Monotone and Numerical-Analytic Methods for Differential Equations

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Abstract—The method of lower and upper solutions combined with monotone iterative techniques is used for ordinary differential equations with integral boundary conditions showing the existence of extremal solutions. Some existence results are also formulated using the numerical-analytic method. Sufficient conditions for existence are given. © 2003 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

In this paper, we shall consider the following differential problem:

\[ x'(t) = f\left(t, x(t), \int_0^T k(s)x(s)\, ds\right) \equiv Fx(t), \quad t \in J = [0, T], \quad T > 0, \]

\[ x(0) = \lambda x(T) + \int_0^T D(s)x(s)\, ds + d \equiv Bx, \quad d \in \mathbb{R}, \]

where \( f \in C(J \times \mathbb{R}^2, \mathbb{R}) \), \( k, D \in C(J, \mathbb{R}) \), and \( \lambda \in \mathbb{R} \).

It is well known that the monotone iterative technique is a powerful method used to approximate solutions of several problems (for details, see [1]; see also [2–4]). The purpose of this paper is to show that it can be applied successfully for problems with integral boundary conditions of type (1). We use this technique assuming that \( f(t, x, y) \) satisfies the one-sided Lipschitz condition in \( x \) and it is nondecreasing with respect to \( y \). Then problem (1) has extremal solutions for a parameter \( \lambda \geq 0 \). Note that the boundary condition from (1) contains, as special cases, periodic boundary conditions and initial conditions too.

The numerical-analytic method is applicable for differential problems with boundary conditions (for details, see [5,6]; see also [7–9]). We use this technique successfully for the integrodifferential problems of Fredholm type. Some existence results are obtained if we assume that \( f \) satisfies the Lipschitz condition with respect to the last two variables with corresponding constants. The
Banach fixed-point theorem is used to show that a comparison integral equation has a unique solution.

2. THE MONOTONE METHOD

A function $u \in C^1(J, \mathbb{R})$ is said to be a lower solution of problem (1) if

$$u'(t) \leq Fu(t), \quad t \in J,$$
$$u(0) \leq Bu,$$

and an upper solution of (1) if the inequalities are reversed.

Functions $\rho, \lambda \in C^1(J, \mathbb{R})$ are called minimal and maximal solutions of problem (1), respectively, if every solution $x \in C^1(J, \mathbb{R})$ of (1) satisfies the relation $\rho(t) \leq x(t) \leq \lambda(t), \ t \in J$. If both minimal and maximal solutions exist, we call them the extremal solutions of (1).

Let $\Omega = \{u : y_0(t) \leq u \leq z_0(t), \ t \in J\}$ and $\Delta = \{w \in C^1(J, \mathbb{R}) : y_0(t) \leq w(t) \leq z_0(t), \ t \in J\}$ be nonempty sets for some functions $y_0$ and $z_0$. The set $\Delta$ we call also the segment $[y_0, z_0]$.

We introduce the following assumptions for later use.

(H1) $f \in C(J \times \Omega \times \mathbb{R}, \mathbb{R}), \ k, D \in C(J, \mathbb{R}_+), \ X \geq 0$.

(H2) $y_0, z_0 \in C^1(J, \mathbb{R})$ are lower and upper solutions of (1), respectively, and such that $y_0(t) \leq z_0(t), \ t \in J$.

(H3) $f$ is nondecreasing in the third variable and there exists a constant $M > 0$ such that

$$f(t, u, v) - f(t, u, w) \leq M|v - w|$$

for $u \leq v, v, w \in \Omega, v \in \mathbb{R}, t \in J$.

**LEMMA 1.** Let Assumptions (H1) and (H2) hold. Assume that $u, v \in \Delta$ are lower and upper solutions of problem (1), respectively, and $u(t) \leq v(t)$ on $J$. If

$$y'(t) = Fu(t) - Mu(t), \quad t \in J, \quad y(0) = Bu,$$
$$z'(t) = Fu(t) - Mz(t), \quad t \in J, \quad z(0) = Bz,$$

then

$$u(t) \leq y(t) \leq z(t) \leq v(t), \quad t \in J,$$

and $y, z$ are lower and upper solutions of (1), respectively.

**PROOF.** Note that there exist unique solutions for $y$ and $z$. Put $p = u - y, q = z - v$, so $p(0) \leq Bu - Bu = 0, q(0) \leq Bv - Bv = 0,$ and

$$p'(t) = Fu(t) - Fu(t) + M[y(t) - u(t)] = -Mp(t), \quad t \in J,$$
$$q'(t) = Fu(t) - M[z(t) - v(t)] - Fu(t) = -Mq(t), \quad t \in J.$$

Hence, $p(t) \leq e^{-Mt}p(0) \leq 0, q(t) \leq e^{-Mt}q(0) \leq 0, \ t \in J$, showing that $u(t) \leq y(t), z(t) \leq v(t), \ t \in J$. Now let $p = y - z$, so $p(0) = Bu - Bv \leq 0$. Assumption (H3) yields

$$p'(t) = Fu(t) - Fu(t) - M[y(t) - u(t) - z(t) + v(t)] \leq -Mp(t), \quad t \in J.$$

Hence, $p(t) \leq 0, \ t \in J$, showing that $y(t) \leq z(t), \ t \in J$. It proves that (2) holds. Now we need to show that $y, z$ are lower and upper solutions of (1), respectively. Again using Assumption (H3), we see

$$y'(t) = Fu(t) - M[y(t) - u(t)] - Fy(t) + Fy(t) \leq Fy(t),$$
$$z'(t) = Fu(t) - M[z(t) - v(t)] - Fz(t) + Fz(t) \geq Fz(t),$$

for $t \in J$ and $y(0) = Bu \leq By, z(0) = Bz \geq Bz$. It shows that $y, z$ are lower and upper solutions of (1), respectively. This ends the proof.
THEOREM 1. Let Assumptions (H1)-(H3) hold. Then there exist monotone sequences \( \{y_n, z_n\} \) such that \( y_n(t) \to y(t), \quad z_n(t) \to z(t), \quad t \in J, \) as \( n \to \infty, \) and this convergence is uniform and monotone on \( J. \) Moreover, \( y, z \) are extremal solutions of (1) in \( \Delta. \)

PROOF. Let

\[
\begin{align*}
y'_{n+1}(t) &= Fy_n(t) - M[y_{n+1}(t) - y_n(t)], \quad t \in J, \quad y_{n+1}(0) = By_n, \\
z'_{n+1}(t) &= Fz_n(t) - M[z_{n+1}(t) - z_n(t)], \quad t \in J, \quad z_{n+1}(0) = Bz_n,
\end{align*}
\]

for \( n = 0, 1, \ldots \). Lemma 1 shows \( y_0(t) \leq y_1(t) \leq z_1(t) \leq z_0(t), \quad t \in J, \) and \( y_1, z_1 \) are lower and upper solutions of (1), respectively.

Assume that

\[
y_0(t) \leq y_1(t) \leq \cdots \leq y_k(t) \leq z_k(t) \leq \cdots \leq z_1(t) \leq z_0(t), \quad t \in J,
\]

for some \( k \geq 1 \) and let \( y_k, z_k \) be lower and upper solutions of (1), respectively. Then, again using Lemma 1, we get \( y_k(t) \leq y_{k+1}(t) \leq z_{k+1}(t) \leq z_k(t), \quad t \in J. \) By mathematical induction, we have

\[
y_0(t) \leq y_1(t) \leq \cdots \leq y_n(t) \leq z_n(t) \leq \cdots \leq z_1(t) \leq z_0(t), \quad t \in J,
\]

for all \( n. \) Hence \( y_n(t) \to y(t), \quad z_n(t) \to z(t), \quad t \in J, \) if \( n \to \infty. \) Indeed, \( y \) and \( z \) are solutions of problem (1).

To finish the proof, it is enough to show that \( y \) and \( z \) are extremal solutions of (1) in \( \Delta. \) To do it, we need to show that if \( w \) is any solution of (1) such that \( y_0(t) \leq w(t) \leq z_0(t), \quad t \in J, \) then \( y_0(t) \leq y(t) \leq w(t) \leq z(t) \leq z_0(t), \quad t \in J. \) Assume that for some \( k, \) \( y_k(t) \leq w(t) \leq z_k(t), \quad t \in J. \) Put \( p = y_{k+1} - w, \quad q = w - z_{k+1}. \) Then \( p(0) = By_k - Bw \leq 0, \quad q(0) = Bw - Bz_k \leq 0, \) and

\[
\begin{align*}
p'(t) &= Fy_k(t) - Fw(t) - M[y_{k+1}(t) - y_k(t)] \leq -Mp(t), \quad t \in J, \\
q'(t) &= Fw(t) - Fz_k(t) + M[z_{k+1}(t) - z_k(t)] \leq -Mq(t), \quad t \in J.
\end{align*}
\]

Hence, \( y_{k+1}(t) \leq w(t) \leq z_{k+1}(t), \quad t \in J. \) This proves, by mathematical induction, that \( y_n(t) \leq w(t) \leq z_n(t), \quad t \in J, \) for all \( n. \) Taking the limit \( n \to \infty, \) we get \( y(t) \leq w(t) \leq z(t) \) on \( J \) so the assertion of Theorem 1 is true. The proof is complete.

EXAMPLE. Consider the following problem:

\[
\begin{align*}
x'(t) &= e^{tsin^2 x(t)}, \quad t \in J = [0,T], \quad \text{with} \quad T = \ln 2, \\
x(0) &= \int_0^T x(s) \, ds.
\end{align*}
\]

Indeed, \( 0 < e^{tsin^2 x} \leq e^t, \quad t \in J, \quad x \in \mathbb{R}. \) Note that \( y_0(t) = 0, \quad z_0(t) = e^t \) on \( J \) are lower and upper solutions of problem (3), respectively. Moreover, \( M = 2 \ln 2. \) By Theorem 1, problem (3) has extremal solutions in the segment \([y_0, z_0].\)

REMARK 1. Put \( \lambda = 1, \quad d = 0, \quad D(t) = 0, \quad t \in J. \) Then \( Bx = x(T). \) Theorem 1 is better in comparison with Theorem 3.1 in [2] where an extra condition is required, namely \( \max_{t \in J} k(t) \leq c; \) the constant \( c \) depends on \( M \) and \( T. \) See also corresponding results of [3] for problems with impulses. Theorem 1 also generalizes Theorem 3.1 in [4].

REMARK 2. Assume that \( f \) does not depend on the third variable. Let \( d = 0, \quad D(t) = 0, \quad t \in J. \) In this case, we have periodic boundary problems or initial problems for differential equations, and Theorem 1 contains some results of [1].
3. THE NUMERICAL-ANALYTIC METHOD

Put

\[ LFx(t) = \left( 1 - \frac{t}{T} \right) \int_0^t Fx(s) ds + \frac{t}{T} \int_t^T Fx(s) ds, \]

\[ B_0 = \int_0^T sD(s) ds, \quad B_1 = \int_0^T D(s) ds, \]

\[ B_2 = (\lambda T + B_0)^{-1}, \quad B_3(\bar{x}_0) = -B_2 \left[ d + (B_1 - 1 + \lambda)\bar{x}_0 \right], \]

assuming that \( \lambda T + B_0 \neq 0 \). This assumption is connected with the numerical-analytic method (for details, see \([5,6]\)). According to this method, find the value of \( \delta \) such that \( x(t) = \bar{x}_0 + \int_0^t Fx(s) ds + \delta t, \ t \in J \) satisfies the boundary condition \( x(0) = Bx \). Note that \( \int_0^t Fx(s) ds = LFx(t) + (t/T) \int_0^T Fx(s) ds \), so problem (1) can be transformed to the following auxiliary equation:

\[ x(t) = \bar{x}_0 + LFx(t) - B_2t \int_0^T D(s)LFx(s) ds + tB_3(\bar{x}_0) \equiv G(t, x; \bar{x}_0), \quad t \in J. \tag{4} \]

Note that if \( x \) satisfies (4), then the boundary condition from (1) is satisfied too. Moreover, \( G(0, x; \bar{x}_0) = \bar{x}_0 \), so \( x(0) = \bar{x}_0 \).

Let us introduce the following assumptions.

(H4) There are constants \( K > 0, L \geq 0 \) such that

\[ |f(t, x, y) - f(t, \bar{x}, \bar{y})| \leq K |x - \bar{x}| + L |y - \bar{y}|, \]

for all \( t \in J, x, \bar{x}, y, \bar{y} \in \mathbb{R} \).

(H5) For any nonnegative function \( h \in C(J \times \mathbb{R}, \mathbb{R}_+) \), there exists a unique solution \( u \in C(J, \mathbb{R}_+) \) of the comparison equation

\[ \Omega u(t) + h(t, x_0) = u(t), \quad t \in J, \tag{5} \]

where

\[ \Omega u(t) = \left( 1 - \frac{t}{T} \right) \int_0^t \left[ Ku(s) ds + L \int_0^T |k(\tau)|u(\tau) d\tau \right] ds \]

\[ + \frac{t}{T} \int_t^T \left[ Ku(s) ds + L \int_0^T |k(\tau)|u(\tau) d\tau \right] ds, \]

\[ \Omega u(t) = \Omega_1 u(t) + B_2 t \int_0^T |D(s)|\Omega_1 u(s) ds. \]

Note that

\[ |LFx(t) - LF\bar{x}(t)| \leq \Omega_1 \ |x - \bar{x}| (t), \]

\[ |G(t, x; \bar{x}_0) - G(t, \bar{x}; \bar{x}_0)| \leq |LFx(t) - LF\bar{x}(t)| + |B_2 t \int_0^T |D(s)|LFx(s) - LF\bar{x}(s)| ds \]

\[ \leq \Omega_1 \ |x - \bar{x}| (t) + |B_2 t \int_0^T |D(s)|\Omega_1 \ |x - \bar{x}| (s) ds. \tag{6} \]

For \( t \in J, n = 0, 1, \ldots \), let us define the sequence \( \{u_n\} \) by

\[ u_{n+1}(t) = \Omega u_n(t), \quad u_0(t) = u(t), \]

where \( u \) is defined as in Assumption (H5) with \( h(t, \bar{x}_0) = |G(t, x_0; \bar{x}_0) - x_0(t)| \).

To obtain a solution of problem (4), we shall first establish some properties for sequence \( \{u_n\} \).
LEMMA 2. Let Assumptions (H₄) and (H₅) be satisfied and let λT + B₀ ≠ 0. Then
\[ u_{n+1}(t) = u_{n}(t) \leq u_{0}(t), \quad t \in J, \quad n = 0, 1, \ldots, \]
and the sequence \{u_{n}\} converges uniformly to zero function, so \( u_{n}(t) \to 0, \ t \in J, \) if \( n \to \infty. \)

PROOF. Note that \( u_{1}(t) = Ωu_{0}(t) \leq Ωu_{0}(t) + h(t, x_{0}) = u_{0}(t), \ t \in J. \) By induction in \( n, \) we are able to prove that \( u_{n+1}(t) \leq u_{n}(t), \ t \in J, \ n = 0, 1, \ldots. \) Now, if \( n \to \infty, \) then \( u_{n} \to u, \) where \( u \) is a solution of the equation \( u(t) = Ωu(t), \ t \in J. \) Hence \( u(t) = 0 \) on \( J, \) by Assumption (H₅). The proof is complete.

LEMMA 3. Assume that \( f \in C(J \times ℝ^{2}, ℝ), \) \( k, D \in C(J, ℝ), \) \( d, λ ∈ ℝ, \) and \( λT + B₀ ≠ 0. \) Let Assumptions (H₄) and (H₅) be satisfied. Then we have the estimates
\[ |x_{n}(t) - x_{0}(t)| \leq u_{0}(t), \quad t \in J, \]
\[ |x_{n+k}(t) - x_{k}(t)| \leq u_{k}(t), \quad t \in J, \]
where \( x_{0} \in C^{1}(J, ℝ), \) and \( x_{n+1}(t) = G(t, x_{n}; x_{0}), \ t \in J. \) Moreover,
\[ x_{n+1}(0) = λx_{n+1}(T) + \int_{0}^{T} D(s)x_{n+1}(s) \, ds + d, \quad n = 0, 1, \ldots \]

PROOF. Put \( R(t; x_{0}) = |G(t, x_{0}; x_{0}) - x_{0}(t)|, \ t \in J. \) Indeed, \( |x_{1}(t) - x_{0}(t)| = R(t; x_{0}) \leq u_{0}(t), \ t \in J. \) Assume that \( |x_{k}(t) - x_{0}(t)| \leq u_{0}(t), \ t \in J, \) for some \( k ≥ 0. \) Then, by (6), we have
\[ |x_{k+1}(t) - x_{0}(t)| \leq |G(t, x_{k}; x_{0}) - G(t, x_{0}; x_{0})| + R(t, x_{0}) \leq Ωu_{0}(t) + R(t, x_{0}) = u_{0}(t). \]

Hence, by mathematical induction, we have \( |x_{n}(t) - x_{0}(t)| \leq u_{0}(t), \ t \in J, \) for \( n = 0, 1, \ldots. \) Based on the above, let us assume that \( |x_{n+k}(t) - x_{k}(t)| \leq u_{k}(t), \ t \in J \) for all \( n \) and some \( k ≥ 0. \) Then, again using (6), we see that
\[ |x_{n+k+1}(t) - x_{k+1}(t)| = |G(t, x_{n+k}; x_{0}) - G(t, x_{k}; x_{0})| \leq σu_{k}(t) = u_{k+1}(t) \]
for \( t \in J. \) Hence, by mathematical induction, (7) holds. It is quite simple to verify that \( x_{n+1} \) satisfies (8) for any \( n = 0, 1, \ldots. \) It ends the proof.

Put \( Λ(x_{0}) = \{ x \in C^{1}(J, ℝ) : |x_{0}(t) - x(t)| \leq u_{0}(t), \ t \in J \}. \) Combining Lemmas 2 and 3, we have the following theorem.

THEOREM 2. Assume that \( f \in C(J \times ℝ^{2}, ℝ), \) \( k, D \in C(J, ℝ), \) \( d, λ ∈ ℝ, \) and \( λT + B₀ ≠ 0. \) Let Assumptions (H₄) and (H₅) be satisfied. Then, for every \( x_{0} ∈ ℝ, \) there exists a solution \( X \) of problem (4) where \( x_{n}(t) → X(t), \ t \in J, \) as \( n → \infty, \) and
\[ |x_{n}(t) - X(t)| ≤ u_{n}(t), \quad t \in J. \]
The function \( X \) is a unique solution of problem (4) in the class \( Λ(x_{0}). \)

Moreover, \( X \) is the solution of problem (1) iff
\[ \frac{1}{T} \int_{0}^{T} F(t, x_{0}) \, ds + B_{2} \int_{0}^{T} D(s)CF(t, x_{0}) \, ds = B_{3} (x_{0}). \]

PROOF. By Lemmas 2 and 3, \( x_{n}(t) → X(t), \ t \in J. \) Indeed, \( X \) is a solution of problem (4). We need to show the uniqueness of \( X. \) Assume that problem (4) has another solution \( X \) such that
\[ |X(t) - x_{0}(t)| \leq u_{0}(t) \] on \( J. \) Then, by (6), we have
\[ |X(t) - X(t)| \leq |X(t) - x_{n+1}(t)| + |G(t, x_{n}; x_{0}) - G(t, x; x_{0})| \]
\[ ≤ u_{n+1}(t) + Ω_{1} |x_{n} - X|(t) + B_{2} |t \int_{0}^{T} Ω_{1} |x_{n} - X|(s) \, ds, \]
for \( t \in J \). Hence, by mathematical induction, we have \( |\bar{x}(t) - X(t)| \leq 2u_{n+1}(t), \ t \in J, \ n = 0, 1, \ldots \) showing that \( \bar{x} = X \) on \( J \). It ends the proof.

**Remark 3.** Let \( \lambda T + B_0 \neq 0 \). Assumption (H₃) is satisfied if

\[
Z < 1, \quad \text{where} \quad Z = \left[ 1 + |B_2| T \int_0^T |D(s)| \, ds \right] \frac{T}{2} A, \quad A = K + L \int_0^T |k(s)| \, ds. \tag{9}
\]

To get condition (9), we need to apply the Banach fixed-point theorem to equation (5). Let \( u, \bar{u} \in C(J, \mathbb{R}_+) \). Then

\[
|\Omega u(t) - \Omega \bar{u}(t)| = \left| \Omega_1 u(t) - \Omega_1 \bar{u}(t) + |B_2| t \int_0^T |D(s)| \left| \Omega_1 u(s) - \Omega_1 \bar{u}(s) \right| \, ds \right| \leq Z \max_{t \in J} |u(t) - \bar{u}(t)|
\]

because

\[
|\Omega_1 u(t) - \Omega_1 \bar{u}(t)| \leq A \left[ \left( 1 - \frac{T}{T} \right) \int_0^t |u(s) - \bar{u}(s)| \, ds + \frac{T}{T} \int_t^T |u(s) - \bar{u}(s)| \, ds \right] \\
\leq 2A \left( 1 - \frac{T}{T} \right) t \max_{t \in J} |u(t) - \bar{u}(t)| \leq \frac{T}{2} A \max_{t \in J} |u(t) - \bar{u}(t)|.
\]

Hence, operator \( \Omega \) is a contraction mapping so problem (5) has a unique solution, by the Banach fixed-point theorem.

**REFERENCES**