{N}=1$. Mirror symmetry implies that similar results are also valid for Kähler deformations. On the superfield theoretical view, this corresponds to adding appropriate superpotential (complex and Kähler) terms in the original $\mathcal{N}=2$ $\mathrm{SYM}_4$. We have studied complex deformations of $\mathcal{N}=2$ supersymmetric quiver theories by using the method of [8] and given amongst others the field expansion of the $\mathcal{N}=1$ running gauge coupling constant $\mathrm{g}_{N=1}$ around $\mathrm{g}_{N=2}$. We have also developed the explicit analysis for Kähler deformations of $\mathcal{N}=2$ supersymmetric quiver theories and shown that such real deformations require massive gauge prepotentials $U^{(\mathrm{mass})}$ implying in turn a spontaneously broken $U(N)$ gauge symmetry down to $SU(N)$. We have worked out this program explicitly and shown amongst others that Kähler deformations are given by the following

$$\delta_{\text{Kähler}}\mathcal{L}_{\mathcal{N}=2} = \int d^4\theta\, \mathcal{R}\big(U^{(\text{mass})}\big), \tag{6.2}$$

where $\mathcal{R}(U^{(\text{mass})})$ is as in Eq. (5.6). This relation, which generalize naturally for all ADE fibered CY3, should be compared with the usual complex deformation involving the chiral superpotential of adjoint matter,

$$\delta_{\text{complex}}\mathcal{L}_{\mathcal{N}=2} = \int d^4\theta\, \mathcal{P}(\Phi) \tag{6.3}$$

with $\mathcal{P}(\Phi)$ as in Eq. (3.3). The analysis we have developed in this Letter has the remarkable property of being explicit. It allows superfield realizations of geometric properties of CY3 and offers a powerful method to deal with 4D $\mathcal{N}=1$ supersymmetric field theories living on wrapped D5. Through this explicit field theoretic study, one also learns that, on the $\mathcal{N}=1$ supersymmetric field theoretical side, mirror symmetry acts by exchanging the roles of adjoint matters $\Phi$ and massive gauge prepotentials $U^{(\text{mass})}$. On the geometric side, we have shown that Kähler deformations, generated by the real superfield $\mathcal{R}(U^{(\text{mass})})$, are given by the real part of the integral of a $(2,1)$ form on CY3 as shown on Eq. (5.10). This analysis may be also extended to incorporate D3-branes by considering affine ADE symmetries. Details on aspects of this study as well as other issues may be found in [14].

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