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Realcompact subspaces of size ω_1

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Abstract

We show that every uncountable compact space has a realcompact subspace of size ω_1, that if there are no S-spaces, then every uncountable Tychonoff space has a realcompact subspace of size ω_1 and that Ostazewski's space has no uncountable realcompact subspace. © 2000 Elsevier Science B.V. All rights reserved.

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At the end of a seminar at Ohio University where Frank Tall had been talking about spaces with Lindelöf subspaces of size ω_1, Arhangel'skii asked him whether all uncountable compact spaces had a realcompact subspace of size ω_1. This paper will answer that question in the affirmative and will also consider the question of whether the compactness hypothesis is necessary, that is, whether every uncountable Tychonoff space has a realcompact subspace of size ω_1. The answer to this latter question is more complicated and, in fact, depends on the model of set theory in which we are working.

I wish to thank Professor Arhangel'skii for not only asking the question but also for many instructive conversations on this subject. In particular, I thank him for pointing out several references and for providing the proof of one of the theorems.

We shall, of course, assume that all spaces are Tychonoff. We denote the continuous real-valued functions on a space X by $C(X)$. We first give two sufficient conditions for an uncountable space to have a realcompact subspace of size ω_1.

Theorem 1. *If there is a subset $A \subset X$ and $f \in C(A)$ with $f^{\to}(A) \geqslant \omega_1$, then X has a realcompact subspace of size ω_1.*

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Proof. Let $A \subset X$ and let $f \in C(A)$ with $f^{\rightarrow}(A) \geqslant \omega_1$. Pick $B \subset f^{\rightarrow}(A)$ with $|B| = \omega_1$ and $R \subset A$ such that f is 1–1 on R. Then $f \restriction R : R \rightarrow B$ is a 1–1 continuous map from R onto B which implies that R is submetrizable (that is, R has a coarser metric topology); hence R is realcompact (see [3, 8.18]). □

Obviously many spaces satisfy the hypothesis of Theorem 1. Spaces with uncountable spread come immediately to mind where the *spread* of a topological space X is $\sup\{|D|\colon D$ is a discrete subspace of $X\}$. We will see in a bit that there are at least two classes of spaces that do not satisfy these hypotheses. An interesting question to me would be to find necessary and sufficient conditions for Tychonoff spaces to satisfy that hypothesis. Note, for example, that all spaces that do not are zero-dimensional.

We give next another sufficient condition for a space to have a realcompact subspace of size ω_1. The proof of this theorem was suggested by Prof. Arhangelskii. We remind the reader that a space X is *right separated* (*in type* β) if there is a well-ordering (of type β) on X with respect to which initial segments are open. All ordinal spaces, for example, are right separated.

Theorem 2. *If X is uncountable, right separated and if X has the property that every subspace of size ω_1 has a complete accumulation point in X, then X has a Lindelöf, hence realcompact, subspace of size ω_1.*

Proof. The proof is by induction on β where X is right separated in type β. So suppose the theorem is true for all $\delta < \beta$ and let $X = \{x_\xi\colon \xi < \beta\}$ be right separated in type β where β is uncountable. It is clear that $\beta > \omega_1$ for if $\beta = \omega_1$, then X itself has no complete accumulation point.

Let $A = \{x_\xi\colon \xi < \omega_1\}$. First we consider the case where A has more than one complete accumulation point. In this case we let x_γ be the one with the least index. Then clearly $\gamma < \beta$, and we can choose a neighborhood G of x_γ in X with $cl_X\, G \subset \{x_\xi\colon \xi \leqslant \gamma\}$ since X is right separated. Clearly since x_γ is a complete accumulation point of A, $|cl_X\, G| \geqslant \omega_1$. Also $cl_X\, G$ is right separated in type δ where $\omega_1 \leqslant \delta \leqslant \gamma < \beta$, and $cl_X\, G$ satisfies the hypothesis that every subset of size ω_1 has a complete accumulation point. Then by the induction hypothesis $cl_X\, G$ has a Lindelöf, hence realcompact, subset of size ω_1.

If, on the other hand, A has only one complete accumulation point, say, x_σ, then $A \cup \{x_\sigma\}$ is itself a Lindelöf subset of size ω_1. □

Recall that a space X is *scattered* if every subspace of X has isolated points. It is well known that scattered spaces are right separated. We sketch a brief proof of that fact for completeness.

Proposition 3. *Scattered spaces are right separated.*

Proof. Let X be scattered. Let X_0 be the isolated points of X and $X_{\alpha+1}$ be the isolated points of $X - X_\alpha$. For a limit ordinal γ, let X_γ be the isolated points of $X - \bigcup_{\beta<\gamma} X_\beta$.

There will be a smallest ordinal κ such that $X = \bigcup_{\beta<\kappa} X_\beta$. Now well order each X_β arbitrarily by $\prec_\beta$ and well order $X = \bigcup_{\beta<\kappa} X_\beta$ by $x < y$ if and only if $x \in X_\beta$, $y \in X_\alpha$ and $\beta < \alpha$ or $x, y \in X_\beta$ and $x \prec_\beta y$. This well ordering is a right separation of X. □

In [6], Pelezynski and Semadeni show that for a compact space X, X is scattered (they call scattered spaces *dispersed*) if and only if X does not map onto the unit interval. Thus we can now draw the following conclusion.

Theorem 4. *Every uncountable compact space has a realcompact subspace of cardinality ω_1.*

Proof. Let X be an uncountable compact space. If X does not satisfy the hypothesis of Theorem 1, then by [6], X is scattered, hence right separated, and so X satisfies the hypotheses of Theorem 2. □

We will show later that it is not possible to reduce the hypothesis of compactness very much, even in the presence of some powerful other properties.

Turning next to the noncompact case we will need to consider two different models of set theory. The first one we will look at is a model in which there are no S-spaces, that is, there are no hereditarily separable spaces that are not Lindelöf. Todorčević in [11] has constructed such a model of ZFC. We will use the following fact from [1].

Proposition 5 (See [1, Remark following Problem 4]). *The following are equivalent in* ZFC.

(1) *Every hereditarily separable space is hereditarily Lindelöf.* (*That is, there are no S-spaces.*)
(2) *Every space with countable spread is hereditarily Lindelöf.*

In [1] Arhangel'skii attributes this proposition to mathematical folklore. We will, however, sketch a proof for completeness.

Proof. We need only prove (1) ⇒ (2). Assume (1) and let X be a space that is not hereditarily Lindelöf. Then X contains a right separated subspace A such that no uncountable subset of A is Lindelöf. (See [4, 2.9(b)].) Now by (1) A is not hereditarily separable and so by [4, 2.9(c) and 2.12], A has an uncountable discrete subspace which implies that X does not have countable spread. □

This proposition easily gives us the next theorem.

Theorem 6 (There are no S-spaces). *Every uncountable Tychonoff space has a realcompact subspace of size ω_1.*

Proof. Let X be uncountable. If X has uncountable spread, then X satisfies the hypothesis of Theorem 1. Otherwise, X is hereditarily Lindelöf by Theorem 5, and hence every subspace is realcompact. □

There is, however, a model of set theory in which it is possible for an uncountable space to have no realcompact subspace of size ω_1. The following ZFC theorem gives us sufficient conditions for such a space.

Theorem 7. *If X is perfectly normal, right separated in type ω_1 and if closed subsets of X are either countable or co-countable, then X has no realcompact subset of size ω_1.*

Proof. Let X be perfectly normal and right separated in type ω_1. Suppose that $A \subset X$ with $|A| = \omega_1$, and A is realcompact. We will show that X cannot satisfy the cardinality hypothesis on closed subsets.

For each $\alpha < \omega_1$, let $F_\alpha = \{x_\xi\colon \xi > \alpha\}$. Since X is perfectly normal and right separated, each F_α is a zero-set of X. Hence there is a z-ultrafilter $\mathcal{F}$ on A with $\{A \cap F_\alpha\colon \alpha < \omega_1\} \subset \mathcal{F}$. Now $\mathcal{F}$ is free and A is realcompact, and so there is a collection $\{Z_i\colon i \in \omega\} \subset \mathcal{F}$ with $\bigcap_{i\in\omega} Z_i = \emptyset$. Since each member of A misses some Z_i and since A is uncountable, there is some $i \in \omega$ such that uncountably many members of A miss Z_i. But Z_i must meet each $A \cap F_\alpha$ as well. Now $Z_i = K \cap A$ for some K closed in X. Clearly K can be neither countable nor co-countable. □

In [5] Ostazewski constructed a space that satisfies the hypotheses of Theorem 7. The space is constructed with underlying set ω_1 using the set-theoretical principle $\diamond$. $\diamond$ is equivalent to the Continuum Hypothesis plus a combinatorial principle called $\clubsuit$ and is known to be consistent with ZFC, and in fact, holds in Gödel's constructible universe. (See [8, IV].) Thus we have proved the next theorem.

Theorem 8 ($\diamond$). *There is an uncountable Tychonoff space with no realcompact subspace of size ω_1.*

We recall that Ostaszewski's space, in addition to satisfying the hypotheses of Theorem 5, is hereditarily separable, locally compact, locally countable, scattered, and almost compact. Thus we cannot reduce the compactness hypothesis in Theorem 3 to almost compactness, even in the presence of these other properties.

We also know, from Theorems 1 and 8, that no uncountable subspace of Ostaszewski's space has a continuous real-valued function on it with uncountable range, but we can actually say more. We say that a real-valued function f defined on a space $X = \{x_\xi\colon \xi < \kappa\}$ for some ordinal κ is *eventually constant* if there is $\alpha < \kappa$ and a real number z such that $f(x_\xi) = z$ for all $\xi \geqslant \alpha$. We say that a subset $A \subset X$ is *z-embedded* in X if every zero-set of A is the trace on A of a zero-set of X. All subsets of a perfectly normal space are z-embedded, and if A is z-embedded in X, then $\upsilon A \subset \upsilon X$. See [2] for a discussion of these matters.

Theorem 9. *Let X be Ostaszewski's space and let A be an uncountable subset of X. Then every real-valued continuous function on A is eventually constant.*

Proof. Let $f \in C(A)$. Since X is perfectly normal, A is z-embedded in X and so $\upsilon A \subset \upsilon X = X \cup \{\omega_1\}$ (see [2, 4.1]). Now no uncountable subset of X is realcompact and so $\omega_1 \in \upsilon A$. Let F be the extension of f to υA and suppose that $F(\omega_1) = r$. It is easy to see that $f = r$ on some final segment of A. □

There are not many known examples of perfectly normal spaces of nonmeasurable cardinality that are not realcompact. It was a conjecture of the late Robert L. Blair that it is consistent with ZFC that every perfectly normal space of small enough cardinality is realcompact. See [2] for a discussion of this question. In particular Blair conjectured that this was true under MA + ¬CH. Some partial results in this direction have been obtained. In [9], it was shown that MA + ¬CH implies that regular spaces of nonmeasurable cardinality in which closed sets have countable character are realcompact. In [10] it was shown that MA + ¬CH implies that every perfectly normal space is p-realcompact (= every zero-set of βX that meets $\beta X - X$ meets $\beta X - \upsilon X$). But the general question as far as this author knows, remains unsolved.

There are several other questions suggested by this work.

Question 1. What are necessary and sufficient conditions for an uncountable space to have the property that all real-valued continuous functions on any of its subsets have countable range, that is, to fail to satisfy the hypothesis of Theorem 1? We know by [6] that if such a space is compact, then it is scattered, but, in fact, we show in the next result that if the space is pseudocompact, then it must be scattered as well.

Theorem 10. *If for all $f \in C(X)$, $|f^{\rightarrow}(X)| < 2^{\omega}$ and if X is pseudocompact, then X is scattered.*

Proof. Suppose that for all $f \in C(X)$, $|f^{\rightarrow}(X)| < 2^{\omega}$ and that X is not scattered. Then βX is not scattered and so by [6], there is $g \in C(\beta X)$ such that $|g^{\rightarrow}(\beta X)| = 2^{\omega}$. Now by the hypothesis on X, there is $r \in g^{\rightarrow}(\beta X) - g^{\rightarrow}(X)$. Then $g^{\leftarrow}\{r\}$ is a zero-set of βX that misses X and so X is not pseudocompact (see [3, 6I(1)]). □

Question 2. Is it consistent with ZFC that every uncountable perfectly normal space has a realcompact subspace of size ω_1?

Question 3. Does every uncountable realcompact space have a realcompact subspace of size ω_1?

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