Uniqueness of traveling wave solutions
for a biological reaction–diffusion equation

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Abstract

The existence of traveling wave solutions for a reaction–diffusion, which serves as models for microbial growth in a flow reactor and for mathematical epidemiology, was previously confirmed. However, the problem on the uniqueness of traveling wave solutions remains open. In this paper we give a complete proof of the uniqueness of traveling wave solutions.

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1. Introduction

In this paper we investigate the uniqueness of traveling wave solutions for a system of reaction–diffusion equations

\[
\frac{\partial S}{\partial t} = \rho \frac{\partial^2 S}{\partial x^2} - \alpha \frac{\partial S}{\partial x} - f(S)P,
\]

\[
\frac{\partial P}{\partial t} = d \frac{\partial^2 P}{\partial x^2} - \alpha \frac{\partial P}{\partial x} + \left[ f(S) - K \right] P,
\]

(1.1)

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that has served in recent years as a mathematical model to study some problems in biology and chemical reaction. For example, Eq. (1.1) with $\alpha = 0$ was introduced in [8] to study a population model with diffusion. [4,5] used this system, with $\alpha = 0$ and $f(S) = S$, as a simple diffusive epidemic model, in which $S$ and $P$ represent the densities of susceptible and infective populations. For $\alpha = 0$ and $K = 0$, Eq. (1.1) has also served as a model for the single-stage reaction of the first order in combustion [11,12]. Most recently Eq. (1.1) has been derived in [3,9] to study a single population microbial growth for a limiting nutrient in a flow reactor, where $\alpha > 0$ is the flow velocity, and $S, P$ denote respectively the concentration of nutrient and microbial population in the reactor. We refer readers to [2–5, 7,8,11,12] and the references therein for further details of model description.

While variety of dynamical properties of Eq. (1.1) has been investigated under the relevant boundary conditions, the problem on existence of traveling wave solutions, which reflect important phenomenon of wave propagation, has been most extensively studied. To best describe this phenomenon [10], let us consider a long flow reactor that we treat it mathematically to be infinitely long. Suppose that the amount $S_0$ of nutrient is input at a constant velocity $\alpha$ at one end of the flow reactor, says at $x = -\infty$. If there is no bacteria population, then the concentration of nutrient keeps a constant and is washed out at the other end of reactor. On the other hand, suppose that $f(S)$, the uptake function (or birth rate) of bacteria cell, is increasing with respect to $S$ and $f(S_0) - K > 0$, and let a small quantity of bacteria be introduced, then the population increases when growth rate $f(S) - K > 0$. The growth rate eventually becomes negative because of the reduction of the nutrient so that the bacteria population declines. Hence one may expect that a hump-shaped bacteria population density $P(x,t)$ moves towards the other end of reactor. That is, we expect that there are constants $c, S_0$, with $f(S_0) < K$, and a nonnegative traveling wave solution $(S(x,t), P(x,t)) = (U(x+ct), V(x+ct))$ such that

$$
\lim_{z \to -\infty} U(z) = S_0, \quad \lim_{z \to -\infty} U(z) = S_0, \\
\lim_{z \to \infty} V(z) = \lim_{z \to \infty} V(z) = 0,
$$

where $z = x + ct$ (we suppose that the space variable $x \in \mathbb{R}$). After several authors’ effort [1,4–8,10], the problem on existence of the traveling wave solutions of above type has recently been completely solved that can be summarized as follows:

**Theorem 1.1.** Suppose $\rho \geq 0$, $d \geq 0$, and $K > 0$ are constants. Also suppose that $f$ is monotone increasing with $f(0) = 0$ and $f(S_K) = K$ for some positive number $S_K$. Then, given $S^0 > S_K$ and $c \in \mathbb{R}$, there exists $S_0 < S_K$ such that Eq. (1.1) has a nonnegative traveling wave solution $(S(x,t), P(x,t)) = (U(x+ct), V(x+ct))$ satisfying the boundary condition (1.2) if and only if

$$
c + \alpha \geq \sqrt{4d[f(S^0) - K]}.
$$

Moreover, $U(z)$ is strictly monotone decreasing and $V(z)$ is strictly positive for $z \in \mathbb{R}$.

**Remark 1.** Theorem 1.1 remains true for a larger class of functions $f$’s that are not necessarily monotone increasing [6].
Remark 2. For \( K = 0 \), the traveling wave has a different boundary condition than the one given in (1.2). For the studies of traveling wave with \( K = 0 \), we refer readers to \[11\].

Now the question remaining open is the uniqueness of traveling solutions. To be specific, given \( S^0 > S_K \) and a real number \( c \) with \( c + \alpha \geq \sqrt{4d[f(S^0) - K]} \), is the corresponding traveling wave solution unique (up to a time translation)? Upon a direct substitution, the equation for traveling wave \((U, V)\) is given by

\[
C \dot{U} = \rho \ddot{U} - f(U) V, \quad C \dot{V} = d \ddot{V} + \left[ f(U) - K \right] V \tag{1.3}
\]

with \( C = c + \alpha \). A straightforward computation shows that at each equilibrium point \((S_0, 0)\) of Eq. (1.3) with \( S_0 < S_K \), the stable manifold is one-dimensional. This fact rules out the possibility of having two different traveling wave solutions connecting the same pair of equilibria \((S_0, 0)\) and \((S^0, 0)\). On the other hand, the corresponding unstable manifold to the equilibrium point \((S^0, 0)\) for \( S^0 > S_K \) is three-dimensional. Then, regarding the uniqueness, the question will be reduced to as

For given \( S^0 > S_K \) and \( C \) with \( C \geq \sqrt{4d[f(S^0) - K]} \), can there be two points \( S_{01} < S_{02} < S_K \), and two nonnegative traveling waves \((U_i(z), V_i(z))\) that joins \((S_{0i}, 0)\) and \((S^0, 0)\) for \( i = 1, 2 \)?

The uniqueness of the traveling wave has only been confirmed for some special cases:

(a) \( \rho = 0 \). In this case Eq. (1.3) can actually be reduced to a two-dimensional system and the relation between \( S^0 \) and \( S_0 \) can be analytically expressed (see \[7,8\]).
(b) \[10\] used a singular perturbation method to study the case when \( \rho > 0 \) is sufficiently small. Hence the existence and uniqueness of traveling wave solutions has been extended to small \( \rho > 0 \) from \( \rho = 0 \).
(c) \( d = 0 \). In this case Eq. (1.3) is three-dimensional. The uniqueness of traveling waves was proved in \[1\] by transforming Eq. (1.3) to a two-dimensional monotone system.

However, all the approaches mentioned above cannot be applied to the general case where \( \rho \) and \( d \) are arbitrary positive constants. A new technique needs to be introduced to study the uniqueness of traveling wave solutions. In this paper we will give a complete proof of the uniqueness of traveling wave solutions by a direct analysis of Eq. (1.3). In addition, we will show that, for a fixed wave speed \( c \), if \((U_i(z), V_i(z)), i = 1, 2,\) are two traveling waves connecting \((S_{0i}, 0)\) and \((S^0, 0)\) with \( S^{02} < S^{01} \), then \( S_{01} < S_{02} \). Complete statements and proofs of our main theorems will be given in Section 3 while Section 2 is devoted to investigate more detailed behaviors of the traveling wave solutions that will be applied to Section 3.

Before ending this section we remark that the main results provided in Section 3 will be used to establish the existence of the traveling wave solutions for a model of microbial growth in a flow reactor with two competing populations introduced in \[3\]. This will be done in a forthcoming paper.
2. Preliminaries

The purpose of this section is to establish Propositions 2.1–2.3 that will serve as the main tools to prove the uniqueness of the traveling waves. Without loss of generality, throughout this paper we suppose $d = 1$, for this can be achieved by a time scaling. For the convenience of discussion let us first reverse the time in Eq. (1.3) by introducing

$$u(t) = U(-t), \quad v(t) = V(-t), \quad t \in \mathbb{R}.$$ 

Then the equations for $u$ and $v$ are given as

$$\rho \dddot{u} = -C \dot{u} + f(u)v, \quad \dddot{v} = -C \dot{v} - \left[ f(u) - K \right]v$$

with the boundary condition

$$u(-\infty) = u_0 < S_K < u(0) = u(\infty), \quad v(-\infty) = v(\infty) = 0.$$  \hspace{1cm} (2.2)

We suppose that

$(A) \quad f(u)$ is Lipschitz continuous, strictly increasing, $f(0) = 0$, and there is a $S_K > 0$ such that $f(S_K) = K$.

We point out that $(A)$ is a common assumption satisfied by all models mentioned in Section 1. From Theorem 1.1 it follows that for each $u_0 > S_K$, if

$$C \geq \sqrt{4[f(u_0) - K]},$$

then Eq. (2.1) has a nonnegative heteroclinic solution $(u(t), v(t))$ satisfying the boundary condition (2.2) for some positive number $u_0 < S_K$. Moreover, $u(t)$ is strictly increasing and $v(t)$ is positive for $t \in \mathbb{R}$. In what follows we always refer to this heteroclinic solution as a positive traveling wave (solution) connecting $(u_0, 0)$ and $(u(0), 0)$.

In the rest of this section, for a positive traveling wave $(u(t), v(t))$ of Eq. (2.1) with $(u(\infty), v(\infty)) = (u_0, 0)$ we always suppose the strict inequality

$$C > \sqrt{4[f(u_0) - K]}$$

without specifying, except at the end of this section where the case of equality

$$C = \sqrt{4[f(u_0) - K]}$$

will be discussed. Let us first provide some useful expressions for the $v$ component. To do so we formally write the equation for $v(t)$ as a linear nonhomogeneous equation

$$\dddot{v} = -C \dot{v} - \left[ f(u(t)) - K \right]v = -C \dot{v} - \left[ f(u_0) - K \right]v + g(t)v(t),$$

or

$$\dddot{v} + C \dot{v} + \left[ f(u_0) - K \right]v = \phi(t),$$

where $\phi(t) = g(t)v(t)$ with

$$g(t) = f(u_0) - f(u(t)) > 0, \quad t \in \mathbb{R},$$

and

$$\phi(t) = f(u_0) - f(u(t)) > 0, \quad t \in \mathbb{R}.$$
because \( u(t) < u^0 \) and \( f \) is increasing. The corresponding linear homogeneous equation of (2.3) has the characteristic equation
\[
\lambda^2 + C\lambda + f(u^0) - K = 0,
\]
which has two negative eigenvalues
\[
\lambda_1 = \frac{-C + \sqrt{C^2 - 4[f(u^0) - K]}}{2}, \quad \lambda_2 = \frac{-C - \sqrt{C^2 - 4[f(u^0) - K]}}{2},
\]
with \( \lambda_1 > \lambda_2 \). Applying the variation-of-constant formula to (2.3), we obtain the expression of \( v(t) \) as
\[
v(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \alpha \left[ \int_0^t e^{\lambda_1 (t-s)} g(s) v(s) ds - \int_0^t e^{\lambda_2 (t-s)} g(s) v(s) ds \right],
\]
where
\[
\alpha = \frac{1}{\lambda_1 - \lambda_2} = \frac{1}{\sqrt{C^2 - 4[f(u^0) - K]}} > 0,
\]
and the constants \( c_1 \) and \( c_2 \) satisfy
\[
v(0) = c_1 + c_2, \quad \dot{v}(0) = c_1 \lambda_1 + c_2 \lambda_2.
\]
Solving this system gives
\[
c_1 = \alpha \left[ -\lambda_2 v(0) + \dot{v}(0) \right], \quad c_2 = \alpha \left[ \lambda_1 v(0) - \dot{v}(0) \right]. \tag{2.5}
\]
Next we give another expression for \( v(t) \) for a positive traveling wave solution \((u(t), v(t))\) that connects \((u_0, 0)\) and \((u^0, 0)\). Noticing that
\[
\lambda_2 - \lambda_1 < 0, \quad g(t) v(t) = (f(u^0) - f(u(t))) v(t) > 0,
\]
it follows from (2.4) that for \( t \geq 0 \),
\[
e^{-\lambda_1 t} v(t) = c_1 + c_2 e^{(\lambda_2 - \lambda_1) t}
\]
\[
\quad + \alpha \left[ \int_0^t e^{-\lambda_1 s} v(s) g(s) ds - \int_0^t e^{(\lambda_2 - \lambda_1) (t-s)} e^{-\lambda_1 s} v(s) g(s) ds \right]
\]
\[
\quad \leq |c_1| + |c_2| + \alpha \int_0^t e^{-\lambda_1 s} v(s) g(s) ds. \tag{2.6}
\]
A straightforward computation shows that the equilibrium \((u^0, 0, 0, 0)\) of Eq. (2.1) (in the phase space \((u, \dot{u}, v, \dot{v})\)) has only a simple zero eigenvalue and the associated center manifold is the line \( v \equiv 0 \). Since \((u(t), v(t))\) converges to \((u^0, 0)\) as \( t \to \infty \) and \( v(t) \neq 0 \), \((u(t), \dot{u}(t), v(t), \dot{v}(t))\) must converge to \((u^0, 0, 0, 0)\) along the stable manifold.
of the equilibrium \((u_0,0,0,0)\). So that \(u(t)\) converges to \(u^0\) exponentially as \(t \to \infty\). Consequently \(g(t) = f(u^0) - f(u(t))\) converges to zero exponentially as \(t \to \infty\). Hence \(\int_0^\infty g(s) \, ds < \infty\). Applying the Gronwall’s inequality to (2.6), we obtain
\[
e^{-\lambda_1 t} v(t) \leq \left( |c_1| + |c_2| \right) e^{\lambda_1 t} \int_0^t g(s) \, ds \leq \left( |c_1| + |c_2| \right) e^{\lambda_1 t} \int_0^\infty g(s) \, ds < \infty, \quad t \geq 0. \quad (2.7)
\]
It follows from (2.7) that
\[
\lim_{t \to \infty} \int_0^t g(s) v(s) e^{-\lambda_1 s} \, ds = 0. \quad (2.8)
\]
Moreover, \(g(t)v(t)e^{-\lambda_1 t} \to 0\) as \(t \to \infty\) implies that
\[
\lim_{t \to \infty} \int_0^t e^{(\lambda_2 - \lambda_1)(t-s)} g(s) v(s) e^{-\lambda_1 s} \, ds = 0. \quad (2.9)
\]
Now by using the first equality in (2.4), we can rewrite the expression for \(v(t)\) as
\[
v(t) = e^{\lambda_1 t} \left[ M + H(t) \right] \quad (2.10)
\]
with
\[
M = c_1 + \alpha \int_0^\infty g(s) v(s) e^{-\lambda_1 s} \, ds \\
= \alpha \left[ -\lambda_2 v(0) + \dot{v}(0) + \int_0^\infty g(s) v(s) e^{-\lambda_1 s} \, ds \right] \quad (2.11)
\]
and
\[
H(t) = c_2 e^{(\lambda_2 - \lambda_1)t} - \alpha \int_0^t g(s) v(s) e^{-\lambda_1 s} \, ds - \alpha \int_0^t e^{(\lambda_2 - \lambda_1)(t-s)} g(s) v(s) e^{-\lambda_1 s} \, ds.
\]
From (2.8)–(2.9) and the expression of \(H(t)\) one easily sees that
\[
\lim_{t \to \infty} H(t) = 0.
\]
Expression (2.10) will be used to prove the uniqueness of traveling wave solutions.

**Lemma 2.1.** Let \((u_i(t), v_i(t)), i = 1, 2,\) be two positive solutions of Eq. (2.1) satisfying
\[
v_i(t_0) = v_2(t_0), \quad \dot{v}_1(t_0) > \dot{v}_2(t_0)
\]
for some \(t_0 \in \mathbb{R}\). If in addition, there is a \(t^* > t_0\) such that
\[
u_1(t) < u_2(t) < u^0, \quad t \in [t_0, t^*),
\]
then
\[
v_1(t) > v_2(t)
\]
for all \(t \in [t_0, t^*]\).
Proof. Without loss of generality we let \( t_0 = 0 \) (otherwise we can consider the translation \((\tilde{u}_i(t), \tilde{v}_i(t)) = (u_i(t + t_0), v_i(t + t_0))\) if necessary). Let \( g_i(t) = f(u^0) - f(u_i(t)) \). Then \( u_1(t) < u_2(t) \) yields that \( g_1(t) > g_2(t) > 0 \) for \( t \in [0, t^*] \). We observe that \( v_1(t) > v_2(t) \) for small \( t > 0 \) since \( \dot{v}_1(0) > \dot{v}_2(0) \). We claim that \( v_1(t) > v_2(t) \) remains true for all \( t \in [0, t^*] \). If this is not the case, then there must be a \( t_1 \in (0, t^*] \) such that \( v_1(t_1) = v_2(t_1) \) and

\[
v_1(t) > v_2(t), \quad t \in [0, t_1).
\]

Hence

\[
g_1(t)v_1(t) - g_2(t)v_2(t) > 0, \quad t \in [0, t_1].
\]

By formulas (2.4), (2.5), and the inequality \( \lambda_1 > \lambda_2 \) we obtain

\[
v_1(t_1) - v_2(t_1) = \alpha \left[ \dot{v}_1(0) - \dot{v}_2(0) \right] \left( e^{\lambda_1 t_1} - e^{\lambda_2 t_1} \right) + \alpha \int_0^{t_1} \left[ e^{\lambda_1(t_1-s)} - e^{\lambda_2(t_1-s)} \right] \left\{ g_1(s)v_1(s) - g_2(s)v_2(s) \right\} ds > 0,
\]

which contradicts the assumption \( v_1(t_1) = v_2(t_1) \).

Lemma 2.2. Suppose that \((u_i(t), v_i(t))\) are two traveling wave solutions connecting \((\xi_i, 0)\) and \((u^0, 0)\) with \( 0 < \xi_1 < \xi_2 < S_K \). If there is a \( t_1 \leq +\infty \) \((t_1 \text{ is allowed to be } +\infty)\) such that \( u_1(t) < u_2(t) \) for \( t \in (-\infty, t_1) \) and \( u_1(t_1) = u_2(t_1) \), then there is a \( t_0 < t_1 \) such that \( v_1(t_0) = v_2(t_0) \) and \( \dot{v}_1(t_0) > \dot{v}_2(t_0) \).

Proof. A straightforward computation shows that the characteristic equation for the equilibrium \( e_i = (\xi_i, 0) \) of Eq. (2.1) is

\[
(\rho \mu^2 + C \mu) \left( \mu^2 + C \mu - [K - f(\xi_i)] \right) = 0
\]

that has one zero eigenvalue, two negative eigenvalue and one positive eigenvalue

\[
\mu_i = \frac{-C + \sqrt{C^2 + 4[K - f(\xi_i)]}}{2}.
\]

Moreover, the eigenvector \( h_i \) (in the phase space \((u, \dot{u}, v, \dot{v})\)) corresponding to \( \mu_i \) is given by

\[
h_i = \left( \frac{f(\xi_i)}{\rho(\mu_i^2 + C \mu_i)}, \frac{\mu_i f(\xi_i)}{\rho(\mu_i^2 + C \mu_i)}, 1, \mu_i \right),
\]

which is a positive vector. Since the center manifold associated with the zero eigenvalue is the line \( \{(\alpha, 0) ; \alpha \in \mathbb{R}\} \) consisting of all equilibrium points, the positive traveling wave \((u_i(t), v_i(t))\) must converge to \( e_i = (\xi_i, 0) \), as \( t \to -\infty \), along the unstable manifold corresponding to the eigenvalue \( \mu_i \). Notice that \( f \) is increasing and \( \xi_1 < \xi_2 \), one has \( \mu_1 > \mu_2 \).

It follows that \( v_1(t) \) goes to zero faster than \( v_2(t) \) does as \( t \to -\infty \). Consequently there is a \( t' \) such that

\[
v_1(t) < v_2(t), \quad t \in (-\infty, t').
\]
On the other hand, we show that there must be a \( t'_0 \in (-\infty, t_1) \) such that \( v_1(t'_0) > v_2(t'_0) \). If this is not the case, then \( v_1(t) \leq v_2(t) \) for all \( t \in (-\infty, t_1) \). Hence

\[
f(u_2(t))v_2(t) \geq f(u_1(t))v_1(t), \quad t \in (-\infty, t_1).
\]  

(2.13)

Applying the variation-of-constant formula to the first equation in Eq. (2.1) (or by a direct verification), we can express a solution \((u(t), v(t))\) that converges to \((u_0, 0)\), as \( t \to -\infty \), in an integral form as

\[
u(t) = u_0 + \frac{1}{C} \int_{-\infty}^{t} \left[ 1 - e^{-C(t-s)/\rho} \right] f(u(s)) v(s) \, ds.
\]  

(2.14)

Applying (2.13) and (2.14) to \( u_1(t) \) and \( u_2(t) \) respectively, we obtain

\[
u_2(t_1) = \xi_2 + \frac{1}{C} \int_{-\infty}^{t_1} \left[ 1 - e^{-C(t_1-s)/\rho} \right] f(u_2(s)) v_2(s) \, ds
\]

\[
> \xi_1 + \frac{1}{C} \int_{-\infty}^{t_1} \left[ 1 - e^{-C(t_1-s)/\rho} \right] f(u_1(s)) v_1(s) \, ds
\]

\[
= u_1(t_1).
\]

The above inequality contradicts the assumption of \( u_1(t_1) = u_2(t_1) \). Hence there exists a \( t'_0 \leq t_1 \) such that \( v_1(t'_0) > v_2(t'_0) \). Thus (2.12) and the continuity yields that there is a \( t_0 < t_1 \) such that

\[
v_1(t) < v_2(t), \quad t \in (-\infty, t_0), \quad v_1(t_0) = v_2(t_0).
\]  

(2.15)

It now remains to show that \( \dot{v}_1(t_0) > \dot{v}_2(t_0) \). First it is obvious that \( \dot{v}_1(t_0) \geq \dot{v}_2(t_0) \). So it is sufficient to show that \( \dot{v}_1(t_0) \neq \dot{v}_2(t_0) \). Suppose in opposite that \( \dot{v}_1(t_0) = \dot{v}_2(t_0) \). Then, with the use of inequality \( f(u_1(t_0)) < f(u_2(t_0)) \), we deduce from the second equation of Eq. (2.1) that

\[
\dot{v}_1(t_0) = -C \dot{v}_1(t_0) + \left[ K - f(u_1(t_0)) \right] v_1(t_0)
\]

\[
= -C \dot{v}_2(t_0) + \left[ K - f(u_1(t_0)) \right] v_2(t_0)
\]

\[
> -C \dot{v}_2(t_0) + \left[ K - f(u_2(t_0)) \right] v_2(t_0)
\]

\[
= \dot{v}_2(t_0).
\]

Thus the function \( v_1(t) - v_2(t) \) has a local minimum zero at \( t_0 \). This is clearly a contradiction to (2.15). \( \square \)

**Proposition 2.1.**

(a) For any positive traveling wave solution \((u(t), v(t))\) connecting \((u_0, 0)\) and \((u^0, 0)\), the corresponding number \( M \) given in (2.11) is positive.
(b) Suppose there are two traveling waves \((u_i(t), v_i(t))\), \(i = 1, 2\), connecting \((\xi_i, 0)\) and \((u^0, 0)\) with \(\xi_1 < \xi_2 < S_K\). If
\[
u_1(t) < \nu_2(t), \quad t \in \mathbb{R},
\]
then \(M_1 > M_2\), where \(M_i\) is the number in (2.11) associated with \((u_i(t), v_i(t))\) for \(i = 1, 2\).

**Proof.** Since \(\nu(t) > 0\) and \(\nu(-\infty) = \nu(\infty) = 0\), there is a time \(t_0\) such that \(\dot{\nu}(t_0) = 0\). Let \((\tilde{u}(t), \tilde{v}(t)) = (u(t + t_0), v(t + t_0))\) be a translation. Then \((\tilde{u}(t), \tilde{v}(t))\) is a traveling wave that connects the same points as \((u(t), v(t))\) does. By (2.10), (2.11), and the equality \(\dot{\tilde{v}}(0) = \dot{\nu}(t_0) = 0\), we obtain
\[
\tilde{v}(t) = e^{\lambda_1 t} \left[ \tilde{M} + \tilde{H}(t) \right]
\]
with
\[
\tilde{M} = \alpha \left[ -\lambda_2 \dot{\nu}(0) + \int_0^\infty \left[ f(u^0) - f(\tilde{u}(s)) \right] \dot{\tilde{v}}(s) e^{-\lambda_1 s} ds \right] > 0
\]
and \(\tilde{H}(t) \to 0\) as \(t \to \infty\). Now we have
\[
\nu(t) = \tilde{v}(t - t_0) = e^{\lambda_1 (t - t_0)} \left[ \tilde{M} + \tilde{H}(t - t_0) \right] = e^{\lambda_1 t} \left[ e^{-\lambda_1 t_0} \tilde{M} + e^{-\lambda_1 t_0} \tilde{H}(t - t_0) \right].
\]
Since \(e^{-\lambda_1 t_0} \tilde{H}(t - t_0) \to 0\) as \(t \to \infty\), by comparing the above equality and (2.10), we conclude that \(M = e^{-\lambda_1 t_0} \tilde{M} > 0\). So that Part (a) holds.

Next, let \((u_i(t), v_i(t))\) be the solutions satisfying the assumptions given in part (b). By Lemma 2.2 there exists a \(t_0 \in \mathbb{R}\) such that
\[
v_1(t_0) = v_2(t_0), \quad \dot{v}_1(t_0) > \dot{v}_2(t_0).
\]
Without loss of generality, otherwise by a translation if necessary, we can suppose \(t_0 = 0\). Then Lemma 2.1 yields that \(v_1(t) > v_2(t)\) for all \(t > 0\). Thus the inequality \(u_1(t) < u_2(t)\) for \(t > 0\) yields that
\[
\left[ f(u^0) - u_1(t) \right] v_1(t) > \left[ f(u^0) - u_2(t) \right] v_2(t), \quad t > 0.
\]
It immediately follows that
\[
M_1 = \alpha \left[ -\lambda_2 v_1(0) + \dot{v}_1(0) + \int_0^\infty \left[ f(u^0) - u_1(s) \right] v_1(s) e^{-\lambda_1 s} ds \right]
> \alpha \left[ -\lambda_2 v_2(0) + \dot{v}_2(0) + \int_0^\infty \left[ f(u^0) - u_2(s) \right] v_2(s) e^{-\lambda_1 s} ds \right]
= M_2.
\]
This complete the proof of Part (b).  \(\square\)
Lemma 2.3. Let $y_i : \mathbb{R} \to \mathbb{R}, i = 1, 2,$ be monotone increasing, differentiable functions and $y_1(-\infty) < y_2(-\infty)$. If there are $t_1 < t_2$ such that

$$y_1(t_1) > y_2(t_1), \quad y_1(t_2) = y_2(t_2),$$

then there exist numbers $t_0$ and $\alpha^* > 0$ such that

$$y_1(t_0 - \alpha^*) = y_2(t_0), \quad \dot{y}_1(t_0 - \alpha^*) = \dot{y}_2(t_0),$$

$$y_1(t - \alpha^*) < y_2(t), \quad t \in (-\infty, t_0).$$

Proof. For $\alpha > 0$, let $y^\alpha : (-\infty, t_2] \to \mathbb{R}$ be defined by $y^\alpha(t) = y_1(t - \alpha), t \in (-\infty, t_2]$. By the continuity of $y_1, y_2$ and the inequality $y_1(-\infty) < y_2(-\infty)$, there is a $T < t_1$ such that

$$y_1(t) < y_2(t), \quad t \in (-\infty, T]. \quad (2.16)$$

Hence the monotonicity of $y_1$ and $y_2$ and (2.16) yield that for all $\alpha \geq 0$,

$$y^\alpha(t) < y_2(t), \quad t \in (-\infty, T]. \quad (2.17)$$

Now the assumption $y_1(t_1) > y_2(t_1)$ implies that $y^\alpha(t_1) = y_1(t_1 - \alpha) > y_2(t_1)$ for all small $\alpha > 0$. On the other hand, if $t \in [T, t_2]$, then $t - t_2 + T \leq T$. Hence by (2.16) one has

$$y^{t_2-T}(t) = y_1(t - t_2 + T) < y_2(t - t_2 + T), \quad t \in [t_2, T].$$

It follows that

$$\inf\{\alpha : y^\alpha(t) < y_2(t), \quad t \in [T, t_2]\} = \alpha^*$$

is well defined and is positive. By the definition of $\alpha^*$ and continuity one easily concludes that $y^{\alpha^*}(t) \leq y_2(t)$ for $t \in [T, t_2]$ and the equality holds for at least a $t \in [T, t_2]$. If we let $t_0 = \min\{t \in [T, t_2] : y^\alpha(t) = y_2(t)\}$. Then $t_0 < t_2$. From the definition of $t_0$ and (2.17) we easily deduce that $y_1(t - \alpha^*) = y^{\alpha^*}(t) < y_2(t)$ for all $t < t_0, y_1(t_0 - \alpha^*) = y_2(t_0)$ and $\dot{y}_1(t_0 - \alpha^*) = y_2(t_0).$ \hfill $\Box$

Lemma 2.4. Suppose there are two points $0 < \xi_1 < \xi_2 < S_K$ and two positive traveling wave solutions $(u_i(t), v_i(t)), i = 1, 2$ such that $(u_i(t), v_i(t))$ connects $(\xi_i, 0)$ and $(u^0, 0)$. If there is a $t_1 \in \mathbb{R}$ such that $u_1(t_1) = u_2(t_1)$ and $u_1(t) < u_2(t)$ for all $t \in (-\infty, t_1)$, then $\dot{u}_1(t_1) > \dot{u}_2(t_1)$.

Proof. It is clear that $\dot{u}_1(t_1) \geq \dot{u}_2(t_1)$. So we only need to exclude the possibility of $\dot{u}_1(t_1) = \dot{u}_2(t_1)$. To this end suppose $\dot{u}_1(t_1) = \dot{u}_2(t_1)$. By Lemma 2.2 there is a $t_0 < t_1$ such that $v_1(t_0) = v_2(t_0)$ and $\dot{v}_1(t_0) > \dot{v}_2(t_0)$. Hence it follows from Lemma 2.1 that $v_1(t) > v_2(t)$ for $t \in (t_0, t_1]$. Let $w(t) = u_2(t) - u_1(t)$. Then

$$w(t) > 0, \quad t \in (-\infty, t_1),$$

and

$$w(t_1) = \dot{w}(t_1) = 0.$$
Since $w(t_1)$ is not a local maximum of $w$, we deduce that
\[ \ddot{u}_2(t_1) - \ddot{u}_1(t_1) = \ddot{w}(t_1) \geq 0. \]
On the other hand, we deduce from the first equation of Eq. (2.1) that
\[
\begin{align*}
\rho \ddot{u}_2(t_1) &= -C \dot{u}_2(t_1) + f(u_2(t_1)) v_2(t_1) \\
&= -C \dot{u}_1(t_1) + f(u_1(t_1)) v_2(t_1) \\
&< -C \dot{u}_1(t_1) + f(u_1(t_1)) v_1(t_1) \\
&= \rho \ddot{u}_1(t_1).
\end{align*}
\]
This yields that
\[ \ddot{w}(t_1) = \ddot{u}_2(t_1) - \ddot{u}_1(t_1) < 0, \]
which gives a contradiction. \(\Box\)

Now we are ready to present our second proposition as follows.

**Proposition 2.2.** Suppose that there are two points $0 < \xi_1 < \xi_2 < S_K$ and two positive traveling waves solutions $(u_i(t), v_i(t))$, $i = 1, 2$ such that $(u_i(t), v_i(t))$ connects $(\xi_i, 0)$ and $(\mu^0, 0)$. If there is a $t_1 \in \mathbb{R}$ such that $u_1(t_1) = u_2(t_1)$, then the following holds:

1. $\dot{u}_1(t_1) > \dot{u}_2(t_1)$.
2. $u_1(t) < u_2(t)$ for $t \in (-\infty, t_1)$ and $u_1(t) > u_2(t)$ for $t \in (t_1, \infty)$.
3. There is at least a $t_2 > t_1$ such that $v_1(t_2) < v_2(t_2)$.

**Proof.** By the assumption the set \{t: $u_1(t) = u_2(t)$\} is nonempty and has a lower bound. Hence $t_m = \inf\{t: u_1(t) = u_2(t)\} \leq t_1$ is a real number. And from the definition of $t_m$ it follows that $u_1(t_m) = u_2(t_m)$ and $u_1(t) < u_2(t)$ for all $t \in (-\infty, t_m)$. By Lemma 2.4 we have $\dot{u}_1(t_m) > \dot{u}_2(t_m)$. We claim that $t_m = t_1$. For, if $t_m < t_1$, then $\dot{u}_1(t_m) > \dot{u}_2(t_m)$ implies that there is a $t' \in (t_m, t_1)$ such that $u_1(t') > u_2(t')$. The fact of $u_1(t_1) = u_2(t_1)$ and Lemma 2.3 yield that there is a $t_0$ and $a^* > 0$ such that
\[
\begin{align*}
u(t_0 - a^*) = u_2(t_0), \quad \dot{u}_1(t_0 - a^*) = \dot{u}_2(t_0), \quad u_1(t - a^*) < u_2(t),
& \quad t \in (-\infty, t_0). \end{align*}
\]
If we let $\tilde{u}_1(t) = u_1(t - a^*)$, $\tilde{v}_1(t) = v_1(t - a^*)$, then $(\tilde{u}_1(t), \tilde{v}_1(t))$ is also a traveling wave connecting $(\xi_1, 0)$ and $(\mu^0, 0)$. Moreover we have $\tilde{u}_1(t) < u_2(t)$ for $t \in (-\infty, t_0)$ and $\tilde{u}_1(t_0) = \tilde{u}_2(t_0)$, $\dot{\tilde{u}}_1(t_0) = \dot{\tilde{u}}_2(t_0)$. This leads to a contradiction to Lemma 2.4. So that $t_m = t_1$. This implies that $u_1(t) < u_2(t)$ for all $t < t_1$ and $\dot{u}_1(t_1) > \dot{u}_2(t_1)$. From above argument we easily conclude that
\[ \tilde{t} = t_m = \inf\{t: u_1(t) = u_2(t)\}, \text{ whenever } u_1(\tilde{t}) = u_2(\tilde{t}). \]

The uniqueness of $t_m$ therefore implies that $u_1(t) \neq u_2(t)$ for all $t > t_1$. It immediately follows from the inequality $\dot{u}_1(t_1) > \dot{u}_2(t_1)$ and the continuity that $u_1(t_2) > u_2(t_2)$ for all $t > t_1$. Hence Parts (1) and (2) of the proposition hold. Next we prove Part (3). Suppose
in opposite that $v_1(t) \geq v_2(t)$ for all $t \in [t_1, \infty)$. Integrating the first equation of Eq. (2.1) from $t_1$ to $\infty$ and with the use of Part (2), we arrive at

\[-\rho \dot{u}_1(t_1) = -C[u_1(\infty) - u_1(t_1)] + \int_{t_1}^{\infty} f(u_1(s))v_1(s) \, ds \]

\[= -C[u_2(\infty) - u_2(t_1)] + \int_{t_1}^{\infty} f(u_1(s))v_1(s) \, ds \]

\[> -C[u_2(\infty) - u_2(t_1)] + \int_{t_1}^{\infty} f(u_2(s))v_2(s) \, ds \]

\[= -\rho \dot{u}_2(t_1).\]

We therefore have $\dot{u}_1(t_1) < \dot{u}_2(t_1)$. This leads to a contradiction to Part (1). Consequently there must be a $t_2 > t_1$ such that $v_1(t_2) < v_2(t_2)$. □

Now let us return to the case of

\[C = \sqrt{4[f(u^0) - K]}.\]  (2.18)

We shall show that Propositions 2.1 and 2.2 still hold, except that in Proposition 2.1 the numbers $M$, $M_1$, and $M_2$ are defined slightly different. Note that under the condition (2.18), the corresponding linear homogeneous equation of (2.3) has a double eigenvalue

\[\lambda_1 = \lambda_2 = -\frac{C}{2} < 0.\]

In this case the application of variation-of-constant formula to (2.3) yields the expression for $v(t)$ as

\[v(t) = e^{\lambda_1 t} \left[ \left(-\lambda_1 v(0) + \dot{v}(0)\right)t + v(0) \right] + \int_{0}^{t} e^{\lambda_1 (t-s)} (t-s)g(s)v(s) \, ds.\]  (2.19)

Then, with the use of exponential convergence to zero of $g(t) = f(u^0) - f(u(t))$ as $t \to \infty$, the expression (2.19), and the Gronwall’s inequality, by arguing in the same way as for the case of $C > \sqrt{4[f(u^0) - K]}$, one is able to show that

\[\int_{0}^{\infty} s e^{-\lambda_1 s} g(s) v(s) \, ds < \infty,\]

\[\int_{t}^{\infty} e^{-\lambda_1 s} g(s) v(s) \, ds \to 0 \quad \text{as } t \to \infty.\]  (2.20)

Similarly to (2.10) and (2.11) we can use (2.19) to further express $v(t)$ as

\[v(t) = e^{\lambda_1 t} \left[M^0 t + H^0(t)\right] = te^{\lambda_1 t} \left[M^0 + \frac{1}{t} H^0(t)\right].\]  (2.21)
with
\[ M^0 = \left[ -\lambda_1 v(0) + \dot{v}(0) + \int_0^\infty e^{-\lambda_1 s} g(s)v(s) \, ds \right] \]  
(2.22)
and
\[ H^0(t) = v(0) - t \int_t^\infty e^{-\lambda_1 s} g(s)v(s) \, ds - \int_0^t se^{-\lambda_1 s} g(s)v(s) \, ds. \]

From (2.20) one easily sees that
\[ \frac{1}{t} H^0(t) \to 0 \text{ as } t \to \infty. \]

Recall that for the case \( C > \sqrt{4[f(u^0) - K]} \), only the proofs of Lemma 2.1 and Proposition 2.1 have used the expressions (2.4) and (2.10). Now it becomes evident that for \( C = \sqrt{4[f(u^0) - K]} \), Lemma 2.1 can be proved in a same fashion by using the expression (2.19). Similarly, Proposition 2.1 remains true if we replace the numbers \( M, M_1, M_2 \) by the numbers \( M^0, M_1^0, M_2^0 \) defined respectively by the expression (2.22). Hence we have

**Proposition 2.3.** Propositions 2.1 and 2.2 remain valid for the case \( C = \sqrt{4[f(u^0) - K]} \), except in Proposition 2.1 the numbers \( M, M_1, M_2 \) are replaced respectively by the numbers \( M^0, M_1^0, M_2^0 \) defined by the expression (2.22).

### 3. Uniqueness of traveling wave solutions

We are now in the position to prove the uniqueness of traveling wave solutions.

**Theorem 3.1.** Let \( u^0 > S_K \) and \( C > 0 \) be given with
\[ C \geq \sqrt{4[f(u^0) - K]}. \]
Then there is a unique \( u_0 < S_K \) such that Eq. (2.1) has a positive traveling wave solution connecting \((u_0, 0)\) and \((u^0, 0)\).

**Proof.** We shall give the proof under the inequality \( C > \sqrt{4[f(u^0) - K]} \). The proof for the case of equality is exactly the same with the use of Proposition 2.3. Suppose in opposite that there are two traveling waves \((u_i(t), v_i(t))\) connecting \((\xi_i, 0)\) and \((u^0, 0)\), \(i = 1, 2\), with \( \xi_1 < \xi_2 < S_K \). Since
\[ u_i(-\infty) = \xi_i < S_K < u^0 = u_i(\infty), \quad i = 1, 2, \]
by a translation if necessary we can suppose \( u_1(0) = u_2(0) = S_K \). Hence Part (1) of Proposition 2.2 yields that \( u_1(-a) > u_2(0) \) for all small \( a > 0 \). The inequality \( u_1(-\infty - a) =
\[ \xi_1 < \xi_2 < u_2(-\infty) \] therefore implies that for each small \( a > 0 \), there is a \( t_a \) such that \( u_1(t_a - a) = u_2(t_a) \). The number \( t_a \) is uniquely defined by Part (2) of Proposition 2.2. Let
\[ J = \{ a \in \mathbb{R} : \text{there is a } t_a \text{ such that } u_1(t_a - a) = u_2(t_a) \}. \]
Then \( J \) is nonempty. If \( a \in J \), then there exists a \( t_a \) such that \( u_1(t_a - a) = u_2(t_a) \). So that \( u_1(t_a - a') > u_2(t_a) \) for any \( a' < a \) by the monotonicity of \( u_1(t) \). Thus there is a \( t_a' \) such that \( u_1(t_a' - a') = u_2(t_a') \). Hence \( a' \in J \). That is, \( (-\infty, a] \subset J \) whenever \( a \in J \). This implies that \( J \) is an interval. We show that \( t_a \) is monotone increasing for \( a \in J \). Let \( a_1, a_2 \in J \) with \( a_1 < a_2 \). Then
\[ u_1(t_{a_i} - a_i) = u_2(t_{a_i}), \quad i = 1, 2. \]
Part (2) of Proposition 2.2 implies
\[ u_1(t - a_1) < u_2(t), \quad t < t_{a_1}. \]
The monotonicity of \( u_1(t) \) yields that
\[ u_1(t - a_2) < u_1(t - a_1) \leq u_2(t), \quad t \leq t_{a_1}. \]
Hence we must have \( t_{a_2} > t_{a_1} \) in order that \( u_1(t_{a_2} - a_2) = u_2(t_{a_2}) \). Now let
\[ a^* = \sup\{a : a \in J\}. \]
Then either \( a^* \) is a real number or \( a^* = \infty \). Note that \( t_a \) is increasing as \( a \) increases. So that \( \lim_{a \to a^*} t_a = t_{a^*} \) is well defined. We claim that \( t_{a^*} = \infty \). For, if \( t_{a^*} < \infty \), then, by letting \( a \to a^* \) in the equality \( u_1(t_a - a) = u_2(t_a) \) we obtain
\[ u_1(t_{a^*} - a^*) = u_2(t_{a^*}) > \xi_2 > \xi_1. \]
It therefore follows that \( t_{a^*} - a^* > -\infty \), or equivalent \( a^* < \infty \). Thus both \( t_{a^*} - a^* \) and \( t_{a^*} \) are real numbers. So that \( \tilde{u}_1(t_{a^*} - a^*) > \tilde{u}_2(t_{a^*}) \) by Part (1) of Proposition 2.2. From this inequality we easily conclude that for each small \( \varepsilon > 0 \), there is a \( t_{a^* + \varepsilon} \) such that
\[ u_1(t_{a^* + \varepsilon} - (a^* + \varepsilon)) = u_2(t_{a^* + \varepsilon}), \]
contradicting the definition of \( a^* \). In what follows we shall show that either \( a^* < \infty \) or \( a^* = \infty \) will lead to a contradiction.

**Case I.** \( a^* < \infty \).

Since \( t_a \to \infty \) as \( a \to a^* \), for each \( t \in \mathbb{R} \) there is an \( a < a^* \) such that \( t < t_a \). Thus we have
\[ u_1(t - a^*) < u_1(t - a) < u_2(t). \]
Let \( (\tilde{u}_1(t), \tilde{v}_1(t)) = (u_1(t - a^*), v_1(t - a^*)) \). Then
\[ \tilde{u}_1(t) < u_2(t), \quad t \in \mathbb{R}. \] (3.1)

With the use of (2.10) we can write
\[ \tilde{v}_1(t) = e^{\lambda_1 t}[M_1 + H_1(t)], \quad v_2(t) = e^{\lambda_1 t}[M_2 + H_2(t)], \]
where \( H_i(t) \to 0 \) as \( t \to \infty \) for \( i = 1, 2 \). In addition, (3.1) and Part (b) of Proposition 2.1 yield that \( M_1 > M_2 \). Thus one easily conclude that there are small number \( \delta > 0 \) and sufficiently large \( T \) such that for any \( \varepsilon \in [0, \delta] \) and all \( t \geq T \),
\[ e^{\lambda_1 \varepsilon} M_1 + e^{\lambda_1 \varepsilon} H_1(t + \varepsilon) > M_2 + H_2(t). \]
Consequently, for any \( \varepsilon \in [0, \delta] \) and all \( t \geq T \) we have
\[
\tilde{v}_1(t + \varepsilon) = e^{\lambda_1 t} e^{\lambda_{1+} \varepsilon} \left[ M_1 + H_1(t + \varepsilon) \right] > e^{\lambda_1 t} \left[ M_2 + H_2(t) \right] = v_2(t). \tag{3.2}
\]
Since \( t_a \to \infty \) as \( a \to a^* \), there is an \( \varepsilon \in (0, \delta] \) such that \( t_{a^* - \varepsilon} > T \). By the definition of \( t_{a^* - \varepsilon} \) we have
\[
\tilde{u}_1(t_{a^* - \varepsilon} + \varepsilon) = u_1(t_{a^* - \varepsilon} + \varepsilon - a^*) = u_1(t_{a^* - \varepsilon} - (a^* - \varepsilon)) = u_2(t_{a^* - \varepsilon}). \tag{3.3}
\]
(3.3) and Part (3) of Proposition 2.2 therefore imply that there is at least a \( t' > t_{a^* - \varepsilon} > T \) such that \( \tilde{v}_1(t' + \varepsilon) < v_2(t') \) (here we have used the fact that \((\tilde{u}_1(t + \varepsilon), \tilde{v}_1(t + \varepsilon))\) is also a traveling wave solution connecting \((\xi_1, 0)\) and \((u^0, 0)\)). This leads to a contradiction to (3.2).

**Case II.** \( a^* = \infty \).

Note that \( t_a \to \infty \) as \( a \to a^* = \infty \). We deduce that
\[
\lim_{a \to \infty} u_1(t_a - a) = \lim_{a \to \infty} u_2(t_a) = u_2(\infty) = u^0 = u_1(\infty).
\]
The monotonicity of \( u_1(t) \) and the above equality therefore yield that \( t_a - a \to \infty \) as \( a \to \infty \). Now by (2.10) we express \( v_1(t) \) and \( v_2(t) \) respectively as
\[
v_i(t) = e^{\lambda_1 t} \left[ M_i + H_i(t) \right],
\]
with \( H_i(t) \to 0 \) as \( t \to \infty \), and \( M_i > 0 \) for \( i = 1, 2 \) by Part (a) of Proposition 2.1. Note that \( \lambda_1 < 0 \), we can pick a sufficiently large number \( \gamma^* > 0 \) such that
\[
\frac{e^{-\lambda_1 \gamma^*} M_1}{2} > 2M_2.
\]
Since \( H_i(t) \to 0 \) as \( t \to \infty \), there is a sufficiently large \( t^* \) such that for all \( t \geq t^* \),
\[
\left| H_1(t) \right| \leq \frac{M_1}{2}, \quad \left| H_2(t) \right| \leq M_2.
\]
It follows that for each \( a \geq \gamma^* \) and all \( t \geq a + t^* \) we have
\[
v_1(t - a) = e^{\lambda_1 t} e^{-\lambda_1 a} \left[ M_1 + H_1(t - a) \right] \geq e^{\lambda_1 t} \frac{e^{-\lambda_1 a} M_1}{2} \geq e^{\lambda_1 t} 2M_2
\]
\[
\geq e^{\lambda_1 t} \left[ M_2 + H_2(t) \right] = v_2(t). \tag{3.4}
\]
Since \( t_a - a \to \infty \) as \( a \to \infty \), there is a \( \tilde{a} > \gamma^* \) such that \( t_{\tilde{a}} - \tilde{a} > t^* \). Let \( \tilde{u}_1(t) = U_1(t - \tilde{a}) \) and \( \tilde{v}_1(t) = v_1(t - \tilde{a}) \). Then we have
\[
\tilde{u}_1(t_{\tilde{a}}) = u_1(t_{\tilde{a}} - \tilde{a}) = u_2(t_{\tilde{a}}). \tag{3.5}
\]
Moreover, \( t_{\tilde{a}} - \tilde{a} > t^* \) implies \( t_{\tilde{a}} > \tilde{a} + t^* \). So that (3.4) yields
\[
\tilde{v}_1(t) \geq v_1(t - \tilde{a}) > v_2(t), \quad t > t_{\tilde{a}}. \tag{3.6}
\]
(3.5) and (3.6) leads to a contradiction to Part (3) of Proposition 2.2. \( \square \)
In Eq. (2.1) let $C > 0$ be fixed and let $\bar{u}^0 > S_K$ such that

$$C \geq \sqrt{4[f(\bar{u}^0) - K]}.$$ 

The monotonicity of $f$ implies that

$$C > \sqrt{4[f(u^0) - K]}, \quad u^0 \in (S_K, \bar{u}^0).$$

By Theorems 1.1 and 3.1, for each $u^0 \in (S_K, \bar{u}^0)$, there exists a unique $\xi(u^0) \in (0, S_K)$ such that Eq. (2.1) admits a positive traveling wave connecting $(\xi(u^0), 0)$ and $(u^0, 0)$. In what follows we shall show that $\xi(u^0)$ is decreasing when $u^0$ increases. Furthermore,

$$\{\xi(u^0); u^0 \in (S_K, \bar{u}^0)\} = [\xi(\bar{u}^0), S_K).$$

To this end we need some additional results. For each $0 < u_0 < S_K$, the equilibrium $(u_0, 0, 0, 0)$ (in the phase space $(u, \dot{u}, v, \dot{v})$) of Eq. (2.1) has a one-dimensional unstable manifold and one branch of it is positive (see [6, p. 749]). Let $(u(t, u_0), \dot{v}(t, u_0))$ be the solution of Eq. (2.1) that stays in the positive part of unstable manifold of the equilibrium $(u_0, 0, 0, 0)$ for sufficiently negative time $t$. Let

$$t_M(u_0) = \sup\{t; v(s, u_0) > 0, s \in (-\infty, t]\}. \quad (3.7)$$

We summarize some results from [6] that are needed to prove our final theorem of this paper.

**Lemma 3.2.** Let $t_M(u_0)$ be defined in (3.7). Then $u(t, u_0)$ is bounded, strictly increasing in $(-\infty, t_M(u_0))$, and

$$u^+(u_0) = \lim_{t \to t_M(u_0)} u(t, u_0) > S_K.$$ 

Moreover, the following holds:

(i) $u^+(u_0) \to S_K$ as $u_0 \to S_K$ from the left.

(ii) For $u_0 < S_K < \bar{u}^0$, Eq. (2.1) admits a positive traveling wave connecting $(u_0, 0)$ and $(\bar{u}^0, 0)$ if and only if $C \geq \sqrt{4[f(u^0) - K]}$ and $u^+(u_0) = \bar{u}^0$.

(iii) For $u^0 > S_K$ with $C \geq \sqrt{4[f(u^0) - K]}$, let

$$u_0 = \inf\{u'_0 \in (0, S_K); u^+(u'_0) < u^0\},$$

Then $u^+(u_0) = u^0$.

(iv) For $u_0 < S_K$, if $C > \sqrt{4[f(u^+(u_0)) - K]}$, then there is a $\delta > 0$ such that

$$C > \sqrt{4[f(u^+(u_0 + \varepsilon)) - K]}, \quad \text{whenever} \quad |\varepsilon| \leq \delta.$$ 

That is, Eq. (2.1) has a positive traveling wave connecting $(u_0 + \varepsilon, 0)$ and $(u^+(u_0 + \varepsilon), 0)$ if $|\varepsilon| \leq \delta$.

(v) For $u_0 \in (0, S_K)$, if $u^+(u_0) > u'$, then there is a $\delta > 0$ such that

$$u^+(u_0 + \varepsilon) > u', \quad \text{whenever} \quad |\varepsilon| \leq \delta.$$ 

(For the proofs of above results, see [6, pp. 751, 753, Lemma 3.4 in p. 642 and the proof of Theorem 1.1 in p. 764].)
Theorem 3.3. Given $C > 0$ and $\bar{u}^0$ with $C \geq \sqrt{4[f(u^0) - K]}$. For each $u^0 \in (S_K, \bar{u}^0]$ let $\xi(u^0)$ be the unique number such that Eq. (2.1) admits a positive traveling wave connecting the equilibria $(\xi(u^0), 0)$ and $(u^0, 0)$. Then

1. $\xi(u^0)$ is monotone decreasing for $u^0 \in (S_K, \bar{u}^0]$.
2. $\xi((S_K, \bar{u}^0]) = [\xi(u^0), S_K)$, where $\xi((S_K, \bar{u}^0])$ denotes the range of $\xi$. Consequently, for each $u_0 \in [\xi(\bar{u}^0), S_K)$, there is a $u^0 \in (S_K, \bar{u}^0]$ such that Eq. (2.1) has a positive traveling connecting $(u_0, 0)$ and $(u^0, 0)$.

Proof. For $S_K < u^0 \leq \bar{u}^0$, let

$$u_0 = \inf\{u'_0 \in (0, S_K) : u^+(u'_0) < u^0\}.$$ Then Eq. (2.1) admits a positive traveling wave connecting $(u_0, 0)$ and $(u^0, 0)$ by (ii) and (iii) of Lemma 3.2. Theorem 3.1 implies that the number $u_0$ is uniquely determined. It follows that

$$\xi(u^0) = u_0 = \inf\{u'_0 \in (0, S_K) : u^+(u'_0) < u^0\}.$$ It is apparent that

$$\inf\{u'_0 \in (0, S_K) : u^+(u'_0) < u^0\} \subset \inf\{u'_0 \in (0, S_K) : u^+(u'_0) < u^0\}$$

if $u_1^0 < u_2^0$. Hence we have $\xi(u_2^0) \leq \xi(u_1^0)$. It is evident that $\xi(u_1^0) \neq \xi(u_2^0)$ by the uniqueness of a positive traveling wave. So that $\xi(u_2^0) < \xi(u_1^0)$. Consequently $\xi(u^0)$ is a decreasing function of $u^0$ for $u^0 \in (S_K, \bar{u}^0]$. This completes the proof of Part (1) of the theorem. Next we show that the range of $\xi$ is an interval:

$$\xi((S_K, \bar{u}^0]) = [\xi(\bar{u}^0), S_K).$$

(3.8)

Since $\xi(\bar{u}^0) \leq \xi(u^0) < S_K$ for all $u^0 \in (S_K, \bar{u}^0]$, we have $\xi((S_K, \bar{u}^0]) \subset [\xi(\bar{u}^0), S_K)$. Moreover (i) and (ii) of Lemma 3.2 yields that

$$[S_K - \delta, S_K) \subset [\xi((S_K, \bar{u}^0])]$$

(3.9)

for a sufficiently small $\delta > 0$. Suppose on the contrary that (3.8) does not hold. Let

$$\Omega = \{\beta \in [\xi(\bar{u}^0), S_K) : \beta \not\in [\xi((S_K, \bar{u}^0)]\}.$$ Then $u^*_0 = \sup\{\beta : \beta \in \Omega\}$ is well defined and is an interior point of $[\xi(\bar{u}^0), S_K]$ by (3.9). We show that this will lead to a contradiction. First suppose $u^*_0 \in (S_K, \bar{u}^0]$. Then there is a $u^{0*} \in (S_K, \bar{u}^0]$ such that $\xi(u^{0*}) = u^*_0 > \xi(\bar{u}^0)$ since $u^*_0$ is an interior point of $[\xi(\bar{u}^0), S_K)$. Hence $u^{0*} < \bar{u}^0$. This yields that $C > \sqrt{4[f(u^{0*}) - K]}$. Thus (iv) of Lemma 3.2 implies that there is a $\delta > 0$ such that $[u^*_0 - \delta, u^*_0 + \delta] \subset [\xi((S_K, \bar{u}^0))]$. It follows from the definition of $\Omega$ that

$$[u^*_0 - \delta, u^*_0 + \delta] \cap \Omega = \emptyset.$$ This obviously contradicts the definition of $u^*_0$. Next suppose $u^*_0 \not\in [\xi((S_K, \bar{u}^0)])$. Then from (ii) of Lemma 3.2 we deduce that $u^+(u^*_0) > \bar{u}^0$. Thus by (v) of Lemma 3.2 there exists a
small positive number $\nu$ such that $u^+(u_0) > \tilde{u}^0$ for all $u_0 \in (u_0^* - \nu, u_0^* + \nu)$. Or equivalently $u_0 \notin \xi((S_K, \tilde{u}^0))$ for all $u_0 \in (u_0^* - \nu, u_0^* + \nu)$. So that $(u_0^* - \nu, u_0^* + \nu) \subset \Omega$. This again contradicts the definition of $u_0^*$. Thus (3.8) is valid. □

References