ADVANCES IN MATHEMATICS 110, 175–190 (1995)

Heat Diffusion on Homogeneous Trees

MAURO PAGLIACCI*

Facoltà di Economia, Università della Calabria, 87036 Arcavacata di Rende (CS), Italy

AND

MASSIMO A. PICARDELLO†

Dipartimento di Matematica, Università di Roma "Tor Vergata," Via della Ricerca Scientifica, 00133 Rome, Italy; and University of Maryland, College Park, Maryland 20742

Let X be a homogeneous tree. We study the heat diffusion process associated with the nearest neighbour isotropic Markov operator on X. In particular it is shown that the heat maximal operator is weak type $(1, 1)$ and strong type (p, p), for every $1 < p < \infty$. We estimate the asymptotic behaviour of the heat maximal function. Moreover, we introduce a family of H^p spaces on X. It is proved that $H^p = l^p(X)$ for $1 < p < \infty$ and is conjectured that H^p, for p less than 1, is trivial.

1. HEAT MAXIMAL OPERATOR ON TREES

This paper deals with the following question. We are given a homogeneous tree whose edges have the same thermal conductivity. Suppose that one vertex is at a positive temperature at time zero, whereas the others are at temperature zero. What is the highest temperature that the generic vertex x attains?

The asymptotic behaviour, for $n \to \infty$, of the temperature distribution $\varphi(x, n)$ at time n is given by the local limit theorem, extensively studied in the literature [Sa, Pi, Ge]. These asymptotic estimates are not sufficient to yield bounds for the highest temperature $\Phi(x) = \sup_{n \geq 0} \varphi(x, n)$. Explicit

* Partially supported by "Gruppo Nazionale di Analisi Funzionale"; e-mail: map@ipguniv.bitnet, map@ccuscl.unical.it.

† Partially supported by "Gruppo Nazionale di Analisi Funzionale" and CNR-Nato Senior Fellowship 117/25; e-mail: picard@irmtvm51.bitnet, picard@vax.mat.utovrm51.bitnet.

0001-8708/95 $9.00

formulas for $\varphi(x, n)$ are given in [Gr, LM, Pa] but these formulas seem inadequate to answer our question. This paper gives an estimate for $\Phi(x)$ and investigates its role as a maximal operator on trees, which can be of interest to introduce and study suitable Hardy spaces.

More precisely, X is a homogeneous tree, that is, a graph without loops, and with the same number of edges joining each vertex. We denote this number by $q+1$, with $q \geqslant 1$. The case $q=1$ corresponds to a linear tree, isomorphic to the integers. The case $q>1$ gives rise to a nontrivial tree (which is a transitive and simply transitive homogeneous space under groups of type $*_{i=1}^{q+1} \mathbb{Z}_2$ and other free products and free groups: see [BP]). We write $x \sim y$ if two vertices x and y are *neighbours*, i.e., if they are joined by an edge of X.

Let us equip X with the isotropic nearest neighbour transition operator P, defined by $p(x, y) = 1/(q+1)$, if $x \sim y$, $p(x, y) = 0$ otherwise. We often regard P as an operator acting on functions defined on the vertices of X, by the rule $Pf(x) = \sum_y p(x, y) f(y)$. By iteration, P gives rise to a semigroup. Regarding P as the generator of a *Brownian motion on X with discrete time*, i.e., a random walk, it is natural to think of its associated semigroups as a heat diffusion semigroup. In other words, we may identify the semigroup with the countabe diffusion process where all the heat is concentrated at a reference vertex o at time 0 and, at time n, the temperature of a vertex x is exactly $p^{(n)}(o, x)$ (here $P^{(n)}$ denotes the nth iterate of P). In other words, the map $(n, x) \mapsto p^{(n)}(o, x)$ is the fundamental solution of the heat equation with singularity in $x=0$ for $n=0$. The reader is referred to [Pa] for more details on the heat equation on homogeneous trees, and for an explicit formula for its fundamental solution; see also [LM].

It is now natural to introduce a *heat maximal operator*. For $f \in l^1(X)$ (the space of summable functions defined on the vertices of X), define

$$\mathscr{M}f(x) = \sup_{n \geqslant 0} (P^{(n)}f(x)) = \sup_{n \geqslant 0} \left(\sum_{y \in X} p^{(n)}(x, y) f(y) \right).$$

It is not easy to obtain explicit formulas for this nonlinear operator. But we can estimate $\mathscr{M}$ by introducing another linear operator M which dominates it,

$$Mf(x) = \sum_{y \in X} \left(\sup_{n \geqslant 0} p^{(n)}(x, y) \right) f(y).$$

The operator M is linear. Recall that X is a simply transitive homogeneous space under a suitable countable group Γ and P is a group invariant transition operator. Then M is naturally identified with the left convolution operator (under the group Γ) by the *heat maximal function*

$$\Phi(x) = \sup_{n \geqslant 0} p^{(n)}(o, x).$$

Obviously,

$$|\mathcal{M}f(x)| \leqslant M(|f|)(x) = (\Phi * |f|)(x).$$

Therefore any l^p or weak l^p estimate that we may be able to prove for M will also hold for $\mathcal{M}$. Actually, with abuse of terminology, from now on we refer to M as the heat maximal operator on X.

Some preliminary estimates on M are essentially known.

PROPOSITION 1.1. *The operator M is of weak type $(1, 1)$, and it is bounded on $l^p(X)$, for every $1 < p < \infty$.*

COROLLARY 1.2. *$\mathcal{M}$ is weak type $(1, 1)$ and strong type (p, p), for every $1 < p < \infty$.*

Proof of Proposition 1.1. Let us denote by $G(x, y)$ the Green function with singularity at $x = 0$. That is, $G(x, y) = \sum_{n \geqslant 0} p^{(n)}(x, y)$ (the expected number of visits to the vertex x of the random walk generated by P, starting at x) is the fundamental solution of the "Laplace operator" $P - \mathbb{I}$ (here $\mathbb{I}$ denotes the identity operator):

$$(P - \mathbb{I})\, Gf = -f, \qquad \text{for every } f \in l^1(X).$$

By Theorem 21 of [RT], the operator G is weak type $(1, 1)$ and bounded in $l^p(X)$ for $1 < p < \infty$. On the other hand,

$$\Phi(x) = \sup_n p^{(n)}(o, x) \leqslant \sum_{n=0}^{\infty} p^{(n)}(o, x) = G(o, x).$$

Hence G dominates M and also M is bounded in l^p, $1 < p < \infty$, and weak type $(1, 1)$. ∎

By its definition, the heat maximal function has a natural interpretation in the heat diffusion model introduced above: $\Phi(x)$ represents the maximum temperature reached at the vertex x if heat is concentrated in the vertex o. By the local limit theorem (see [Sa, Pi]), the temperature of the vertex x decays, when the time n grows, as $C(x)(2\sqrt{q}/(q+1))\, n^{-3/2}$, where $C(x)$ is independent of n. But this does not say anything about the *maximum* temperature attained by x.

In Section 2 we determine exactly the asymptotic behaviour of $\Phi(x)$. In particular, we show that $\Phi \in l^p$, for every $p > 1$, but $\Phi \notin l^1$. Here is a sketch of our approach towards an estimate for Φ. We reduce the diffusion process to an analogous process on $\mathbb{N}$ with transition rules $r(n, n-1) = (1/(q+1), r(n, n+1) = q/(q+1)$ for $n \geqslant 1$, and $r(0, 1) = 1$. This process, in turn, is approximated with the shift invariant process on $\mathbb{Z}$ given by $r(n, n-1) = 1/(q+1)$, $r(n, n+1) = q/(q+1)$, for every n. The latter process can be studied explicitly, and by a careful control of the approximation errors, we are able to determine how much time elapses before x reaches its maximum temperature, after being reached by the "heat wave" generated at the vertex o at time 0. The exponential growth of X might suggest that heat disperses very quickly, and therefore this time delay should be small, much smaller than the time needed for the heat to reach x (which is exactly $|x| = \mathrm{dist}(o, x)$, of course). But, surprisingly, this is not so: the time delay is of the order of $|x|/(q-1)$, linear in $|x|$. Because of this fact, Φ is not so small as to belong to $l^1(X)$, although it does belong to l^p, for every $1 < p < \infty$.

A potential application of this estimate is a new theory of H^p spaces on X. H^p spaces associated to a large class of transitive operators on general (not necessarily homogeneous) trees were studied in [KPT, DP], as spaces of harmonic functions on the vertices of the tree. The boundary values of these functions may be regarded either as functions or as "distributions" on the boundary Ω of the tree. Our maximal operator gives rise to a different notion of H^p, whose functions are defined on the vertices of X but are not harmonic. In a natural sense they may be extended to "harmonic" functions on the "half-space" $X \times \mathbb{N}$. In other words, H^p-functions on X now play the role of boundary values of a suitable class of harmonic function defined on a larger space, whose boundary is X.

We say that a function $f \in l^1(X)$ belongs to H^p if Mf belongs to $l^1(X)$.

Since $\Phi \in l^p(X)$, for every $p > 1$, the Dirac masses all belong to H^p, for $p > 1$, hence $H^p = l^p(X)$, for $p > 1$. But $\Phi \notin l^1(X)$ and a function f may belong to H^1 only if it has enough cancellation to allow $\Phi * f$ to decay at infinity at a sufficiently fast rate. By analogy with some symmetric spaces, one may expect H^p to be trivial (that is, $H^p = \{0\}$), for $p < 1$. In Section 3 we gather some computational evidence in support of this conjecture, by producing a large class of typical test functions in $l^p(X)$ that do not belong to H^p.

2. An Estimate for the Heat Maximal Function

For two functions f and g of a variable x we use the notation $f \approx g$ if there exist constants C_1, C_2 with $0 < C_1 < C_2$ such that

$$C_1 g(x) \leqslant f(x) \leqslant C_2 g(x), \qquad \text{for every } x.$$

THEOREM 2.1 (An estimate for the Heat Maximal Function). *There exists a sequence* $\{h_j\}$, *with* $h_j \approx j^{-1/2}q^{-j}$ *such that, if* $|x| = j$, *we have*

$$\Phi(x) = h_j + O(j^{-3/2}q^{-j}).$$

Proof. We split the proof into several lemmas. The basic idea of the proof is that, because of the isotropy, the random walk generated by μ_1 can be considered as a random walk on $\mathbb{N}$ with one-step transition probabilities given by

$$r(0, 1) = 1, \qquad r(n, n+1) = \frac{q}{q+1}, \qquad r(n, n-1) = \frac{1}{q+1} \qquad (\text{for } n \in \mathbb{N}).$$

Let $r_+ = r(n, n+1)$ and $r_- = r(n, n-1)$. Let R be the operator on $\mathbb{N}$ given by the above transition probabilities. Then $P^{(n)}(x)$, the nth convolution power of P, is given by

$$p^{(n)}(o, x) = \frac{r^{(n)}(0, |x|)}{w_n},$$

where $w_n = (q+1)\, q^{n-1}$ is the cardinality of the set W_n of vertices at distance n from the origin of X. The random walk on $\mathbb{N}$ generated by R was studied in [Gr].

We consider the transition invariant operator $\tilde{R}$ on $\mathbb{Z}$ given by $\tilde{r}(n, n+1) = r_+$, $\tilde{r}(n, n-1) = r_-$ and we compare the probabilities of getting from 0 to j under the operators R and $\tilde{R}$, both regarded as operators on $\mathbb{Z}$. Observe that all paths from 0 to j in n steps, with $n \geqslant j$, must include loops. That is, there are k edges in $\mathbb{Z}$ that are crossed at least twice in opposite directions. Note that there are exactly $\binom{n}{k}$ such paths and $n = j + 2k$. Each such path yields the same contribution to the probability $\tilde{r}^{(n)}(0, j)$: namely $r_+^{j+k} r_-^k$. Therefore

$$\tilde{r}^{(j+2k)}(j) = \binom{j+2k}{k} r_+^{j+k} r_-^k = \binom{j+2k}{k} \frac{q^{j+k}}{(q+1)^{j+2k}}.$$

LEMMA 2.2. *For each* x, *with* $|x| = j$, *one has*

$$\sup_{n \geqslant 0} r^{(n)}(j) \approx \sup_{n \geqslant 0} \tilde{r}^{(n)}(j).$$

Proof. To prove the lemma we make use of an explicit formula for $r^{(n)}(j)$ given in [Gr, Lemma 4.1]. We have

$$r^{(j+2k)}(j) = (r_+ r_-)^{j/2+k} \left(\frac{r_+}{r_-}\right)^{j/2}$$

$$\times \left\{ \binom{j+2k}{k} + \frac{r_+}{r_-} \binom{j+2k}{j+k} - \frac{r_+ - r_-}{r_+ r_-} \sum_{t=1}^{k} \left(\frac{r_+}{r_-}\right)^t \binom{j+2k}{j+k+t} \right\}$$

$$= \frac{q^{j+k}}{(q+1)^{j+2k}} \left\{ \frac{q+1}{q} \binom{j+2k}{k} - \frac{q^2-1}{q} \sum_{t=1}^{k} q^{-t} \binom{j+2k}{j+k+t} \right\}$$

$$\leqslant \frac{q+1}{q} \tilde{r}^{(j+2k)}(j).$$

Moreover,

$$r^{(j+2k)}(j) = \frac{q^{j+k}}{(q+1)^{j+2k}} \left\{ \binom{j+2k}{k} + \frac{1}{q} \binom{j+2k}{k} - \frac{q^2-1}{q} \sum_{t=1}^{k} q^{-t} \binom{j+2k}{1+k+t} \right\}$$

$$\geqslant \frac{q^{j+k}}{(q+1)^{j+2k}} \left\{ \binom{j+2k}{k} - \frac{q^2-1}{q} \sum_{t=1}^{k} q^{-t} \binom{j+2k}{j+k+t} \right\}.$$

But, since $t \geqslant 1$ and $j+k > n/2 = (j+2k)/2$, we have

$$\binom{j+2k}{j+k+t} < \binom{j+2k}{j+k} = \binom{j+2k}{k}.$$

Therefore

$$r^{(j+2k)}(j) \geqslant \frac{q^{j+k}}{(q+1)^{j+2k}} \binom{j+2k}{k} \left\{ 1 - \frac{q^2-1}{q} \sum_{t=1}^{k} q^{-t} \right\}$$

$$= \tilde{r}^{(j+2k)}(j) \left\{ 1 - \frac{q+1}{q} (1 - q^{-k}) \right\}$$

$$\geqslant \frac{1}{q} \tilde{r}^{(j+2k)}(j). \quad \blacksquare$$

To prove the theorem we must find the value of $n = n(j)$ for which $\tilde{r}^{(n)}(j)$ attains the maximum, i.e., we must find the maximum point, denoted by k_0, for the function

$$k \mapsto \binom{n}{k} r_+^{j+k} r_-^k .$$

For such k_0 we must have

$$\binom{n+2}{k+1} r_+^{j+k+1} r_-^{k+1} < \binom{n}{k} r_+^{j+k} r_-^k , \tag{1}$$

but

$$\binom{n}{k} r_+^{j+k} r_-^k > \binom{n-2}{k-1} r_+^{j+k-1} r_-^{k-1}.$$

Set

$$B(j, k) = \binom{n}{k} \Big/ \binom{n+2}{k+1} = \frac{(k+1)(j+k+1)}{(j+2k+2)(j+2k+1)} .$$

Then (1) is equivalent to

$$\frac{q}{(q+1)^2} = r_- r_+ < B(j, k). \tag{2}$$

LEMMA 2.3. *For $j \geqslant 2$, the function*

$$k \mapsto B(j, k), \qquad \textit{for} \quad k \in \mathbb{N}$$

has a unique extreme point.

Proof. For $k \in \mathbb{Z}$

$$\frac{d}{dk} B(j, k) = \frac{-2k^2 + 2(j^2 - j + 2) k + j^3 + j^2 - 2j - 2}{(j+2k+2)^2 (j+2k+1)^2} .$$

Therefore, for $k = 0$,

$$\operatorname{sgn}\left(\frac{d}{dk} B(j, k)\right) = \operatorname{sgn}(j^3 + j^2 - 2j - 2)$$

and this is positive if $j \geqslant 2$. Moreover, if $k \to \pm\infty$, the numerator in $(d/dk)\, B(j, k)$ tends to $-\infty$. Therefore $(d/dk)\, B(j, k) = 0$ has exactly two real solutions, only one of which is positive. ∎

Let $c = q/(q+1)^2$.

COROLLARY 2.4. *For each $j \geq 2$, the equation*

$$B(j, k) = c \tag{3}$$

has only one positive (real) solution.

Proof. For every $q \geq 1$,

$$c < \tfrac{1}{4} = \lim_{k \to +\infty} B(j, k).$$

In the proof of Lemma 2.3 we showed that for $j \geq 2$, $B(j, k)$ is monotonically decreasing for large k. Therefore, if $B(j, k_0) = c$, then $B(j, k)$ must have a maximum point for some $k > k_0$. Now, if there were two solutions k_1, k_2 of the equation in the statement, then $(d/dk)\, B(j, k) = 0$ for some $k' \in [k_1, k_2]$ and for some other $k'' > k_2$. ∎

This proves that there exists exactly one value k_0 of k such that (2) is true for k_0 but not for $k_0 - 1$.

Let us denote by $\tilde{k}_0 \in \mathbb{R}^+$ the positive solution of (3), which is given by

$$\tilde{k}_0 = \tilde{k}_0(j) = -\frac{1}{2\beta}(\alpha + \beta j - \sqrt{\gamma + \beta j^2}),$$

where $\alpha = 2(1 - 3c)$, $\beta = 1 - 4c$, and $\gamma = 4c^2$.

Now we ask the following question: How long does it take for a vertex of the tree to reach the maximum temperature, after having been hit by a "heat wave" originating at the reference vertex o? The next lemma gives an approximate answer (the expected delay computed in its statement is real, but not necessarily integer).

LEMMA 2.5. $\tilde{k}_0 = \tilde{k}_0(j) = j/(q-1) + K_q + O(j^{-1})$, *where*

$$K_q = \frac{(q^2 - q + 1)}{(q-1)^2}.$$

Proof. Observe first that

$$\frac{j}{q-1} = -\frac{1}{2\beta}(\beta j - \sqrt{\beta j^2}).$$

Hence,

$$\tilde{k}_0(j) - \frac{j}{q-1} = \tilde{k}_0(j) - \left(-\frac{1}{2\beta}(\beta j - \sqrt{\beta j^2})\right)$$

$$= \frac{1}{2\beta}(\sqrt{\beta}\, j(\sqrt{1 + \gamma/(\beta j^2)} - 1) - \alpha)$$

$$= -\frac{\alpha}{2\beta} + \frac{1}{2\beta}\left(\sqrt{\beta}\, j\left(\frac{\gamma}{2\beta j^2}\right)\right) + O(j^{-4})$$

$$= -\frac{\alpha}{2\beta} + O(j^{-1}) = \frac{q^2-q+1}{(q-1)^2} + O(j^{-1}). \quad \blacksquare$$

We want to estimate $\Phi(x) = \sup_{n \geqslant 0} p^{(n)}(o, x)$. By Lemma 2.2, if $|x| = j$,

$$\Phi(x) = p^{(j+2k_0)}(o, x) \approx \frac{\tilde{r}^{(j+2k_0)}(j)}{w_n} = \frac{\tilde{r}^{(j+2k_0)}(j)}{(q+1)\, q^{j-1}}.$$

But

$$\tilde{r}^{(j+2k_0)}(j) = \binom{j+2k_0}{k_0} \frac{q^{j+k_0}}{(q+1)^{j+2k_0}}. \tag{4}$$

Let $\vartheta_j = [j/(q-1)]$. Then

$$\tilde{r}^{(j+2k_0)}(j) = \binom{j+2\vartheta_j}{\vartheta_j} \frac{q^{j+\vartheta_j}}{(q+1)^{j+2\vartheta_j}}.$$

With notation as in (4), it is easy to check that

$$\frac{\tilde{r}^{(j+2(k_0+1))}(j)}{\tilde{r}^{(j+2k_0)}(j)} = \frac{1}{B(j, k_0)} \frac{q}{(q+1)^2},$$

and

$$\frac{\tilde{r}^{(j+2(k_0-1))}(j)}{\tilde{r}^{(j+2k_0)}(j)} = B(j, k_0 - 1) \frac{(q+1)^2}{q}.$$

Therefore, by Lemma 2.3, $\tilde{r}^{(j+2(k_0(j)) \pm 1))}(j)/\tilde{r}^{(j+2k_0(j))}(j)$ is bounded above and below with respect to j. By Lemma 2.5, $k_0 = \vartheta_j + K_q + O(j^{-1})$. Iterating the above argument we see that

$$\tilde{r}^{(j+2k_0)}(j) \approx \tilde{r}^{(j+\vartheta_j)}(j).$$

Now it is enough to give an estimate of the binomial coefficient $\binom{j+2\vartheta_j}{\vartheta_j}$. We achieve this by making use of the Stirling formula

$$\sqrt{2\pi n}\, n^n e^{-n} < n! < \sqrt{2\pi n}\, n^n e^{-n} e^{1/4n}.$$

Let

$$\lambda(j) = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{j+2\vartheta_j}}{\sqrt{(j+\vartheta_j)\,\vartheta_j}} \frac{(1+2\vartheta_j)^{1+2\vartheta_j}}{(1+\vartheta_j)^{1+\vartheta_j}\, \vartheta_j^{\vartheta_j}}. \tag{5}$$

The Stirling formula yields

$$\lambda(j)\exp\left(\frac{1}{4}\left(\frac{1}{j+\vartheta_j}+\frac{1}{\vartheta_j}\right)\right)<\binom{j+2\vartheta_j}{\vartheta_j}<\lambda(j)\exp\left(\frac{1}{4(j+2\vartheta_j)}\right). \qquad (6)$$

LEMMA 2.6. *Let*

$$A(s)=\frac{(1+2s)^{1+2s}}{(1+s)^{1+s}\,s^s}.$$

Then $A(s)\approx A[s]$, *if* $0<s<K$, *with constants depending only on* K.

Proof. It is enough to show that, for every $s>0$, $(1+s)^{1+s}\approx(1+[s])^{1+[s]}$, and $s^s\approx[s]^{[s]}$. On the other hand, $(1+s)^{1+s}$ is monotone increasing for $s>0$, and s^s is bounded if $0\leqslant s\leqslant 1$. Therefore it suffices to show that, for $s>0$,

$$(1+s)^{1+s}\approx(1+(s+1))^{1+(s+1)}=(2+s)^{2+s}.$$

But

$$\frac{(2+s)^{2+s}}{(1+s)^{1+s}}=\left(1+\frac{1}{1+s}\right)^{1+s}(2+s),$$

and the lemma follows by the fact that $(1+1/(1+s))^{1+s}$ is bounded with respect to s. ∎

For j large enough, Lemma 2.6 allows one to replace ϑ_j with $j/(q-1)$ in (5) and (6). Therefore we obtain the following corollary.

COROLLARY 2.7.

$$\lambda(j)=C_q\frac{1}{\sqrt{j}}\frac{(q+1)^{j(q+1)/(q-1)}}{q^{jq/(q-1)}},$$

where $C_q=(1/\sqrt{2\pi})\sqrt{(q^2-1)/q}$ *and*

$$\lambda(j)\exp\left(\frac{1}{4j}\frac{q-1}{q+1}\right)<\binom{1+2\vartheta_j}{\vartheta_j}<\lambda(j)\exp\left(\frac{1}{4j}\frac{q^2-1}{q}\right).$$

End of the proof of the Theorem. We show that

$$\lambda(j)(1-\varepsilon(j))<\binom{1+2\vartheta_j}{\vartheta_j}<\lambda(j)(1+\varepsilon(j)),$$

where $\varepsilon(j) = O(j^{-1})$. Indeed,

$$\exp\left(\frac{1}{4j}\frac{q^2-1}{q}\right) = 1 + \frac{1}{4j}\frac{q^2-1}{q} + O(j^{-2}),$$
$$\exp\left(\frac{1}{4j}\frac{q-1}{q+1}\right) = 1 + \frac{1}{4j}\frac{q-1}{q+1} + O(j^{-2}),$$

hence

$$\left|\binom{1+2\vartheta_j}{\vartheta_j} - \lambda(j)\right| \leqslant \lambda(j)\left(\exp\left(\frac{1}{4j}\frac{q^2-1}{q}\right) - \exp\left(\frac{1}{4j}\frac{q-1}{q+1}\right)\right)$$
$$= \lambda(j)\left(\frac{1}{4j}\frac{q^3-1}{q(q+1)} + O(j^{-2})\right).$$

For j large enough, these estimates imply that

$$\lambda(j)\frac{q^{j+k_0}}{(q+1)^{j+2k_0}}(1-\varepsilon(j)) < \tilde{r}^{(j+2k_0)}(j) < \lambda(j)\frac{q^{j+k_0}}{(q+1)^{j+2k_0}}(1+\varepsilon(j)),$$

with $\varepsilon(j) = O(j^{-1})$. In other words,

$$\tilde{r}^{(j+2k_0)}(j) = \lambda(j)\frac{q^{j+k_0}}{(q+1)^{j+2k_0}} + O\left(\frac{\lambda(j)}{j}\frac{q^{j+k_0}}{(q+1)^{j+2k_0}}\right)$$
$$= C_q\frac{1}{\sqrt{j}}\frac{(q+1)^{j(q+1)/(q-1)}}{q^{jq/(q-1)}} + O(C_q j^{-3/2})$$
$$= \frac{C_q}{\sqrt{j}} + O(j^{-3/2}).$$

Hence

$$\Phi(x) \approx \frac{\tilde{r}^{(j+2k_0)}(j)}{q^j} = \frac{C_q}{j^{1/2}q^j} + O(j^{-3/2}q^{-j}),$$

and the theorem is proved. ▮

COROLLARY 2.8 (l^p-behaviour). *$\Phi \in l^p$, for every $p > 1$, but $\Phi \notin l^1$.*

3. A CONJECTURE ON THE RADIAL H^p SPACE ASSOCIATED WITH THE HEAT MAXIMAL FUNCTION

Let

$$H_h^p(X) = \{f \in l^p(X) : \Phi * f \in l^p(X)\}.$$

Equipped with the l^p-norm, H_h^p is a closed subspace of l^p. Note that this is not the H^p space of [KPT].

For $p > 1$, $\Phi \in l^p(X)$, so $\delta_x \in H_h^p(X)$, for every $x \in X$. Hence H_h^p contains all finitely supported functions, for $p > 1$. Therefore $H_h^p = l^p$, for every $p > 1$.

But $\Phi \notin l^1$. What is H_h^1? More generally, what is $H_h^p(X)$, for $0 < p \leqslant 1$?

We conjecture that, at least for $p < 1$, the only radial function in $H_h^p(X)$ is the zero function. Here is some evidence.

Let f be a radial function in l^p. If $|x| = n$, write

$$f(x) = \frac{1}{q^n} h_n.$$

Therefore $h = \{h_n\}$ is a sequence in l^p. Now, by writing f_n instead of $f(x)$ when $|x| = n$, a direct computation shows that, for $|x| = n$,

$$\Phi * f(x) = \sum_{j=1}^{n-1} \left(f_{n-j} + \frac{q-1}{q} \sum_{l=1}^{j-1} q^l f_{n-j+2l} + q^j f_{n+j} \right) \Phi(j)$$
$$+ \sum_{k=n}^{\infty} \left(q^{k-n} f_{k-n} + \frac{q-1}{q} \sum_{l=1}^{n-1} q^{k-n+l} f_{k-n+2l} + q^k f_{n+k} \right) \Phi(k).$$

By Theorem 2.1 this amounts to

$$\Phi * f(n) \approx f_n + \frac{1}{q^n} \sum_{j=1}^{n-1} \frac{1}{\sqrt{j}} \left(h_{n-j} + \frac{q-1}{q} \sum_{l=1}^{j-1} q^{-l} h_{n-j+2l} + q^{-j} h_{n+j} \right)$$
$$+ \frac{1}{q^n} \sum_{k=n}^{\infty} \frac{1}{\sqrt{k}\, q^{k-n}} \left(h_{k-n} + \frac{q-1}{q} \sum_{l=1}^{n-1} q^{-l} h_{k-n+2l} + q^{-k} h_{n+k} \right)$$
$$= f_n + I_1(n) + I_2(n).$$

Hence

$$\Phi * f \in l^p \Leftrightarrow \sum_{n \geqslant 0} q^n |I_1(n) + I_2(n)|^p < \infty.$$

For the sake of concreteness, the sequence h is in l^p. We confine ourselves to the case where $|h|$ is non-increasing. Then

$$\left| \sum_{l=1}^{j-1} q^{-l} h_{n-j+2l} \right| \approx h_{n-j}.$$

By the same token,

$$q^{-j}h_{n+j} \approx h_{n-j}.$$

Hence,

$$I_1(n) \approx \frac{1}{q^n} \sum_{j=1}^{n-1} \frac{1}{\sqrt{j}} h_{n-j}. \tag{7}$$

Now, h_n must vanish at infinity faster than $1/n$. Suppose, to begin with, that h_n decays slowly, that is,

$$|h_n| = n^{-(1/p+\beta)} \ (\beta > 0), \qquad \text{or} \qquad |h_n| = n^{-1/p}(\log n)^{-(1/p+\beta)} \ (\beta > 0),$$

or something similar. Then in the former case,

$$\begin{aligned}
\left| \sum_{j=1}^{n-1} \frac{1}{\sqrt{j}} h_{n-j} \right| &\leqslant \sum_{j=1}^{n-1} j^{-1/2}(n-j)^{-1/p-\beta} \\
&= n^{-1/2-1/p-\beta} \sum_{j=1}^{n-1} \left(\frac{j}{n}\right)^{-1/2} \left(1-\frac{j}{n}\right)^{-1/p-\beta} \\
&\leqslant n^{-3/2-1/p-\beta} \int_0^{(n-1)/n} x^{-1/2}(1-x)^{-1/p-\beta}\, dx \\
&\leqslant n^{-3/2-1/p-\beta} \left(C_1 + \frac{1}{\sqrt{2}} \int_{1/2}^{1-1/n} (1-x)^{-1/p-\beta}\, dx \right) \\
&= n^{-3/2-1/p-\beta}(C_1 - C_2 n^{-1/p-\beta-1}) \\
&\approx n^{-3/2-1/p-\beta},
\end{aligned}$$

where C_1 and C_2 are constants.

Therefore,

$$|q^n I_1^p(n)| \approx q^{n(1-p)} n^{-(\beta+3/2)p-1},$$

where C is a constant. The other cases yield the same estimate.

On the other hand, by the same argument, we see that

$$I_2(n) \approx \frac{1}{q^n} \sum_{k=n}^{\infty} \frac{1}{\sqrt{k}\, q^{k-n}} h_{k-n}. \tag{8}$$

As we are assuming that h_n decays polynomially, it follows from (8) that

$$I_2(n) \approx n^{-1/2} q^{-n}.$$

Therefore $I_1(n)/I_2(n) \to 0$ when $n \to \infty$, and

$$\sum_{n \geq 0} q^n |I_1(n) + I_2(n)|^p \approx \sum_{n \geq 0} q^n |I_2(n)|^p = \infty,$$

for every $p \leq 1$.

Hence $\Phi * f \notin l^p$, for $p \leq 1$, if $h_n = q^n f_n$ decays polynomially.

Now let us assume that h_n decays faster, say

$$|h_n| = \frac{1}{q^{\alpha n}} \frac{1}{n^\beta}, \qquad \text{for some} \quad \alpha > 0,\ \beta > 0.$$

Actually, since α is arbitrary, it is enough to look at the case $|h_n| = 1/q^{\alpha n}$. Then, by (7),

$$\begin{aligned} |I_1(n)| &\approx \frac{1}{q^n} \sum_{j=1}^{n-1} \frac{q^{\alpha(j-n)}}{\sqrt{j}} \left(1 + \frac{q-1}{q} \sum_{l=1}^{j-1} q^{-(1+2\alpha)l} + q^{-(1+2\alpha)j}\right) \\ &\approx \frac{1}{q^n} \sum_{j=1}^{n-1} \frac{q^{\alpha(j-n)}}{\sqrt{j}} \approx \frac{1}{q^{(1+\alpha)n}} \frac{q^{\alpha n}}{\sqrt{n}} \\ &\approx \frac{1}{\sqrt{n}\, q^n}. \end{aligned}$$

Therefore,

$$\sum_{n \geq 0} q^n I_1(n) = \infty.$$

On the other hand, (8) yields

$$I_2(n) \approx \frac{1}{q^n} \sum_{k=n}^{\infty} \frac{1}{\sqrt{k}\, q^{(1+\alpha)(k-n)}} \approx \frac{1}{\sqrt{n}\, q^n}.$$

Indeed,

$$\begin{aligned} \left| \sum_{k=n}^{\infty} \frac{h_{k-n}}{\sqrt{k}\, q^{k-n}} \right| &= \left| \frac{h_0}{\sqrt{n}} + \sum_{k=n+1}^{\infty} \frac{h_{k-n}}{\sqrt{k}\, q^{k-n}} \right| \\ &\geq \frac{|h_0|}{\sqrt{n}} - \sum_{k=n+1}^{\infty} \frac{|h_{k-n}|}{\sqrt{k}\, q^{k-n}}. \end{aligned}$$

But

$$\sum_{k=n+1}^{\infty} \frac{|h_{k-n}|}{\sqrt{k}\, q^{k-n}} \leqslant |h_1| \sum_{k=n+1}^{\infty} \frac{1}{\sqrt{k}\, q^{k-n}}$$
$$= |h_1| \sum_{k=n+1}^{\infty} \frac{1}{\sqrt{n-j}\, q^{j}}$$
$$\leqslant \frac{|h_1|}{\sqrt{n}} \sum_{j=1}^{\infty} q^{-j} \leqslant C \frac{|h_0|}{\sqrt{n}},$$

where C is a constant independent of n. Therefore,

$$\left| \sum_{k=n}^{\infty} \frac{h_{k-n}}{\sqrt{k}\, q^{k-n}} \right| \geqslant \frac{C}{\sqrt{n}}.$$

Now $q^n I_2^p(n) \geqslant C q^{n(1-p)} n^{-p/2}$, hence $I_1(n)$ and $I_2(n)$ are of the same order of magnitude. It does not seem to be possible, however, to construct a sequence h such that the dominant terms in $I_1(n)$ and $I_2(n)$ cancel out for each n. At least, it appears that this cannot happen in the case $p < 1$, where both $I_1(n)$ and $I_2(n)$ diverge, and there does not seem to be a way to find a function f such that $I_1(n) + I_2(n)$ decays sufficiently fast. This motivates our conjecture. (Of course, such a system of cancellations, if available, should necessarily arise from oscillating terms in h_n, for instance, alternating signs, because $\Phi * f \notin l^p$, if $f \geqslant 0$).

Acknowledgments

It is a pleasure to acknowledge many valuable suggestions by Michael Cowling, Vadim Kaimanovich and Mitchell H. Taibleson. A part of this research was completed while both authors were visiting the Department of Mathematics of Washington University in St. Louis. The second-named author was supported by a CNR–Nato Senior Fellowship.

References

[BP] W. Betori and M. Pagliacci, Harmonic analysis for groups acting on trees, *Boll. Un. Mat. Ital. B (6)* **3** (1984), 333–345.

[DP] F. Di Biase and M. A. Picardello, The Green formula and H^p spaces on trees, *Math. Zeitsch.*

[Ge] P. Gerl, Über die Anzahl der Darstellungen von Worten, *Monatsh. Math.* **75** (1971), 205–214.

[Gr] R. I. Grigorchuk, Symmetrical random walk on discrete groups, *in* "Multicomponent Random System" (R. L. Dobrushin and Ya. G. Sinai Eds.), pp. 285–325, Dekker, New York/Basel, 1980.

[KPT] A. Korányi, A. M. Picardello, and M. H. Taibleson, Hardy spaces on non-homogeneous trees, *Sympos. Math.* **29** (1987), 205–254.

[LM] B. Ya. Levit and S. A. Molchanov, Invariant chains on a free group with a finite number of generators, *Moscow Univ. Math. Bull.* **26**, Nos. 3/4 (1971), (1973), 131–138.

[Pa] M. Pagliacci, Heat and wave equations on homogeneous trees, *Boll. Un. Mat. Ital. A* **7-A** (1993), 37–45.

[Pi] M. A. Picardello, Spherical functions and local central limit theorems on free groups, *Ann. Mat. Pura Appl.* **133** (1983), 177–191.

[RT] R. Rochberg and M. H. Taibleson, Factorization of the Green's operator and weak-type estimates for a random walk on a tree, *Publ. Mat.* **35** (1991), 187–207.

[Sa] S. Sawyer, Isotropic random walk in a tree, *Z. Wahrsch.* **42** (1978), 279–292.