Indag. Mathem., N.S., 18 (1), 135–146 March 26, 2007

Semigroups in symmetric Lie groups

by Laércio J. dos Santos [1] and Luiz A.B. San Martin [2]

Imecc, Unicamp, Caixa Postal 6065, 13.083-970, Campinas, SP, Brazil

Communicated by Prof. J.J. Duistermaat at the meeting of May 29, 2006

ABSTRACT

Let G be a Lie group and $L \subset G$ a Lie subgroup. We give necessary and sufficient conditions for a family of cosets of L to generate a subsemigroup with nonempty interior in G. We apply these conditions to symmetric pairs (G, L) where L is a subgroup of G such that $G_0^\tau \subset L \subset G^\tau$ and τ is an involutive automorphism of G. As a consequence we prove that for several τ the fixed point group G^τ is a maximal semigroup.

1. INTRODUCTION

Let G be a Lie group and $L \subset G$ a Lie subgroup. In this article we look at semigroups containing L or some of its cosets. One of the objectives is to decide when L is a maximal subsemigroup of G, that is, to verify whether L is properly contained in a proper subsemigroup of G.

Our approach consists in finding necessary and sufficient conditions ensuring that a semigroup generated by cosets of L has nonempty interior in G. Then we apply known results on semigroups in Lie groups (specially when G is semi-simple) to check that certain subsemigroups with nonempty interior must coincide with the whole group.

Key words and phrases: Semigroups, Subgroup of fixed points, Symmetric Lie groups, Involutive automorphisms, Flag manifolds, Semi-simple Lie groups

E-mails: lsantos@ime.unicamp.br (L.J. dos Santos), smartin@ime.unicamp.br (L.A.B. San Martin).

[1] Supported by FAPESP grant n° 01/11345-1.

[2] Supported by CNPq grant n° 305513/2003-6 and FAPESP grant n° 02/10246-2.

This method works well in case G/L is an effective irreducible symmetric space, that is, there exists an involutive automorphism τ of G such that $G_0^\tau \subset L \subset G^\tau$, where G^τ is the group of τ-fixed points and G_0^τ its identity component. In this context we can prove that in most of the cases G^τ is not contained in a proper subsemigroup of G. Such semigroups where extensively studied in the literature in connection with causal symmetric spaces (see for instance the monograph by Hilgert and Ólafsson [4], and references therein).

This way we let B be a collection of cosets of L and write $S(L, B)$ for the subsemigroup of G generated by B. To give necessary and sufficient conditions for $S(L, B)$ to have nonempty interior we consider product maps

$$Lx_1 \times \cdots \times Lx_n \longrightarrow G, \quad n \geqslant 1,$$

with $x_1, \dots, x_n \in B$. The semigroup $S(L, B)$ is the union of the images of these maps. We work out our conditions in terms of the differentials of these product maps (see Theorem 2.4). As a result we can prove that $S(L, B)$ has nonempty interior in case B is not contained in the normalizer $N(\mathfrak{l})$ of the Lie algebra $\mathfrak{l}$ of L and the quotient representation of L on $\mathfrak{g}/\mathfrak{l}$ is irreducible (see Theorem 2.6).

In the case of a symmetric Lie group its Lie algebra $\mathfrak{g}$ decomposes as

$$(1) \qquad \mathfrak{g} = \mathfrak{l} \oplus \mathfrak{q},$$

where $\mathfrak{l}$ and $\mathfrak{q}$ are the eigenspaces associated with the eigenvalues 1 and -1 of τ, respectively. This decomposition is sometimes called the *canonical decomposition* of $\mathfrak{g}$. Note that

$$(2) \qquad [\mathfrak{l}, \mathfrak{l}] \subset \mathfrak{l}, \qquad [\mathfrak{l}, \mathfrak{q}] \subset \mathfrak{q}, \quad \text{and} \quad [\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{l}.$$

Also $\mathfrak{l}$ is the Lie algebra of G^τ.

Let L be a subgroup of G such that $G_0^\tau \subset L \subset G^\tau$ and $x \notin N(\mathfrak{l})$. If $\mathrm{Ad}(L)$ is irreducible on $\mathfrak{q}$, then the semigroup generated by a coset Lx has nonempty interior in G by the general result. An example of the irreducible case is a Cartan involution in a simple Lie group G. In this case the fixed point group K is maximal as a subsemigroup of G since any semigroup S containing K properly has nonempty interior and acts transitively on the flag manifolds of G. For general involutions we combine our results with those of [9] to conclude that if the pair $(\mathfrak{g}, \tau)$ is not regular (see Section 4 below), then the only semigroups which contain L are those contained in $N(\mathfrak{l})$.

For the reducible case a detailed study of the structure of $\mathfrak{g}$ is necessary. In this case we have found a subset Θ of the simple system of roots such that if $x \notin N_\Theta^+ N(\mathfrak{l}_\Theta) \cup N_\Theta^- N(\mathfrak{l}_\Theta)$ (for the notation see Section 4), then the semigroup generated by Lx has nonempty interior. Consequently, the semigroups that contain L are the same as those contained in $N_\Theta^+ N(\mathfrak{l}_\Theta) \cup N_\Theta^- N(\mathfrak{l}_\Theta)$, when $(\mathfrak{g}, \tau)$ is not regular.

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Let $L \subset G$ be a Lie subgroup with Lie algebra $\mathfrak{l}$. We assume throughout that G and L are paracompact. In this section we consider semigroups generated by cosets of L. The goal is to get necessary and sufficient conditions for the semigroups to have nonempty interior.

Given a subset $B \subset G$ we denote by $S(L, B)$ the semigroup generated by the cosets Lx, $x \in B$. The elements of $S(L, B)$ are finite products $s = s_1 \cdots s_k$ with $s_i \in \bigcup_{x \in B} Lx$. Also, we write $G(L, B)$ for the subgroup of G generated by the cosets Lx, $x \in B$. Note that $B \subset G(L, B)$ hence B^{-1} and $L = Lxx^{-1}$ $(x \in B)$ are contained in $G(L, B)$. Moreover, it is easy to see that $G(L, B) = S(L, B \cup B^{-1})$ so that we can give a unified treatment for the semigroups and subgroups generated by sets of cosets. If $B = \{x\}$ is a singleton we write simply $S(L, x)$ and $G(L, x)$.

Let us denote by p_n, $n \geqslant 2$, the product map $p_n : G^n \to G$:

$$p_n : (s_1, \dots, s_n) \longmapsto s_1 \cdots s_n.$$

Given a n-tuple $\mathbf{x} = (x_1, \dots, x_n) \in B^n$ the subset $L\mathbf{x} := Lx_1 \times \cdots \times Lx_n$ is a submanifold of G^n. We denote by $q_{\mathbf{x}} : L\mathbf{x} \to G$ the restriction of p_n to $L\mathbf{x}$ and let $\operatorname{im} q_{\mathbf{x}}$ be the image of $q_{\mathbf{x}}$. Clearly, $S(L, B) = \bigcup \operatorname{im} q_{\mathbf{x}}$ with $\mathbf{x}$ running through the n-tuples of elements of B, $n \geqslant 1$. Presently we shall apply Baire's categories to give necessary and sufficient conditions to have $\operatorname{int} S(L, B) \neq \emptyset$.

First let us recall that if M and N are finite-dimensional manifolds with M paracompact then for a smooth (C^∞) map $f : M \to N$ there are two possibilities:

1. For some $x \in M$ the rank of the differential df_x is $\dim N$. In this case the implicit function theorem ensures that the image $f(M)$ of f has nonempty interior in N.
2. $\operatorname{rank} df_x < \dim N$ for all $x \in M$. Then the $f(M)$ is a set of first category (meager set) in N, that is, is the union of at most countable closed sets with empty interior. In fact, M is the union of at most countable compact subsets and if $K \subset M$ is compact then $f(K)$ is compact and by Sard's theorem has empty interior. In this case $f(M)$ has empty interior.

Now it is obvious that $\operatorname{int} S(L, B) \neq \emptyset$ if $\operatorname{im} q_{\mathbf{x}}$ has nonempty interior for some n-tuple $\mathbf{x}$. The converse is also true in case B is countable.

Proposition 2.1. *Suppose that $B \subset G$ is at most countable. Then $\operatorname{int} S(L, B) \neq \emptyset$ if and only if there exists $n \geqslant 1$ and $\mathbf{x} \in B^n$ such that $\operatorname{im} q_{\mathbf{x}}$ has nonempty interior.*

Proof. The condition is obviously sufficient. The converse is a consequence of Baire's categories theorem and the above comments. In fact, each $q_{\mathbf{x}}$ is a smooth map and if all the images $\operatorname{im} q_{\mathbf{x}}$ were meager sets then $S(L, B)$ would be meager, contradicting the assumption that $\operatorname{int} S(L, B) \neq \emptyset$. □

Now we compute the image of differential of the maps $q_{\mathbf{x}}$.

Lemma 2.2. *Given an n-tuple* $\mathbf{x} \in B^n$ *the image of the differential* $d(q_{\mathbf{x}})_\sigma$ *at* $\sigma = (s_1, \ldots, s_n) \in L\mathbf{x}$ *is the subspace*

$$d(\mathcal{R}_s)_1\big(\mathfrak{l} + \mathrm{Ad}(s_1)(\mathfrak{l}) + \cdots + \mathrm{Ad}(s_1 \cdots s_{n-1})(\mathfrak{l})\big),$$

where $s = s_1 \cdots s_n$ *and* $\mathcal{R}$ *is the right action.*

Proof. Denote by $\mathcal{L}$ the left action. Then

$$q_{\mathbf{x}}(s_1, \ldots, s_n) = s_1 \cdots s_n = \mathcal{L}_{s_1 \cdots s_{i-1}} \circ \mathcal{R}_{s_{i+1} \cdots s_n}(s_i).$$

The tangent space to a coset Lx at a point r is $d(\mathcal{R}_r)_1(\mathfrak{l})$. Therefore the image of the i-th partial derivative of $q_{\mathbf{x}}$ at $(s_1, \ldots, s_n)$ is given by

$$\begin{aligned}
\partial_i q_{\mathbf{x}} &= d(\mathcal{L}_{s_1 \cdots s_{i-1}} \circ \mathcal{R}_{s_{i+1} \cdots s_n})_{s_i}\big(d(\mathcal{R}_{s_i})_1(\mathfrak{l})\big) \\
&= d(\mathcal{L}_{s_1 \cdots s_{i-1}} \circ \mathcal{R}_{s_{i+1} \cdots s_n} \circ \mathcal{R}_{s_i})_1(\mathfrak{l}) \\
&= d(\mathcal{R}_{s_i \cdots s_n} \circ \mathcal{L}_{s_1 \cdots s_{i-1}})_1(\mathfrak{l}) \\
&= d(\mathcal{R}_s)_1 \circ \mathrm{Ad}(s_1 \cdots s_{i-1})(\mathfrak{l}).
\end{aligned}$$

Adding up on i the lemma follows. □

Take $s_1, \ldots, s_n \in \bigcup_{x \in B} Lx$. In the sequel we write

(3) $$V(s_1, \ldots, s_n) = \mathfrak{l} + \mathrm{Ad}(s_1)(\mathfrak{l}) + \cdots + \mathrm{Ad}(s_1 \cdots s_n)(\mathfrak{l})$$

for the subspaces of $\mathfrak{g}$ whose right translations give the images of the differentials of the maps $q_{\mathbf{x}}$. It follows easily from the definition that

(4) $$V(s_1, \ldots, s_r, t_1, \ldots, t_m) = V(s_1, \ldots, s_{r-1}) + \mathrm{Ad}(s)V(t_1, \ldots, t_m),$$

where $s = s_1 \cdots s_r$. Now let d be the maximum of the dimensions of these subspaces and take two of them, say $V(s_1, \ldots, s_{r-1})$ and $V(t_1, \ldots, t_m)$, having dimension d. If we fix (an arbitrary) $s_r \in \bigcup_{x \in B} Lx$ and apply (4) we conclude that

$$V(s_1, \ldots, s_{r-1}) = \mathrm{Ad}(s_1 \cdots s_r)V(t_1, \ldots, t_m),$$

that is, $V(t_1, \ldots, t_m) = \mathrm{Ad}(s_1 \cdots s_r)^{-1}V(s_1, \ldots, s_{r-1})$. This means that all the subspaces having maximal dimension are equal to one and the same $\mathrm{Ad}(s_1 \cdots s_r)^{-1}$ $V(s_1, \ldots, s_{r-1})$. We denote this subspace by $V(L, B)$. We have

Lemma 2.3. *The subspace* $V = V(L, B)$ *satisfies the following properties*:

(1) $\mathfrak{l} \subset V$.
(2) $\mathrm{Ad}(x)V = V$ *for all* $x \in B$.
(3) $\mathrm{Ad}(l)V = V$ *for all* $l \in L$.

Furthermore $V(L, B)$ *is the smallest subspace of* $\mathfrak{g}$ *with these properties.*

Proof. Clearly $\mathfrak{l} \subset V(L,B)$ by definition of the subspaces $V(s_1,\ldots,s_n)$. Now write $V(L,B) = V(t_1,\ldots,t_m)$ and take $x \in B$. Then by (4) we have $V(x,t_1,\ldots,t_m) = \mathfrak{l} + \mathrm{Ad}(x)V(t_1,\ldots,t_m)$, so that

$$V(L,B) = V(x,t_1,\ldots,t_m) = \mathrm{Ad}(x)V(t_1,\ldots,t_m),$$

showing the second statement. For the third property note that if $l \in L$ then by (3) we have $V(lt_1,\ldots,t_m) = \mathrm{Ad}(l)V(t_1,\ldots,t_m)$. So that $V(L,B)$ is $\mathrm{Ad}(l)$-invariant for every $l \in L$. Finally, if $V(L,B) = V(t_1,\ldots,t_m)$ then any subspace of $\mathfrak{g}$ satisfying the above properties must contain $V(t_1,\ldots,t_m)$, by definition of this subspace. □

Clearly, $V(L,B) = \mathfrak{g}$ if and only if there exists an n-tuple $\mathbf{x} \in B^n$ (some $n \geqslant 1$) such that the rank of $q_{\mathbf{x}}$ at some point is $\dim G$. Therefore, combining Lemma 2.3 and Proposition 2.1 we get at once the following criteria for $S(L,B)$ to have nonempty interior.

Theorem 2.4. *Suppose that $B \subset G$ is at most countable. Then $\operatorname{int} S(L,B) \neq \emptyset$ if and only if $V(L,B) = \mathfrak{g}$. Equivalently, $\operatorname{int} S(L,B) \neq \emptyset$ if and only if every subspace V of $\mathfrak{g}$ satisfying the three properties of Lemma 2.3 is equal to $\mathfrak{g}$.*

Corollary 2.5. *Suppose that $B \subset G$ is at most countable. Then $\operatorname{int} S(L,B) \neq \emptyset$ if and only if $\operatorname{int} G(L,B) \neq \emptyset$.*

Proof. Of course $\operatorname{int} G(L,B) \neq \emptyset$ if $\operatorname{int} S(L,B) \neq \emptyset$. On the other hand $G(L,B) = S(L, B \cup B^{-1})$ so that if $\operatorname{int} G(L,B) \neq \emptyset$ then $V(L, B \cup B^{-1}) = \mathfrak{g}$. But by finite dimensionality a subspace $V \subset \mathfrak{g}$ is $\mathrm{Ad}(x)$-invariant if and only if it is $\mathrm{Ad}(x^{-1})$-invariant. Hence $V(L,B) = \mathfrak{g}$ as well so that $\operatorname{int} S(L,B) \neq \emptyset$. □

Remark. The hypothesis that B is at most countable in Theorem 2.4 is required in the proof of the only if part alone. Note however that the proof of the corollary uses that the conditions are necessary and sufficient so that it needs the hypothesis that B is at most countable. We do not know if the corollary is true without this assumption.

As a consequence of the above criteria we derive an algebraic sufficient condition to be used later. Recall that there is a representation of L in the quotient space $\mathfrak{g}/\mathfrak{l}$ given by factorizing the adjoints $\mathrm{Ad}(l)$, $l \in L$. Also, let

$$N(\mathfrak{l}) = \{x \in G\colon \mathrm{Ad}(x)(\mathfrak{l}) = \mathfrak{l}\}$$

be the normalizer of $\mathfrak{l}$ in $\mathfrak{g}$.

Theorem 2.6. *Suppose that the quotient representation of L on $\mathfrak{g}/\mathfrak{l}$ is irreducible. Take $x \in G \setminus N(\mathfrak{l})$. Then $\operatorname{int} S(L,x) \neq \emptyset$.*

Proof. Let $V \subset \mathfrak{g}$ be a subspace satisfying the properties of Lemma 2.3. Since $x \notin N(\mathfrak{l})$, it follows that $V \neq \mathfrak{l}$. Hence its projection $\overline{V}$ on $\mathfrak{g}/\mathfrak{l}$ is not $\{0\}$ and is

L-invariant. By irreducibility we have $\overline{V} = \mathfrak{g}/\mathfrak{l}$, that is, $V = \mathfrak{g}$, concluding the proof. □

Corollary 2.7. *With the assumptions as in the theorem suppose that S is a semigroup of G containing L and not contained in $N(\mathfrak{l})$. Then* $\operatorname{int} S \neq \emptyset$.

Proof. Just apply the theorem to the coset Lx with $x \in S \setminus N(\mathfrak{l})$. Since $L \subset S$ we have $S(L, x) \subset S$ showing that $\operatorname{int} S \neq \emptyset$. □

Remark. A special case which will be discussed later is when there exists a subspace $\mathfrak{q} \subset \mathfrak{g}$ with $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{q}$ and such that $\mathfrak{q}$ is $\mathrm{Ad}(L)$-invariant. In this case the representations of L on $\mathfrak{q}$ and $\mathfrak{g}/\mathfrak{l}$ are isomorphic and hence the result of the above theorem holds if the adjoint representation of L on $\mathfrak{q}$ is irreducible.

Remark. By applying the inversion $x \mapsto x^{-1}$ of G it follows easily that the results above are also true for cosets xL, $x \in G$.

3. FLAG MANIFOLDS AND SEMIGROUPS

In the next section we apply Theorem 2.6 to semigroups in symmetric Lie groups. For that we recall here some concepts and results about flag manifolds and semigroups in semi-simple Lie groups. For more details we refer to [2,3,13] and [14] (for flag manifolds) and to [8] and [12] (for semigroups).

Let $\mathfrak{g}$ be a real noncompact semi-simple Lie algebra and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be a Cartan decomposition with $\mathfrak{k}$ a maximal compactly embedded subalgebra of $\mathfrak{g}$ and $\mathfrak{s}$ its orthogonal complement with respect to the Cartan–Killing form $\langle \cdot, \cdot \rangle$. Let $\mathfrak{a} \subset \mathfrak{s}$ be a maximal abelian subspace and denote by Π the set of restricted roots of the pair $(\mathfrak{g}, \mathfrak{a})$. Also let

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}\colon \mathrm{ad}(H)(X) = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}$$

be the root space corresponding to the root α. Select a simple system of roots $\Sigma \subset \Pi$ and let Π^+ and $\mathfrak{a}^+$ denote the corresponding set of positive roots and Weyl chamber, respectively. The subalgebras

$$\mathfrak{n}^+ = \sum_{\alpha \in \Pi^+} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{n}^- = \sum_{\alpha \in \Pi^-} \mathfrak{g}_\alpha$$

(where $\Pi^- = -\Pi^+$) are nilpotent and the Iwasawa decomposition of $\mathfrak{g}$ reads $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}^+$. Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. The standard minimal parabolic subalgebra of $\mathfrak{g}$ is $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}^+$. More generally let Θ be a subset of Σ, $\langle \Theta \rangle$ the set of all linear combinations of Θ, and $\langle \Theta \rangle^\pm = \langle \Theta \rangle \cap \Pi^\pm$.

Also, take the subalgebras

$$\mathfrak{n}^\pm(\Theta) = \sum_{\alpha \in \langle \Theta \rangle^+} \mathfrak{g}_{\pm\alpha} \quad \text{and} \quad \mathfrak{n}^\pm_\Theta = \sum_{\alpha \in \Pi^+ \setminus \langle \Theta \rangle^+} \mathfrak{g}_{\pm\alpha}.$$

The standard parabolic subalgebra associated with Θ is

$$\mathfrak{p}_{\Theta} = \mathfrak{n}^{-}(\Theta) + \mathfrak{p}.$$

Let $\mathfrak{a}(\Theta)$ be the subspace of $\mathfrak{a}$ generated by H_{α}, where $\alpha \in \Theta$ and H_{α} is defined by $\alpha(\cdot) = \langle H_{\alpha}, \cdot \rangle$. Furthermore, let $\mathfrak{a}_{\Theta}$ denote the orthogonal complement of $\mathfrak{a}(\Theta)$, in $\mathfrak{a}$, with respect to $\langle \cdot, \cdot \rangle$. Then we have the following decomposition:

$$\mathfrak{p}_{\Theta} = \mathfrak{l}_{\Theta} + \mathfrak{n}_{\Theta}^{+},$$

where $\mathfrak{m}_{\Theta} = \mathfrak{m} + \mathfrak{a}(\Theta) + \mathfrak{n}^{+}(\Theta) + \mathfrak{n}^{-}(\Theta)$ and $\mathfrak{l}_{\Theta} = \mathfrak{m}_{\Theta} + \mathfrak{a}_{\Theta}$.

If G is a Lie group with Lie algebra $\mathfrak{g}$ we put $K = \exp \mathfrak{k}$. The *parabolic subgroup* P_{Θ} is the normalizer of $\mathfrak{p}_{\Theta}$ in G. The *flag manifold* associated with Θ is defined by $\mathbb{F}_{\Theta} = G/P_{\Theta}$. The subgroup $P = P_{\emptyset}$ is a minimal parabolic subgroup, and $\mathbb{F} = G/P$ is a maximal flag manifold.

We consider now subsemigroups of G having nonempty interior and recall the results to be used later. To this purpose we give a rough picture of their actions on the flag manifolds of G. We do this by analogy with the action of one-parameter subgroups of G.

Thus let us take an element $H \in \mathfrak{a}$. It induces vector fields on the flag manifolds $\mathbb{F}_{\Theta}$ which we denote indistinctly by v_H. Clearly, the flow of v_H is the one-parameter subgroup $\exp tH$. Recall that v_H is a gradient vector field with respect to some K-invariant Riemannian metric on $\mathbb{F}_{\Theta}$ (see [2, Proposition 3.3]). As such its flow has either fixed points or noncompact orbits linking asymptotically the fixed points. There is recurrence inside the set of fixed points and the flow is transient outside it.

For instance, if H is split regular (that is, $\alpha(H) \neq 0$ for all root α) then v_H has a finite number of isolated singularities, namely the points $b_w = \overline{w} b_1$ with w running through the Weyl group $\mathcal{W}$. Here b_1 is the origin corresponding to the Weyl chamber $\mathfrak{a}^{+}$ containing H and $\overline{w}$ is a representative of w in the normalizer M^{*} of $\mathfrak{a}$ in K. In general we take $H \in \mathrm{cl}\mathfrak{a}^{+}$. Then the set of fixed points is given by the union of the orbits $K_H \cdot b_w$, $w \in \mathcal{W}$, where K_H is the centralizer of H in K. Each orbit $K_H \cdot b_w$ is a connected component of the set of fixed points and these components are parametrized by the set of double cosets $\mathcal{W}_H \backslash \mathcal{W} / \mathcal{W}_{\Theta}$, where $\mathcal{W}_H$ is the subgroup of $\mathcal{W}$ fixing H and $\mathcal{W}_{\Theta}$ is the subgroup generated by the simple reflections in Θ (see [2, Section 1]).

A fixed point component $K_H \cdot b_{w_1}$ precedes $K_H \cdot b_{w_2}$ if there exists a noncompact orbit flowing from the first component to the second one. This order the components in such a way that there is just one maximal component (that is, an attractor) which is the orbit $K_H \cdot b_1$ as well as a unique minimal component (a repeller) which is the orbit $K_H \cdot b_{w_0}$ where w_0 is the element of maximal length of $\mathcal{W}$.

Now, let $S \subset G$ be a semigroup with $\mathrm{int}\, S \neq \emptyset$. Then we obtain a similar picture for the S-action on the flag manifolds. Here, instead of the fixed point components we take the control sets of S (by definition a subset D with $\mathrm{int}\, D \neq \emptyset$ is a control set if $D \subset \mathrm{cl}(S \cdot x)$ for all $x \in D$ and D is maximal with this property (see [8] or [12])). In fact, the main results of [8] and [12] show the following facts:

1. The control sets for the S-action on a flag manifold $\mathbb{F}_\Theta$ are parametrized by a set of double cosets $\mathcal{W}_{H(S)}\backslash\mathcal{W}/\mathcal{W}_\Theta$ for some $H(S) \in \mathfrak{a}$, depending on S.
2. There exists just one invariant control set $C_\Theta \subset \mathbb{F}_\Theta$ (i.e. $S \cdot x \subset C_\Theta$ if $x \in C_\Theta$). The invariant control set plays the role of the attractor for the S-action.
3. In particular consider the maximal flag manifold $\mathbb{F}$ and the flag manifold $\mathbb{F}_{\Theta(S)} \approx K/K_{H(S)}$ (the K-adjoint orbit of $H(S)$) where $H(S)$ is given by (1). Then we have the following properties:
   (a) The invariant control set $C_{\Theta(S)} \subset \mathbb{F}_{\Theta(S)}$ is contractible in the sense that for every $h \in \operatorname{int} S$ the iterations $h^n C_{\Theta(S)}$ shrink to a single point (the attractor of h in $\mathbb{F}_{\Theta(S)}$).
   (b) The invariant control set $C \subset \mathbb{F}$ on the maximal flag manifold is $\pi^{-1}(C_{\Theta(S)})$, where $\pi : \mathbb{F} \to \mathbb{F}_{\Theta(S)}$ is the canonical fibration.

   These two properties together characterize the flag manifold $\mathbb{F}_{\Theta(S)}$ and a fortiori determines a choice of $H(S)$. The flag manifold $\mathbb{F}_{\Theta(S)}$ (or rather the conjugacy class of the corresponding parabolic subgroup $P_{\Theta(S)}$) is called the parabolic or flag type of S (see [11]). (Note that similar properties hold for the attractors on $\mathbb{F}$ and $\mathbb{F}_{\Theta(S)}$ for the single flow $\exp tH(S)$.)

In the particular case when $H(S) = 0$ the flag manifold $\mathbb{F}_{\Theta(S)}$ degenerates to a point. Then the above characterization of $\mathbb{F}_{\Theta(S)}$ ensures that the invariant control set C of S in $\mathbb{F}$ is the whole $\mathbb{F}$. This means that S acts transitively on $\mathbb{F}$. In this case and when G has finite center one can exploit the compactness of K to prove that S must be G. Let us describe the main ideas of this proof.

First note two elementary and general facts: (i) $S = G$ if G is a connected group and the indentity 1 of G belongs to $\operatorname{int} S$; (ii) An open semigroup of a compact group contains its identity component. Hence one gets $S = G$ if $\operatorname{int} S$ intersects a compact subgroup of G. If G is a semi-simple Lie group and has finite center there are easy ways of getting compact subgroups. Namely the closure of a one-parameter subgroup $\exp tX$ is compact if the eigenvalues of $\operatorname{ad}(X)$ are purely imaginary, i.e., X is elliptic. Here the finiteness of the center is essential.

On the other hand, let $Y \in \mathfrak{g}$ be a nilpotent element, i.e., $\operatorname{ad}(Y)$ is a nilpotent operator. Then an application of the Borel–Morozov theorem permits to prove that there exists an elliptic element X close enough to Y (see [8]). Therefore the strategy reduces to show the existence of a nilpotent element $Y \in \mathfrak{g}$ with $\exp Y \in \operatorname{int} S$, implying that there exists an elliptic element X with $\exp X \in \operatorname{int} S$ as well.

Here is where the transitivity of S on $\mathbb{F}$ comes in. In fact, by transitivity it follows the existence of $h = \exp H \in \operatorname{int} S$ with $H \in \mathfrak{a}^+$ as well as the existence of $h_1 n \in \operatorname{int} S$ with $h_1 = \exp H_1$, $H_1 \in -\mathfrak{a}^+$ and $n \in N = \exp \mathfrak{n}^+$. Then one can manage to get rid of the hyperbolic parts h and h_1 and show that $N \cap \operatorname{int} S \neq \emptyset$ (see [8], Theorem 3.5). Thus one concludes that $S = G$ if S is transitive on maximal flag manifold. Furthermore, there is the following result, proved in [12], exploiting the parabolic type of the semigroup and the above description of the action of S on an arbitrary flag manifold $\mathbb{F}_\Theta$.

Theorem 3.1. *Suppose that G is simple and has finite center. Let S be a semigroup of G with nonempty interior. If S is transitive on $\mathbb{F}_\Theta$, then $S = G$.*

Note that the assumption that G is simple is essential since semigroups of the form $S = G_1 \times S_2$ act transitively on flag manifolds of the type $(G_1 \times G_2)/(P_1 \times G_2)$, where G_1 and G_2 are simple Lie groups and P_1 is a parabolic subgroup of G_1.

Finally we mention that the above theorem is part of a series of similar results holding for semi-simple Lie groups. In fact, as shown in [10] it rarely occurs that a proper semigroup $S \subset G$ (with $\operatorname{int} S \neq \emptyset$ and G semi-simple with finite center) act transitively on a homogeneous space G/H.

4. SEMIGROUPS IN SYMMETRIC GROUPS

Let (G, τ) be a symmetric Lie group, where G is connected semi-simple and noncompact with Lie algebra $\mathfrak{g}$. Let G^τ denote the subgroup of τ-fixed points. Consider the canonical decomposition of $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{q}. \tag{5}$$

Also, let L be a subgroup of G such that $G_0^\tau \subset L \subset G^\tau$. We deal separately with the cases where the adjoint representation of L on $\mathfrak{q}$ is irreducible or not.

4.1. The irreducible case

If $\operatorname{Ad}(L)$ is irreducible on $\mathfrak{q}$, then by Corollary 2.7 (and the remark following it) a semigroup that contains L and is not contained in $N(\mathfrak{l})$ has nonempty interior in G. We will see now some cases where S cannot be proper.

We start by considering the case of Riemannian symmetric pairs. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be a Cartan decomposition of $\mathfrak{g}$ and θ the corresponding Cartan involution. Let K be the connected subgroup of G with Lie algebra $\mathfrak{k}$.

Theorem 4.1. *Suppose that G is simple and let $x \notin K$. Then the coset Kx generates G as a semigroup.*

Proof. Let S be a semigroup generated by Kx, $x \notin K$. Note that K is the normalizer of $\mathfrak{k}$ in G and $\operatorname{Ad}(K)$ is irreducible on $\mathfrak{s}$ (see [7]). Hence, Theorem 2.6 implies that $\operatorname{int} S \neq \emptyset$.

Now there exists $X \in \mathfrak{s}$ and $k \in K$ such that $x = k \exp X$, so that $Kx = K \exp X$. Without loss of generality we can assume that $X \in \mathfrak{a}$. Then by the Iwasawa decomposition $KxP = G$, which implies that S is transitive on the maximal flag manifold $\mathbb{F}$. Therefore by Theorem 3.1 we have $S = G$ if G has finite center.

To get the proof for general G we check first that the center $Z(G)$ of G is contained in S. The quotient $G/Z(G)$ is centerless and the semigroup $S/Z(G) \subset G/Z(G)$ contains the coset $(K/Z(G))x'$ with $x' = xZ(G)$ not in $K/Z(G)$. By the first part of the proof it follows that $S/Z(G) = G/Z(G)$.

Now, $xZ(G) = Z(G)x \subset S$ because $Z(G) \subset K$ (see [3]). Also, $x^{-1}Z(G) \cap S \neq \emptyset$ because $S/Z(G) = G/Z(G)$. Let $u_0 \in Z(G)$ be such that $x^{-1}u_0 \in S$ and take $u \in Z(G)$. Then

$$u = (x^{-1}u_0)(u_0^{-1}ux) \in S,$$

so that $Z(G) \subset S$.

Finally take $y \in G$. Then $yZ(G) \in S/Z(G)$ so that there exists $s \in S$ such that $yZ(G) = sZ(G)$, i.e., $s^{-1}y = \bar{s} \in Z(G) \subset S$. Hence $y = s\bar{s} \in S$, showing that $S = G$. □

If S is a semigroup that contains K properly, then there exists an element $x \in S$ such that $x \notin K$. Since S should contain the coset Kx we have the following consequence of Theorem 4.1.

Corollary 4.2. *If G is simple, then K is maximal as a semigroup of G.*

The extension of the above results to semi-simple groups is easy. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$ be the decomposition of $\mathfrak{g}$ into simple ideals and $\mathfrak{k}_i \subset \mathfrak{g}_i$ a maximal compactly embedded subalgebras in $\mathfrak{g}_i$ such that $\mathfrak{k} = \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_s$. Let K_i denote the connected subgroup with Lie algebra $\mathfrak{k}_i$.

Corollary 4.3. *If S is a semigroup containing K, then $S = A_1 \cdots A_s$ with $A_i = K_i$ or G_i, where G_i is the subgroup corresponding to $\mathfrak{g}_i$.*

Now we consider the affine symmetric spaces. Let θ be a Cartan involution that commutes with τ, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be the corresponding Cartan decomposition. Put $\mathfrak{k}_+ = \mathfrak{k} \cap \mathfrak{l}$, $\mathfrak{k}_- = \mathfrak{k} \cap \mathfrak{q}$, $\mathfrak{s}_+ = \mathfrak{s} \cap \mathfrak{l}$, $\mathfrak{s}_- = \mathfrak{s} \cap \mathfrak{q}$, and $\mathfrak{l}^a = \mathfrak{k}_+ + \mathfrak{s}_-$.

We say that the pair $(\mathfrak{g}, \tau)$ is *regular* if $\mathfrak{z}(\mathfrak{l}^a) \cap \mathfrak{s}_- \neq 0$, where $\mathfrak{z}(\mathfrak{h})$ denotes the center of $\mathfrak{h}$.

The following theorem is the main result of [9].

Theorem 4.4. *Let G be a simple Lie group with finite center. Suppose that $(\mathfrak{g}, \tau)$ is not regular. If S is a semigroup of G with $L \subset S$ and $\operatorname{int} S \neq \emptyset$, then $S = G$.*

Combining Theorem 4.4 and Theorem 2.6 we get at once the following result.

Theorem 4.5. *Let G be a simple Lie group with finite center. Suppose that $\mathrm{Ad}(L)$ is irreducible on $\mathfrak{q}$, and that $(\mathfrak{g}, \tau)$ is not regular. If S is a semigroup such that $L \subset S$ and S is not contained in $N(\mathfrak{l})$ then $S = G$.*

4.2. The reducible case

The reducible symmetric spaces such that the L-representation on $\mathfrak{q}$ is reducible have been classified in [5]. The properties of these space to be used here are contained in Lemma 1.3.4 of [4], which we reproduce here.

Lemma 4.6. *Let $(\mathfrak{g}, \tau)$ be an irreducible effective semi-simple symmetric pair. If $\mathfrak{q}$ is reducible then the following holds.*

(1) *$\mathfrak{q}$ splits into two irreducible components $\mathfrak{q}^+$ and $\mathfrak{q}^-$.*
(2) *The invariant subspaces $\mathfrak{q}^+$ and $\mathfrak{q}^-$ are isotropic for the Cartan–Killing form and are abelian subalgebras.*
(3) *The subalgebras $\mathfrak{p}^+ = \mathfrak{l} + \mathfrak{q}^+$ and $\mathfrak{p}^- = \mathfrak{l} + \mathfrak{q}^-$ are maximal parabolic and $\mathfrak{q}^{\pm}$ is the nilradical of $\mathfrak{p}^{\pm}$.*
(4) *The representations of $\mathfrak{l}$ on $\mathfrak{q}^+$ and $\mathfrak{q}^-$ are not isomorphic. In particular $\mathfrak{q}^+$ and $\mathfrak{q}^-$ are the only nontrivial invariant subspaces of $\mathfrak{q}$.*

The aim here is to determine conditions on $x \in G$ so that the subspace $V(L, x)$, defined in Section 2, is equal to $\mathfrak{g}$. Before stating the results we note that in the notations of Section 3 we have $\mathfrak{l} = \mathfrak{l}_\Theta$, $\mathfrak{q}^+ = \mathfrak{n}_\Theta^+$ and $\mathfrak{q}^- = \mathfrak{n}_\Theta^-$ if we take $\mathfrak{p}^+$ to be the standard parabolic subalgebra $\mathfrak{p}_\Theta$.

Put $N_\Theta^+ = \exp(\mathfrak{n}_\Theta^+)$ and $N_\Theta^- = \exp(\mathfrak{n}_\Theta^-)$ for the subgroups corresponding to the irreducible components of $\mathfrak{q}$.

Lemma 4.7. *Let $x \in G$ be such that $\mathrm{Ad}(x)(\mathfrak{l}) \subset \mathfrak{p}^+$ (resp. $\mathrm{Ad}(x)(\mathfrak{l}) \subset \mathfrak{p}^-$). Then $x \in N_\Theta^+ N(\mathfrak{l})$ (resp. $x \in N_\Theta^- N(\mathfrak{l})$).*

Proof. We have $\mathrm{Ad}(x)(\mathfrak{a}_\Theta) \cap \mathfrak{n}_\Theta^+ = \{0\}$, because the elements of $\mathfrak{a}_\Theta$ are semi-simple ($\mathfrak{a}_\Theta \subset \mathfrak{a} \subset \mathfrak{s}$), while the elements of $\mathfrak{n}_\Theta^+$ are nilpotent. Moreover, $\mathrm{Ad}(x)(\mathfrak{m}_\Theta) \cap \mathfrak{n}_\Theta^+ = \{0\}$. In fact, if $X \in \mathrm{Ad}(x)(\mathfrak{m}_\Theta) \cap \mathfrak{n}_\Theta^+$, then we have

$$\langle X, Y \rangle = 0 \quad \text{for all } Y \in \mathfrak{p}^+,$$

because $\mathfrak{p}^+ = (\mathfrak{n}_\Theta^+)^\perp$. Since the Cartan–Killing form of $\mathfrak{g}$ when restricted to $\mathrm{Ad}(x)(\mathfrak{m}_\Theta)$, is nondegenerate, we have $X = 0$. Hence, if $x \in G$ is such that $\mathrm{Ad}(x)(\mathfrak{l}) \subset \mathfrak{p}^+$, then $\mathfrak{p}^+ = \mathrm{Ad}(x)(\mathfrak{l}) + \mathfrak{n}_\Theta^+$. Consequently, there exists $n \in N_\Theta^+$ such that $\mathrm{Ad}(x)(\mathfrak{l}) = \mathrm{Ad}(n)(\mathfrak{l})$. Then $\mathrm{Ad}(n^{-1}x)(\mathfrak{l}) = \mathfrak{l}$. Consequently, $n^{-1}x \in N(\mathfrak{l})$ and $x \in N_\Theta^+ N(\mathfrak{l})$. Analogously, it is easy to see that if $\mathrm{Ad}(x)(\mathfrak{l}) \subset \mathfrak{p}^-$ then $x \in N_\Theta^- N(\mathfrak{l})$. □

Theorem 4.8. *Suppose that G is simple. Take $x \in G \setminus (N_\Theta^+ N(\mathfrak{l}) \cup N_\Theta^- N(\mathfrak{l}))$. Then the semigroup $S(L, x)$ generated by the coset Lx has nonempty interior in G.*

Proof. Keep the notation of the proof of Theorem 2.6. It is enough to show that if $x \notin N_\Theta^+ N(\mathfrak{l}) \cup N_\Theta^- N(\mathfrak{l})$ then $V(L, x) \cap \mathfrak{q} = \mathfrak{q}$.

It follows by Lemma 4.7 that $\mathrm{Ad}(x)(\mathfrak{l})$ is not contained in $\mathfrak{p}^+$ neither in $\mathfrak{p}^-$. Hence $V(L, x) \cap \mathfrak{q}$ is not contained in $\mathfrak{n}_\Theta^{\pm}$. Moreover, $V(L, x) \cap \mathfrak{q} \neq \{0\}$ because $x \notin N(\mathfrak{l})$. Since $V(L, x) \cap \mathfrak{q}$ is $\mathrm{ad}(\mathfrak{l})$-invariant we have by Lemma 4.6 (4) that $V(L, x) \cap \mathfrak{q} = \mathfrak{q}$. □

Combining Theorem 4.4 and Theorem 4.8 we get at once the following result.

Theorem 4.9. *Assume that G is simple, has finite center and $(\mathfrak{g}, \tau)$ is not regular. If S is a semigroup of G such that $L \subset S$ and S is not contained in $N_\Theta^+ N(\mathfrak{l}) \cup N_\Theta^- N(\mathfrak{l})$, then $S = G$.*

ACKNOWLEDGEMENT

The authors express their gratitude to Prof. J.J. Duistermaat for the helpful suggestions made during the reviewing process.

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(Received July 2005)