Tilting up algebras of small homological
dimensions

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Abstract

We consider algebras \( A \) which satisfy the property that for each indecomposable module \( A X \), either its projective dimension \( \text{pd}_A X \) is at most one or its injective dimension \( \text{id}_A X \) is at most one and that the global dimension \( \text{gl dim} A \) is three. We will show that this class is in bijective correspondence with a class of algebras of global dimension two admitting a special tilting torsion pair. © 2002 Elsevier Science B.V. All rights reserved.

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0. Introduction

Let \( A \) be an artin algebra over a commutative artin ring \( R \) and let \( \text{mod} A \) be the category of finitely generated left \( A \)-modules. Following [4] an algebra \( A \) is called an algebra of small homological dimensions (shod) provided each indecomposable module \( A X \) satisfies that its projective dimension \( \text{pd}_A X \leq 1 \) or that its injective dimension \( \text{id}_A X \leq 1 \). It is easy to see (cf. [16]) that in this case the global dimension \( \text{gl dim} A \leq 3 \). If \( \text{gl dim} A \leq 2 \) then \( A \) is a quasitilted algebra in the sense of [16]. If \( \text{gl dim} A = 3 \) then \( A \) is said to be strict shod. Quasitilted algebras were introduced in [16] in order to give a common treatment of the class of tilted algebras [17] and the

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class of canonical algebras [9,21]. They are usually defined as endomorphism algebras of tilting objects in hereditary abelian categories. The possible hereditary abelian categories have recently been characterized in [11,15]. The investigations presented here grew out of the question whether or not there is a relationship via tilting of strict shod algebras to hereditary abelian categories with tilting object. It is easy to see that there are strict shod algebras which are not piecewise hereditary in the sense of [10]. For a concrete example we refer to the later sections. Also it is easy to see that a strict shod algebra which is piecewise hereditary never will be derived equivalent to a category of coherent sheaves with at least four nontrivial weights. For details we refer to Section 5.

However, we will show here that a strict shod algebra $A$ always admits a tilting module $\theta T$ such that the endomorphism algebra $\Gamma = \text{End}_A T$ satisfies $\text{gl.dim} \, \Gamma = 2$.

The first two sections will contain a review of tilting theory and of the relevant properties of (strict) shod algebras. In Section 3, we will show the existence of a canonical tilting module for a strict shod algebra. To be more precise recall that for shod algebras the following two subcategories are important. For a shod algebra $A$, denote by $\mathcal{L}_A$ the additive subcategory of $\text{mod} \, A$ whose indecomposable objects have the property that all predecessors have projective dimension at most one. Dually $\mathcal{R}_A$ denotes the additive subcategory of $\text{mod} \, A$ whose indecomposable objects have the property that all successors have injective dimension at most one. We will show in Section 3 that for a strict shod algebra $A$ the subcategory $\mathcal{L}_A$ always is contravariantly finite (cf. [2]). This will ensure that the direct sum of the indecomposable Ext-injectives in $\mathcal{L}_A$ together with the direct sum of the indecomposable projectives in $\mathcal{R}_A \setminus \mathcal{L}_A$ will be a tilting module. Note that for quasitilted algebras such a tilting module will not exist in general. This tilting module $\theta T$ will be called the canonical tilting module for a strict shod algebra. A similar version of this module was also considered in [22] in connection with the so-called separating slice of Assem [1].

In Section 4, we will investigate in more detail the results from Section 3. Recall that a cotilting module $\rho T$ induces a torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ on $\text{mod} \, A$ where $\mathcal{Y}(T) = \text{Sub} \, T$, the full subcategory of $\text{mod} \, A$ whose objects are cogenerated by $\rho T$. We consider the set of pairs

$$\mathcal{G} = \{ (\Gamma, \rho T) \mid \text{gl.dim} \, \Gamma = 2, \rho T \text{ cotilting module, pd} \rho X \leq 1 \text{ for } X \in \mathcal{X}(T), \text{id}_\rho Y \leq 1 \text{ for all nonprojective } Y \in \text{ind} \, \mathcal{Y}(T) \text{ and } \text{Ext}^2_\rho (\mathcal{Y}(T), \mathcal{X}(T)) \neq 0 \},$$

where for an arbitrary subcategory $\mathcal{C}$ of $\text{mod} \, \Gamma$, we have denoted by $\text{ind} \, \mathcal{C}$ the indecomposable objects in $\mathcal{C}$.

We will show in Section 4 that $\mathcal{G}$ corresponds to a set $\mathcal{S}$ of pairs $(\Lambda, \theta T)$, where $\Lambda$ is a strict shod algebra and $\theta T$ is a tilting module satisfying some specific properties which will be given later. The pair $(\Lambda, \theta T) \in \mathcal{S}$ for the canonical tilting module $\theta T$ over a strict shod algebra $\Lambda$. We will provide an example that shows that for a fixed strict shod algebra $\Lambda$ there may exist more tilting modules $\theta T$ such that $(\Lambda, \theta T) \in \mathcal{S}$. The correspondence is obtained by classical tilting. This will enable us to determine that certain known classes of algebras $\Gamma$ of global dimension two will not admit a cotilting module $\rho T$ such that $(\Gamma, \rho T) \in \mathcal{G}$, hence will not admit a cotilting module whose endomorphism algebra is strict shod.
In Section 5, we will give a detailed investigation of algebras $I$ which admit a cotilting module $T$ such that $(T, T) \in \mathcal{G}$. Amongst other things we will show that such a cotilting module will have a nonzero injective direct summand such that the corresponding idempotent factor algebra of $I$ is a hereditary artin algebra.

For unexplained representation-theoretic terminology, we refer to [21] or [3]. We denote the composition of morphisms $f : X \to Y$ and $g : Y \to Z$ in a given category $\mathcal{K}$ by $fg$.

1. Review of tilting theory

In this section, we will briefly recall the basic definitions and results from tilting theory we will use in the main part of this article. Let $A$ be an artin algebra. Following [17] a $A$-module $T$ is called a tilting module if $\text{pd}_A T \leq 1, \text{Ext}_A^1(T, T) = 0$, and there exists a short exact sequence $0 \to A \to T^0 \to T^1 \to 0$, where $T^0, T^1 \in \text{add} T$, the full subcategory of $\text{mod} A$ formed by direct sums of direct summands of $T$. If $A$ is a tilting module we consider the endomorphism algebra $\text{End}_A T$. Then $A$ induces torsion pairs $(\mathcal{T}(T), \mathcal{F}(T))$ on $\text{mod} A$ and $(\mathcal{X}(T), \mathcal{Y}(T))$ on $\text{mod} I$, which are defined as follows:

$$\mathcal{T}(T) = \{ X \in \text{mod} A | \text{Ext}_A^1(T, X) = 0 \},$$
$$\mathcal{F}(T) = \{ X \in \text{mod} A | \text{Hom}_A(T, X) = 0 \},$$
$$\mathcal{X}(T) = \{ Y \in \text{mod} I | T \otimes Y = 0 \},$$
$$\mathcal{Y}(T) = \{ Y \in \text{mod} I | \text{Tor}_1^I(T, Y) = 0 \}.$$

The theorem of Brenner–Butler (for details we refer to [17] or [21]) asserts that $\mathcal{T}(T)$ and $\mathcal{Y}(T)$ are equivalent under the restrictions of the functors $\text{Hom}_A(T, -)$ and $T \otimes -$, and that $\mathcal{T}(T)$ and $\mathcal{X}(T)$ are equivalent under the restrictions of the functors $\text{Ext}_A^1(T, -)$ and $\text{Tor}_1^I(T, -)$.

The notion of a cotilting $A$-module is defined dually. Unless stated otherwise tilting modules will have projective dimension at most one and cotilting modules will have injective dimension at most one. If $D$ denotes the standard duality on $\text{mod} A$ and $\text{mod} I$, we have that $D(A \otimes -) \in \mathcal{T}(T)$ and that $\text{Hom}_A(T, D(A \otimes -)) = D(T \otimes -)$ is a $I$-cotilting module. Moreover $A = \text{End}_I D(T \otimes -)$. The subcategories $\mathcal{X}(T)$ and $\mathcal{Y}(T)$ can be identified as follows:

$$\mathcal{X}(T) = \{ Y \in \text{mod} I | \text{Hom}(Y, D(T \otimes -)) = 0 \},$$
$$\mathcal{Y}(T) = \{ Y \in \text{mod} I | \text{Ext}_I^1(Y, D(T \otimes -)) = 0 \}.$$

Following [10] we have a triangle equivalence $F$ from the bounded derived category $D^b(A)$ to the bounded derived category $D^b(I)$ with the property that $F(T) = \text{Hom}(T, I)$ and that $F(F(T) [1]) = \text{Ext}_I^1(T, I)$, where $[1]$ the standard shift on $D^b(I)$.

For some of the calculations in Section 4, the following lemma is useful.
Lemma 1.1. Let $\Lambda$ be an artin algebra and $_A T$ be a tilting module with $I = \text{End}_A T$. Then the triangle equivalence $F$ induces functorial isomorphisms

(i) $\text{Ext}^i_A(\mathcal{F}(T), \mathcal{F}(T)) \simeq \text{Ext}^i_I(\mathcal{Y}(T), \mathcal{X}(T))$ for all $i \geq 0$.
(ii) $\text{Ext}^i_A(\mathcal{F}(T), \mathcal{F}(T)) \simeq \text{Ext}^i_I(\mathcal{X}(T), \mathcal{X}(T))$ for all $i \geq 0$.
(iii) $\text{Ext}^i_A(\mathcal{F}(T), \mathcal{F}(T)) \simeq \text{Ext}^{i-1}_I(\mathcal{Y}(T), \mathcal{X}(T))$ for all $i \geq 1$.
(iv) $\text{Ext}^i_A(\mathcal{F}(T), \mathcal{F}(T)) \simeq \text{Ext}^{i+1}_I(\mathcal{X}(T), \mathcal{Y}(T))$ for all $i \geq 0$.

Proof. The first two assertions follow immediately from the fact that $F$ is a triangle equivalence. For the third assertion let $i \geq 1$, then

$$\text{Ext}^i_A(\mathcal{F}(T), \mathcal{F}(T)) \simeq \text{Hom}_{D^b(\Lambda)}(\mathcal{F}(T), \mathcal{F}(T)[i])$$

$$\simeq \text{Hom}_{D^b(I)}(F(\mathcal{F}(T)), F(\mathcal{F}(T))[i])$$

$$\simeq \text{Hom}_{D^b(I)}(F(\mathcal{F}(T)), F(\mathcal{F}(T))[i])$$

$$\simeq \text{Ext}^{i-1}_I(\mathcal{Y}(T), \mathcal{X}(T)) = 0.$$ For the fourth assertion let $i \geq 0$, then

$$\text{Ext}^i_A(\mathcal{F}(T), \mathcal{F}(T)) \simeq \text{Hom}_{D^b(\Lambda)}(\mathcal{F}(T), \mathcal{F}(T)[i])$$

$$\simeq \text{Hom}_{D^b(I)}(F(\mathcal{F}(T)), F(\mathcal{F}(T))[i])$$

$$\simeq \text{Hom}_{D^b(I)}(F(\mathcal{F}(T)), F(\mathcal{F}(T))[i])$$

$$\simeq \text{Hom}_{D^b(I)}(F(\mathcal{F}(T)[1]), F(\mathcal{F}(T)[i + 1]))$$

$$\simeq \text{Hom}_{D^b(I)}(\mathcal{X}(T), \mathcal{Y}(T))[i + 1]$$

$$\simeq \text{Ext}^{i+1}_I(\mathcal{X}(T), \mathcal{Y}(T)).$$

The following lemma will be used frequently in the later sections. The proof is straightforward. For the proof of (ii) observe that a tilting module $T$ over an algebra of finite global dimension is a cotilting module with possibly $\text{id}_{_A T} > 1$ (cf. 1.3 in [18]).

Lemma 1.2. Let $\Lambda$ be an artin algebra with $\text{gl.dim} \Lambda = d < \infty$. Let $_A T$ be a tilting module with $\text{id}_{_A T} = s$. Then the following holds.

(i) For $X \in \mathcal{F}(T)$ there exists an exact sequence $0 \rightarrow T_s \rightarrow \cdots \rightarrow T_0 \rightarrow X \rightarrow 0$, with $T_0, \ldots, T_s \in \text{add} T$.

(ii) If $\text{Ext}^i_A(X, T) = 0$ for all $i > 0$, then there exists a short exact sequence $0 \rightarrow X \rightarrow T_0 \rightarrow T_1 \rightarrow 0$, with $T_0, T_1 \in \text{add} T$.

In Section 5, we will need some information about tilted algebras. If $H$ is a hereditary artin algebra and $_H T$ is a tilting module then $I = \text{End}_H T$ is called a tilted algebra. We refer to [17] and [21] for characterizations of tilted algebras in terms of admitting a complete slice. Note that in this case it follows from 1.1.(iii) that the induced torsion
pair \((\mathcal{X}(T), \mathcal{Y}(T))\) splits. Before we characterize a rather special class of tilted algebras we will need two preliminary assertions. We denote by \(\tau\) and \(\tau^-\) the Auslander–Reiten translations.

**Lemma 1.3.** Let \(H\) be a hereditary artin algebra and let \(_HT\) be a tilting module with \(\Gamma = \text{End}_H T\). For \(Y \in \mathcal{F}(T)\) the following are equivalent:

(i) \(\tau_H Y \in \mathcal{F}(T)\).

(ii) \(\text{Ext}^1_H(Y, T) = 0\).

(iii) \(\text{pd}_{\mathcal{F}_H} \text{Ext}^1_H(T, Y) \leq 1\).

**Proof.** Clearly, (i) and (ii) are equivalent. Let \(Y \in \mathcal{F}(T)\). If \(\text{pd}_{\mathcal{F}_H} \text{Ext}^1_H(T, Y) \leq 1\), let \(0 \to P_1 \to P_0 \to \text{Ext}^1_H(T, Y) \to 0\) be a projective resolution over \(H\). Applying \(T \otimes -\) to this sequence yields the exact sequence

\[
0 \to Y \to T \otimes P_1 \to T \otimes P_0 \to 0
\]

with \(T \otimes P_1, T \otimes P_0 \in \text{add} T\). Apply \(\text{Hom}_H(-, T)\) to the second exact sequence. This then shows that \(\text{Ext}^1_H(Y, T) = 0\).

Conversely, consider the universal extension

\[
(*) \quad 0 \to Y \to E \to \tilde{T} \to 0
\]

with \(\tilde{T} \in \text{add} T\). Then by construction we infer that \(E \in \mathcal{F}(T)\). Now \(\text{Ext}^1_H(Y, T) = 0\) implies that \(\text{Ext}^1_H(E, T) = 0\). Since \(E \in \mathcal{F}(T)\) there exists an exact sequence

\[
(**) \quad 0 \to T_1 \to T_0 \to E \to 0
\]

with \(T_0, T_1 \in \text{add} T\), hence \((**)\) splits and so \(E \in \text{add} T\). Now applying \(\text{Hom}_H(T, -)\) to \((*)\) shows the assertion.

**Lemma 1.4.** If \(A\) is an artin algebra such that every indecomposable noninjective module \(_AX\) satisfies \(\text{pd}_A X \leq 1\), then \(A\) is a tilted algebra and the Auslander–Reiten quiver of \(A\) contains a unique preinjective component containing a complete slice.

**Proof.** In fact, let \(I\) be an indecomposable injective \(A\)-module and let

\[
0 \to X \to P \to I \to 0
\]

be exact with \(P\) projective. Let \(X'\) be an indecomposable summand of \(X\). If \(X'\) is injective, then \(X'\) is a direct summand of \(P\). If \(X'\) is not injective then by assumption \(\text{pd}_A X' = 1\), hence \(\text{pd}_A X \leq 1\), and so \(\text{pd}_A I = 2\). In particular, \(\text{gl dim} A = 2\), and therefore, \(A\) is a quasitilted algebra which has only finitely many indecomposable modules of projective dimension two. Hence, the assertion follows from 6.2 in [8]. Note that a preinjective containing a complete slice has to be unique.

In the following lemma, we recall from [17] some information about the indecomposable injective modules over a tilted algebra. If \(S\) is a simple module, we denote by \(I(S)\) its injective envelope and by \(P(S)\) its projective cover.

**Lemma 1.5.** Let \(A = \text{End}_H T\) be a tilted algebra where \(H\) is a hereditary artin algebra and \(_HT\) is a tilting module. Let \(_AI\) be an indecomposable injective \(A\)-module. Then
(i) \( I \in \mathcal{I}(T) \) iff \( I = \text{Hom}_H(T, I(S)) \) for \( I(S) \) an indecomposable \( H \)-injective such that \( P(S) \) is a direct summand of \( _HT \).
(ii) \( I \in \mathcal{I}(T) \) iff \( I = \text{Ext}^1_H(T, \tau T') \) for an indecomposable nonprojective direct summand \( T' \) of \( _HT \).

In the following proposition we will give a more detailed description of algebras satisfying the assumptions of Lemma 1.4. For this we need some additional notation. Let \( A \) be a tilted algebra and let \( \mathcal{S} \) be a complete slice (cf. for example [21]). Then there exists a hereditary artin algebra \( H \) and a tilting module \( _HT \) such that \( A = \text{End}_H T \) and that \( \mathcal{S} = \text{Hom}_H(T, D(H_T)) \). If \( H \) is a basic hereditary artin algebra let \( \Delta_0 \) be the index set of the isomorphism classes of the simple \( H \)-modules. Let \( S \subset \Delta_0 \) be the subset corresponding to the simple projective \( H \)-modules. If \( L \subset \Delta_0 \), then we denote by 
\[
P_L = \bigoplus_{x \in L} P(x)
\]
where \( P(x) \) is the projective cover of the simple corresponding to \( x \). We denote by \( e_L \) the idempotent in \( H \) with \( P_L = He_L \). If \( L \subset \Delta_0 \), then we denote by \( H^L \) the hereditary artin algebra \( H/He_L H \). Note that \( \text{mod } H^L = \{ X \in \text{mod } H \mid \text{Hom}_H(P_L, X) = 0 \} \).

**Proposition 1.6.** The following are equivalent for a basic connected artin algebra \( A \).
(i) Each indecomposable noninjective \( A \)-module \( X \) satisfies \( \text{pd}_A X \leq 1 \).
(ii) There is a basic hereditary artin algebra \( H \) such that:
   (a) There is \( S \subset L \subset \Delta_0 \) where no indecomposable \( H^L \)-projective module is \( H \)-injective,
   (b) \( _HT = H P_L \bigoplus e_H \tau_H H^L \) is a tilting module, and
   (c) \( A = \text{End}_H T \).

**Proof.** Let \( A \) be an algebra satisfying (i). By Lemma 1.4, we infer that \( A \) is a tilted algebra with a unique preinjective component containing a complete slice. Let \( \mathcal{S} \) be the unique preinjective component containing a complete slice. Then \( \mathcal{S} \) also contains all indecomposable injective \( A \)-modules, for a complete slice is sincere [21]. Then \( \mathcal{S} \) admits a complete slice \( \mathcal{F} \) with the property that the sources of \( \mathcal{S} \) correspond to injective \( A \)-modules. Set \( H = \text{End}_A \mathcal{S} \) and let \( _HT = D \text{Hom}_A(A, H, \mathcal{S}) \). So \( H \) is a basic hereditary artin algebra and \( _HT \) is a tilting module such that \( A = \text{End}_H T \) and that \( \mathcal{S} = \text{Hom}_H(T, D(H_T)) \). By construction and Lemma 1.5, each simple projective \( H \)-module is a direct summand of \( _HT \). So there exists \( L \subset \Delta_0 \) with \( S \subset L \) and \( _HT = P_L \bigoplus C \), where \( C \) has no indecomposable projective direct summand. Since \( _HT \) is a tilting module, we infer that \( \tau_H C \) is a tilting module in \( \text{mod } H^L \).

Next we claim that \( \mathcal{F}(T) = \text{add } \tau_H C \). For this let \( X \in \mathcal{F}(T) \). So \( X \in \text{mod } H^L \). Let
\[
0 \to P_1 \to P_0 \to X \to 0
\]
be a minimal projective resolution of \( X \). Since \( X \in \mathcal{F}(T) \) we clearly have that \( P_0 \not\in \mathcal{F}(T) \). Since \( P_L \) contains all simple projective \( H \)-modules we infer that \( \text{Hom}_H(P_L, P_0) \neq 0 \), hence \( P_0 \not\in \mathcal{F}(T) \). Consider the canonical torsion exact sequence
\[
0 \to t(P_0) \to P_0 \to P_0/t(P_0) \to 0,
\]
with \( 0 \neq t(P_0) \in \mathcal{F}(T) \) and \( 0 \neq P_0/t(P_0) \in \mathcal{F}(T) \). Since \( P_0 \) is projective and \( H \) is hereditary, we infer that \( t(P_0) \) is projective, and hence in \( \text{add } P_L \). Since \( ** \) is nonsplit we
see that $\text{Ext}^1_H(P_0/t(P_0), T) \neq 0$, and therefore we see that $\text{pd}_A \text{Ext}^1_H(T, P_0/t(P_0)) = 2$ by Lemma 1.3. By assumption we have that $\text{Ext}^1_H(T, P_0/t(P_0))$ is $A$-injective, hence $P_0/t(P_0) \in \text{add } \tau_H C$ by Lemma 1.5. Applying $\text{Hom}_H(-, X)$ to (**) and using (*) shows that we obtain a surjective map $P_0/t(P_0) \twoheadrightarrow X$. So $X$ is generated by $\tau_H C$, hence $\text{Ext}^1_H(\tau_H C, X) = 0$. Let

$$0 \rightarrow \text{Ext}^1_H(T, X) \rightarrow_A I_0 \rightarrow_A I_1 \rightarrow 0$$

be a minimal $A$-injective resolution of $\text{Ext}^1_H(T, X)$. Applying $T_A \otimes -$ to (***) yields by observing Lemma 1.5 the following exact sequence

$$0 \rightarrow X \rightarrow \tau_H C_0 \rightarrow \tau_H C_1 \rightarrow 0$$

with $C_0, C_1 \in \text{add } C$. By the previous considerations we see that the last sequence splits, hence $X \in \text{add } \tau_H C$.

Finally, we show that $\tau_H C$ is $H^2$-projective. Otherwise, there exists an indecomposable $H^2$-projective $P$ which is not a direct summand of $\tau_H C$. Since $\tau_H C$ is a tilting module for $H^2$ there exists an injective map $P \rightarrow \tau_H \tilde{C}$, for some $\tilde{C} \in \text{add } C$. Since $\tau_H C \in \mathcal{F}(T)$, also $P \in \mathcal{F}(T)$, but this contradicts the fact that $\mathcal{F}(T) = \text{add } \tau_H C$. Thus, $H^2 T$ is of the form required in (b).

Conversely, let $H$ be a basic hereditary artin algebra and $H T$ a tilting module satisfying (a) and (b). Let $A = \text{End}_H T$. Note that assumption (a) ensures that $H T$ is a tilting module. We claim that $\mathcal{F}(T) = \text{add } H^2 L$. This follows from the following easy identifications:

$$\mathcal{F}(T) = \{ X \in \text{mod } H^2 | \text{Hom}_H(\tau_H^-(H^2 L), X) = 0 \},$$

$$= \{ X \in \text{mod } H^2 | \text{Ext}^1_H(X, H^2 L) = 0 \},$$

$$= \{ X \in \text{mod } H^2 | \text{Ext}^1_H(X, H^2 L) = 0 \},$$

$$= \text{add } H^2 L.$$

Let $Y$ be an indecomposable $A$-module. If $Y \in \mathcal{Y}(T)$, then it always holds that $\text{pd}_A Y \leq 1$. If $Y \in \mathcal{X}(T)$, then $Y = \text{Ext}^1_H(T, X)$ for some indecomposable $X \in \mathcal{F}(T)$. By the previous considerations we have that $X = P$ for an indecomposable direct summand $P$ of $H^2 L$. By Lemma 1.5 we infer that $Y$ is injective.

This finishes the proof of the proposition. □

2. Review of (strict) shod algebras

In this section, we will briefly recall the basic definitions and results for (strict) shod algebras we will use in the main part of the article. Furthermore, we will fix some notation which will be used in the remaining sections.

Let $A$ be an artin algebra. Following [4] an algebra, $A$ is called an algebra of small homological dimensions (shod) provided each indecomposable module $A X$ satisfies that its projective dimension $\text{pd}_A X \leq 1$ or that its injective dimension $\text{id}_A X \leq 1$. It is easy to see (cf. [16]) that in this case the global dimension $\text{gl dim } A \leq 3$. If $\text{gl dim } A \leq 2$
then $A$ is a quasitilted algebra in the sense of [16]. If $\text{gl.dim } A = 3$ then $A$ is said to be strict shod.

For shod algebras the following two subcategories and their properties are quite important. For this recall that a path in mod $A$ is a sequence $(X_0, \ldots, X_s)$ for some $s \geq 0$ of (isomorphism classes of) indecomposable $A$-modules $X_i$, $0 \leq i \leq s$ such that $\text{Hom}(X_{i-1}, X_i) \neq 0$ and $X_{i-1} \not= X_i$ for all $1 \leq i \leq s$. We will say that $(X_0, \ldots, X_s)$ is a path from $X_0$ to $X_s$ of length $s$, and we write $X \preceq X'$, or $X \preceq_A X'$ to indicate that a path from $X$ to $X'$ exists. We say that $X$ is a predecessor of $X'$ or $X'$ is a successor of $X$.

Let $(X_0, \ldots, X_s)$ for $s > 0$ be a path of irreducible maps from $X_0$ to $X_s$ (i.e. for all $0 \leq i \leq s - 1$ there exists an irreducible map from $X_i \to X_{i+1}$). If $X_{i-1} = \tau_A X_{i+1}$ for some $1 \leq i \leq s - 1$ we say that $X_i$ is a hook of the given path. We say that two hooks $X_i$ and $X_j$ are consecutive if $j = i + 1$.

Let $\mathcal{L}_A$ denote the subset of $\text{ind } A$ given by $\mathcal{L}_A = \{X \in \text{ind } A \mid \text{ for all } Y \preceq X \text{ we have } \text{pd}_A Y \leq 1\}$, and let $\mathcal{R}_A$ denote the subset of $\text{ind } A$ given by $\mathcal{R}_A = \{X \in \text{ind } A \mid \text{ for all } X \preceq Y \text{ we have } \text{id}_A Y \leq 1\}$. When there is no danger of confusion we simply write $\mathcal{L}$ for $\mathcal{L}_A$ and $\mathcal{R}$ for $\mathcal{R}_A$.

The following result from [4] characterizes shod algebras in terms of these subcategories. For further characterizations we refer to [4].

**Theorem 2.1.** For an artin algebra $A$ the following are equivalent.

(i) $A$ is a shod algebra.

(ii) $\mathcal{L}_A \cup \mathcal{R}_A = \text{ind } A$.

(iii) (add $\mathcal{R}_A$, add $(\mathcal{L}_A \setminus \mathcal{R}_A)$) is a split torsion pair in mod $A$.

(iv) (add $(\mathcal{R}_A \setminus \mathcal{L}_A)$, add $\mathcal{L}_A$) is a split torsion pair in mod $A$.

(v) Any path from an indecomposable injective module to an indecomposable projective module can be refined to a path of irreducible maps and any such refinement has at most two hooks, and in case there are two, they are consecutive.

Since we are mainly interested in strict shod algebras we will also need the following proposition from [4].

**Proposition 2.2.** Let $A$ be a shod algebra. Then the following are equivalent.

(i) $A$ is a strict shod algebra.

(ii) $\mathcal{L}_A \setminus \mathcal{R}_A$ contains an indecomposable injective $A$-module.

(iii) $\mathcal{R}_A \setminus \mathcal{L}_A$ contains an indecomposable projective $A$-module.

Let $A$ be a basic shod algebra. The following $A$-modules will play an important role in the subsequent sections. Let $P_1, \ldots, P_r$ be (isomorphism classes of) the indecomposable projective $A$-modules contained in $\mathcal{L}_A$. Let $P' = \bigoplus_{i=1}^r P_i$. Let $P_{r+1}, \ldots, P_s$ be (isomorphism classes of) the indecomposable projective $A$-modules contained in $\mathcal{R}_A \setminus \mathcal{L}_A$. Let $P'' = \bigoplus_{i=r+1}^s P_i$. According to Proposition 2.2, we have that $P'' \neq 0$ if $A$ is a strict shod algebra. Let $A' = \text{End } P'$. By [19] we have that $A'$ is a tilted algebra. By construction we infer that $\mathcal{L}_A \subseteq \text{mod } A'$. Recall that an indecomposable $A$-module $J \in \mathcal{L}_A$ is called Ext-injective in $\mathcal{L}_A$ if $\text{Ext}_A^1(\mathcal{L}_A, J) = 0$. By Theorem 2.1(iv) this is
clearly equivalent to the fact that $J \in \mathcal{L}_A$ and $\tau^{-1}J \in \mathcal{P}_A \setminus \mathcal{L}_A$. Let $J_1, \ldots, J_s$ be (isomorphism classes of) the indecomposable Ext-injective $A$-modules contained in $\mathcal{L}_A$. Let $J' = \bigoplus_{i=1}^s J_i$. By Proposition 2.2(ii), we have that $J' \neq 0$ iff $A$ is a strict shod algebra. Also let $Q_1, \ldots, Q_t$ be (isomorphism classes of) the indecomposable Ext-projective $A$-modules contained in $\mathcal{P}_A \setminus \mathcal{L}_A$. Let $Q'' = \bigoplus_{i=1}^t Q_i$. According to Proposition 2.2, we have that $Q'' \neq 0$ iff $A$ is a strict shod algebra.

Let $\mathcal{T}$ be a tilting module. Then $\mathcal{T}$ admits the following canonical decomposition

$$\mathcal{T} = T_l \bigoplus T_r,$$

where $T_l \in \text{add} \mathcal{L}_A$ and $T_r \in \text{add} (\mathcal{P}_A \setminus \mathcal{L}_A)$.

Lemma 2.3. If $A$ is strict shod algebra then both $T_l$ and $T_r$ are nonzero.

Proof. Since $A$ is strict shod, we infer that $P'' \neq 0$. By definition of a tilting module, we have that $\text{Hom}_A(P'', T) \neq 0$, hence $T_r \neq 0$. The other assertion follows dually by observing that $\text{Hom}_A(T, \text{D}(A_A)) \neq 0$.  

In the next section we will exhibit a canonical tilting module for a strict shod algebra while in Section 4 we will investigate further properties of this canonical tilting module and related ones.

We will now give two examples.

The first example which we borrow from [22] shows that a strict shod algebra will in general not be piecewise hereditary in the sense of [10] or equivalently will not be an iterated tilted algebra. An example is given by the algebra $A$ which is defined as the path algebra over a field $k$ of the following quiver with relations, where the relations are indicated by the dotted lines.

Let $S$ be the simple injective module and $S'$ be the simple projective module, then it holds that $\text{Ext}^2(S, S') \neq 0 \neq \text{Ext}^3(S, S')$. Then it follows from [10] that $A$ cannot be piecewise hereditary. The algebra is strict shod as the following Auslander–Reiten quiver of $A$ easily shows:
The second example will be needed in Section 4. Let \( A \) be the algebra defined over a field \( k \) by the following quiver with relations, where again the relations are indicated by the dotted lines.

The following shows the part of the component of the Auslander–Reiten quiver of \( A \) containing the indecomposable summands of \( J' \) and \( P'' \).

Let \( A \) be a strict shod algebra which we assume to be connected. Then the proof of following proposition can be found in [20], or in [5,6].

**Proposition 2.4.** Let \( A \) be a connected strict shod algebra. Then there exists a unique connected component \( \mathcal{C} \) of the Auslander–Reiten quiver of \( A \) which satisfies the following properties:

(i) \( \mathcal{C} \) does not contain an oriented cycle and is generalized standard,
(ii) \( \mathcal{C} \) contains all indecomposable summands of \( J' \), and
(iii) \( \mathcal{C} \) contains all indecomposable summands of \( P'' \).

This component replaces the connected component for a tilted algebra. In the next section we will need the following lemma, where we use notation above.

**Lemma 2.5.** Let \( A \) be a connected strict shod algebra and let \( A_1', \ldots, A_s' \) be the connected components of \( A' \). Then for each \( 1 \leq i \leq s \) there exists an indecomposable summand \( J_i \) of \( J' \) such that \( J_i \in \text{mod } A_i' \).

**Proof.** Since \( A \) is connected there is for all \( 1 \leq i \leq s \) an indecomposable \( A_i' \)-module \( R_i \) which is a direct summand of the radical \( \text{rad } P \) of some indecomposable direct summand \( P \) of \( P'' \). Consider the short exact sequence

\[
0 \to \text{rad } P \to P \to S \to 0.
\]

Since \( \mathcal{L}_A \subseteq \text{mod } A' \) we infer that \( \text{Hom}_A(\mathcal{L}_A, S) = 0 \). Since \( \text{Ext}_A^1(\mathcal{L}_A, P) = 0 \) by Theorem 2.1(iii) we see that \( \text{Ext}_A^1(\mathcal{L}_A, \text{rad } P) = 0 \). Thus, \( R_i \) is Ext-injective if \( R_i \in \mathcal{L}_A \). Thus consider the case that \( R_i \notin \mathcal{L}_A \). Then \( R_i \) is not a projective \( A \)-module. Otherwise, since \( R_i \in \text{mod } A_i' \), the module \( R_i \) would be projective as \( A_i' \)-module. But by assumption all projective \( A_i' \)-modules are contained in \( \mathcal{L}_A \). We consider \( \tau_A R_i \). We claim that \( \tau_A R_i \notin \mathcal{L}_A \). Otherwise \( \tau_A R_i \) has a predecessor \( Z \) with \( \text{pd } A Z = 2 \), hence there exists an indecomposable injective \( A \)-module \( I \) such that \( \text{Hom}_A(I, \tau_A Z) \neq 0 \). So there exists a path \( I \preceq \tau Z \preceq Z \preceq \tau R_i \preceq R_i \preceq P \) having two nonconsecutive hooks contradicting 2.1. So \( \tau_A R_i \in \mathcal{L}_A \) and clearly is Ext-injective, since by assumption \( R_i \in \mathcal{R}_A \setminus \mathcal{L}_A \). Now \( \tau_A R_i \)
is a $A'$-module. In fact, $\tau_{A}R_{i}$ is a $A'$-module, since otherwise there would exist $j \neq i$ such that $\tau_{A}R_{i} \in \text{mod } A'_{i}$. But then the Auslander–Reiten sequence would give a path in $\text{ind } A'$ between modules lying in different connected components of $\text{mod } A'$.

We will also need the following easy lemma.

**Lemma 2.6.** Let $A$ be a strict shod algebra and let $X \in \mathcal{L}_{A}$ be indecomposable. If every proper successor of $X$ has injective dimension at most one, then $\text{id}_{A}X \leq 2$.

**Proof.** Indeed, let $X \in \mathcal{L}_{A}$ be noninjective and let $0 \to X \to I \to \Omega^{-}X \to 0$ be exact with $I$ injective. Then each indecomposable summand $X'$ of $\Omega^{-}X$ is a proper successor of $X$, hence $\text{id}_{A}\Omega^{-}X \leq 1$, and so $\text{id}_{A}X \leq 2$.

### 3. Existence of canonical tilting modules

We keep the notation from the previous section. In particular, recall that we have introduced the modules $J'$ and $P''$ for a shod algebra $A$. Now we will be interested in the module $T = J' \bigoplus P''$. Let $\mathcal{C}$ be a full subcategory of $\text{mod } A$, which we assume to be closed under isomorphisms, direct sums and direct summands. We say that $\mathcal{C}$ is contravariantly finite if for each $X \in \text{mod } A$ there exists $FX \in \mathcal{C}$ and a map $f_{X} : FX \to X$ such that the induced map $\text{Hom}_{A}(C,f_{X}) : \text{Hom}_{A}(C,FX) \to \text{Hom}_{A}(C,X)$ is surjective for all $C \in \mathcal{C}$. For further information we refer to [2].

**Proposition 3.1.** For a shod algebra $A$ the following are equivalent.

(i) The module $T = J' \bigoplus P''$ is a tilting module.

(ii) The subcategory $\mathcal{L}_{A}$ is contravariantly finite in $\text{mod } A$.

**Proof.** First assume that $T$ is a tilting module. Now $J'$ is a $A'$-module with $\text{pd}_{A}J' = \text{pd}_{A}J' \leq 1$ and $\text{Ext}_{A}^{1}(J',J') \simeq \text{Ext}_{A}^{1}(J',J') = 0$. Since $J'$ has the correct number of indecomposable direct summands we infer that $J'$ is a $A'$-tilting module. Next we claim that $\text{id}_{A'}J' \leq 1$. For this let $J_{i}$ be an indecomposable noninjective direct summand of $J'$. Since $J_{i}$ is Ext-injective in $\mathcal{L}_{A}$ we see that $\tau_{A}J_{i} \notin \mathcal{L}_{A}$. But then there is no path in $\text{ind } A$ from $\tau_{A}J_{i}$ to $P'$. We consider the Auslander–Reiten sequences starting in $J_{i}$ in $\text{mod } A$ and in $\text{mod } A'$.

This yields the following commutative diagram:

$$
\begin{array}{cccccc}
0 & \longrightarrow & J_{i} & \longrightarrow & L & \longrightarrow & \tau_{A}J_{i} & \longrightarrow & 0 \\
\text{id} & = & \text{id} & & f & & \text{id} & & \text{id} \\
0 & \longrightarrow & J_{i} & \longrightarrow & L' & \longrightarrow & \tau_{A'}J_{i} & \longrightarrow & 0
\end{array}
$$

The upper sequence does not split, hence $f \neq 0$. If $\text{Hom}_{A'}(\tau_{A}J_{i},P') \neq 0$, then there exists a path from $\tau_{A}J_{i}$ to $P'$. Thus $\text{Hom}_{A'}(\tau_{A}J_{i},P') = 0$, or equivalently $\text{id}_{A'}J_{i} \leq 1$. 

This implies that $J'$ is a $A'$-cotilting module. Since $\text{Ext}_A^1(\mathcal{L}_A,J')=0$, we infer that for each $X \in \mathcal{L}_A$ there exists an exact sequence $0 \to X \to J_0' \to J_1' \to 0$ with $J_0', J_1' \in \text{add} J'$.

Next we show that $\mathcal{L}_A$ is contravariantly finite. Clearly it is enough to show that each $Z \notin \mathcal{L}_A$ admits a $\mathcal{L}_A$-approximation. For such a $Z$, consider the minimal $\text{add} J'$-approximation $f_Z: J' \to Z$ of $Z$. We claim that this is also a $\mathcal{L}_A$-approximation. For this let $X \in \mathcal{L}_A$ and let $f: X \to Z$ be a map. As observed above we have an exact sequence

$$0 \to X \xrightarrow{\mu} J_0' \to J_1' \to 0$$

with $J_0', J_1' \in \text{add} J'$. Since $Z \notin \mathcal{L}_A$ we conclude from 2.1(iv) that $\text{Ext}_A^1(J_1', Z) = 0$. Thus there exists $g: J_0' \to Z$ with $\mu g = f$. Since $f_Z$ is an $\text{add} J'$-approximation there exists $h: J_0' \to J'$ such that $hf_Z = g$. Thus $f = \mu g = (\mu h)f_Z$, and therefore we see that $\mathcal{L}_A$ is contravariantly finite.

Conversely, assume that $\mathcal{L}_A$ is contravariantly finite. We consider the subcategory $\mathcal{C} = \text{add}(\mathcal{L}_A, P')$. The subcategory $\mathcal{C}$ is trivially also contravariantly finite. Since $\text{Ext}_A^1(\mathcal{L}_A, P') = 0 = \text{Ext}_A^1(P', \mathcal{L}_A)$ and $\mathcal{L}_A$ is closed under extensions, we see that $\mathcal{C}$ is closed under extensions. Moreover, $\mathcal{C}$ contains all projective $A$-modules, hence each $\mathcal{C}$-approximation is surjective. Let $X \notin \mathcal{C}$ and consider the exact sequence

$$0 \to K_X \to F_X \to X \to 0,$$

where $F_X$ is the minimal $\mathcal{C}$-approximation. By Wakamatsu’s lemma [2] we see that $\text{Ext}_A^1(\mathcal{C}, K_X) = 0$. Thus each indecomposable summand of $K_X$ which is contained in $\mathcal{L}_A$ is Ext-injective. Clearly we have that $\text{Ext}_A^1(\mathcal{C}, X) = 0$. Thus also $\text{Ext}_A^1(\mathcal{C}, F_X) = 0$. Thus each indecomposable summand of $F_X$ which is contained in $\mathcal{L}_A$ is Ext-injective. The indecomposable summands of $F_X$ which are not contained in $\mathcal{L}_A$ are direct summands of $P'$. But then it follows that $F_X \in \text{add} T$. Let $I$ be an indecomposable injective $A$-module. If $I \in \mathcal{C}$, then $I \in \text{add} T$. Otherwise we consider the minimal $\mathcal{C}$-approximation $T_0$ of $I$. This gives an exact sequence

$$0 \to K_0 \to T_0 \to I \to 0,$$

As above we see that $\text{Ext}_A^1(\mathcal{L}_A, K_0) = 0$. Iterating yields a long exact sequence

$$0 \to K_3 \to T_3 \xrightarrow{f_3} T_2 \xrightarrow{f_2} T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} I \to 0,$$

with $T_j \in \text{add} T$ and $\text{Ext}_A^1(\mathcal{C}, K_j) = 0$ for $0 \leq j \leq 3$, where $K_j = \ker f_j$. Since $\text{pd}_A I \leq 3$ we see that $0 = \text{Ext}_A^1(I, K_3) = \cdots = \text{Ext}_A^1(K_2, K_3)$. Thus, $K_2 \in \text{add} T$. Since $\text{pd}_A T \leq 1$ and $\text{Ext}_A^1(T, T) = 0$ we see that $T$ is cotilting module, possibly with $\text{id}_A T > 1$. But then $T$ is a tilting module.

Next, we will show that for a strict shod algebra the module $T$ defined above is a tilting module. We start by series of lemmas which are inspired by II, 3.3 in [16].

**Lemma 3.2.** Let $Y$ be an indecomposable direct summand of $J'$ and let $X \to Y$ be irreducible. Then $X \in \text{add} J'$, or $X$ is noninjective and $\tau_* X \in \text{add} J'$.

**Proof.** If $X \to Y$ is irreducible we have that $X \in \mathcal{L}_A$. Assume that $X \notin \text{add} J'$. Then $X$ is not injective and $\tau_* X \in \mathcal{L}_A$. If $Y$ is injective, then the irreducible map $Y \to
Lemma 3.3. Let \( X \) be an indecomposable direct summand of \( J' \) and let \( X \to Z \) be irreducible. If \( Z \in \mathcal{L}_A \), then \( Z \in \text{add} \, J' \).

**Proof.** If \( X \to Z \) is surjective, then the irreducible map yields a surjective map \( \text{Ext}^1_A(\mathcal{L}_A, X) \to \text{Ext}^1_A(\mathcal{L}_A, \tau^{-} X) \). Thus, \( \tau^{-} X \) is Ext-injective, hence \( \tau^{-} X \in \text{add} \, J' \). If \( Y \) is not injective, we have that \( \tau^{-} Y \notin \mathcal{L}_A \). But then \( \text{Ext}^1_A(\mathcal{L}_A, \tau^{-} Y) = 0 \). This and \( \text{Ext}^1_A(\mathcal{L}_A, Y) = 0 \) imply that \( \text{Ext}^1_A(\mathcal{L}_A, \tau^{-} X) = 0 \), hence \( \tau^{-} X \in \text{add} \, J' \).

Lemma 3.4. If \((X, Z, Y)\) is a path in \text{ind} \, A with \( X, Y \in \text{add} \, J' \) then \( Z \in \text{add} \, J' \).

**Proof.** By Proposition 2.4 there are no infinite chains of nonzero, nonisomorphisms between indecomposable direct summands of \( J' \). So the assertion follows from the previous lemma.

Lemma 3.5. If \( \text{Hom}_A(J', C') \neq 0 \) for some \( C' \notin \mathcal{L}_A \) and \( C' \in \text{ind} \, A' \), then \( \text{Hom}_A(\tau^{-} J', C') \neq 0 \).

**Proof.** Assume to the contrary that \( \text{Hom}_A(\tau^{-} J', C') = 0 \). Let \( J_i \) be an indecomposable direct summand of \( J' \) with \( \text{Hom}_A(J_i, C') \neq 0 \). Since \( J_i \in \mathcal{L}_A \) and \( C' \notin \mathcal{L}_A \), we see that \( J_i \) is not a direct summand of \( C' \). In particular \( J_i \) is not simple injective. We will now construct an infinite chain of nonzero, nonisomorphisms between indecomposable direct summands of \( J' \), which contradicts Proposition 2.4. Suppose that for \( r > j > 0 \) we have constructed a path of irreducible maps

\[
J_1 \preceq X_0 \preceq \cdots \preceq X_j \preceq \cdots \preceq X_{r-1}
\]

with the property that for all \( j \) with \( r > j > 0 \) we have \( X_j \in \text{add} \, J' \) and \( \text{Hom}_A(X_j, C') \neq 0 \). Clearly, also \( X_{r-1} \) is not simple injective. So we consider the left almost split map

\[
X_{r-1} \to \bigoplus_{i=1}^s Y_i.
\]

We may assume that \( \text{Hom}_A(Y_1, C') \neq 0 \). We claim that \( Y_1 \in \text{add} \, J' \). Otherwise we have by 3.3 that \( Y_1 \notin \mathcal{L}_A \). Now if \( Y_1 \) is a projective \( A \)-module, then \( Y_1 \) is an indecomposable direct summand of \( P'' \). But \( \text{Hom}_A(P'', C') = 0 \), since by assumption \( C' \in \text{mod} \, A' \). By 3.2 we infer that \( \tau Y_1 \in \text{add} \, J' \). But \( Y_1 = \tau^{-} \tau Y_1 \) and \( \text{Hom}_A(Y_1, C') \neq 0 \) contradict the assumption \( \text{Hom}_A(\tau^{-} J', C') = 0 \), hence \( Y_1 \in \text{add} \, J' \). So for \( X_r = Y_1 \) we have obtained a path of length \( r \).

Theorem 3.6. Let \( A \) be a strict self-injective algebra. Then \( T \) is a tilting module and \( \mathcal{L}_A \) is contravariantly finite.
**Proof.** By Proposition 3.1 it is enough to show that $T$ is a tilting module. We consider the algebra $A'$. In order to show that $T$ is a tilting module it is clearly enough to show that $J'$ is a $A'$-cotilting module. We consider $A_1', \ldots, A_r'$ the connected components of $A'$. Let $J' = \bigoplus_{i=1}^r J'_i$ be the corresponding decomposition of $J'$, so $J_i \in \text{mod } A'_i$ for all $1 \leq i \leq r$. By 2.4, we see that $J_i \neq 0$ for all $i$. Clearly it is enough to show that each $J'_i$ is a $A'_i$-cotilting module. For this we consider the universal extension

$$0 \rightarrow J'_i \rightarrow C_i \rightarrow D(A'_{i,0}) \rightarrow 0$$

with $J'_i \in \text{add } J'_i$. Set $T'_i = J'_i \bigoplus C_i$. It is well-known that $T'_i$ is a $A'_i$-cotilting module. Since $\mathcal{L}'_A \subset \text{mod } A'$ we get that $\text{Ext}^1_A(\mathcal{L}'_A, D(A'_{i,0})) = 0$, hence $\text{Ext}^1_A(\mathcal{L}'_A, C_i) = 0$. Since $A'_i$ is connected we infer that $\text{End } T'_i$ is connected. We claim that $C_i \notin \text{add } J'_i$. For this let $C'_i \in \mathcal{L}'_A$ be an indecomposable direct summand of $C_i$. If $C'_i \in \mathcal{L}'_A$ then it follows from $\text{Ext}^1_A(\mathcal{L}'_A, C_i) = 0$ that $C'_i \notin \text{add } J'_i$. If $C'_i \notin \mathcal{L}'_A$ then there is an indecomposable direct summand $C''_i$ of $C_i$ such that $\text{Hom}_A(J'_i, C''_i) \neq 0$. In fact, consider the decomposition $T'_i = T_l \bigoplus T_r$, where all indecomposable direct summands of $T_l$ are contained in $\mathcal{L}'_A$ and $T_l$ is maximal with this property. Now $\text{Hom}_A(T_l, T_l) = 0$ shows that $\text{Hom}_A(T_l, T_r) \neq 0$, since $A'_i$ is connected. Since $\tau_A J'_i \notin \mathcal{L}'_A$ we have that $\text{Hom}_A(\tau_A J'_i, P') = 0$ and $\text{Hom}_A(P'', C''_i) = 0$, since $C''_i \in \text{mod } A'$. But then by the Auslander–Reiten formula we infer that $\text{Ext}^1((C''_i, J'_i)) \simeq D\text{Hom}_A(\tau_A J'_i, C''_i)$. By Lemma 3.5 we see that $\text{Hom}_A(\tau_A J'_i, C''_i) \neq 0$, hence $\text{Ext}^1((C''_i, J'_i)) \neq 0$, contradicting the fact that $T'_i$ is a $A'_i$-cotilting module. Hence $C_i \in \text{add } J'_i$ and $J'_i$ is a $A'_i$-cotilting module. This finishes the proof of the theorem.

We will call $A$ the canonical tilting module for a strict shod algebra. We conclude this section by some remarks about the module $T$ for quasitilted algebras. For a quasitilted algebra the module $P'' = 0$, hence $T = J'$. For a quasitilted algebra which is not a tilted algebra also $J' = 0$. Thus, if $A$ is a quasitilted algebra such that $T$ is a tilting module, then $A$ is a tilted algebra. We refer to II, 3.4 in [16] for more details.

4. The correspondence

In this section, we will investigate certain tilting modules over strict shod algebras and establish a correspondence to a particular class of artin algebras of global dimension two. We keep the notation from the previous sections.

Let $A$ be a strict shod algebra. If $A$ is a tilting module, then we have introduced in 2.3 a canonical decomposition $A = T_l \bigoplus T_r$.

**Lemma 4.1.** Let $A = T_l \bigoplus T_r$ be a tilting module over a strict shod algebra $A$. Then the following are equivalent.

(i) $\text{add } (\mathcal{R}_A \setminus \mathcal{L}_A) \subset \mathcal{F}(T)$.
(ii) $T_r$ is Ext-projective in $\text{add } (\mathcal{R}_A \setminus \mathcal{L}_A)$.
(iii) $\mathcal{F}(T) \subset \text{add } \mathcal{L}_A$.

In this case $P''$ is a direct summand of $T_r$. 

Proof. Since $\text{Ext}^1_B(T_r, \mathcal{F}(T))=0$, we clearly have that (i) implies (ii). Conversely, if $T_r$ is Ext-projective in $\text{add}(\mathcal{A}\setminus \mathcal{L}_A)$, then $\text{Ext}^1_B(T_r, \text{add}(\mathcal{A}\setminus \mathcal{L}_A))=0$ and by Proposition 2.1(iv) we have that $\text{Ext}^1_B(T_r, \text{add}(\mathcal{A}\setminus \mathcal{L}_A))=0$. So $\text{add}(\mathcal{A}\setminus \mathcal{L}_A) \subset \mathcal{I}(T)$. Also by Proposition 2.1 we see that (i) and (iii) are equivalent. 

The following example shows that $T_r$ may not be projective or equivalently $T_l$ is not necessarily a $\mathcal{A}'$-tilting module. Also note that this example shows that the converse of 2.6 will not hold. The algebra $\mathcal{A}$ is given as the path algebra over a field $k$ with relations indicated by the dotted lines.

The Auslander–Reiten quiver is given below. The direct sum of the indecomposable modules corresponding to the vertices marked by * is a tilting module with the desired property.

Lemma 4.2. Let $A$ be a strict shod algebra and let $\mathcal{A}T = T_l \oplus T_r$ be a tilting module such that the torsion pair $(\mathcal{I}(T), \mathcal{F}(T))$ splits. Then $\mathcal{F}(T) \subset \text{Sub}T$. Moreover, for each $X \in \mathcal{F}(T)$ there exists an exact sequence $0 \rightarrow X \rightarrow T_0 \rightarrow T_1 \rightarrow 0$, with $T_0, T_1 \in \text{add} T$.

Proof. First note that $T$ is a $A$-cotilting module with possibly id$_A T > 1$. Since $(\mathcal{I}(T), \mathcal{F}(T))$ splits, we have $\text{Ext}^i_A(\mathcal{F}(T), \mathcal{F}(T)) = 0$. Since $\mathcal{I}(T)$ is a torsion class containing all the injective $A$-modules this implies that $\text{Ext}^i_A(\mathcal{F}(T), \mathcal{I}(T)) = 0$, for all $i > 0$. In particular, we see that $\text{Ext}^i_A(\mathcal{F}(T), T) = 0$, for all $i > 0$. But then by tilting theory we infer that $\mathcal{F}(T) \subset \text{Sub}T$. Also by tilting theory we infer that we get for $X \in \mathcal{F}(T)$ an exact sequence $0 \rightarrow X \rightarrow T_0 \rightarrow \cdots \rightarrow T_s \rightarrow 0$, with $T_0, \ldots, T_s \in \text{add} T$. Since pd$_A T \leq 1$ we see that we may choose $s = 1$ (cf. 1.2). 

Proposition 4.3. Let $A$ be a strict shod algebra and let $\mathcal{A}T = T_l \oplus T_r$ be a tilting module such that
(a) $T_r$ is Ext-projective in $\text{add}(\mathcal{A}\setminus \mathcal{L}_A)$,
(b) the torsion pair $(\mathcal{I}(T), \mathcal{F}(T))$ splits, and
(c) $\text{id}_A X \leq 1$ for each indecomposable $X \in \mathcal{F}(T)$ which is not a direct summand of $T$.

Let $\Gamma = \text{End}_A T$. Then $\Gamma$ satisfies:

(i) $\text{gl.dim} \Gamma = 2$ and $D(T_\Gamma)$ is a cotilting module.

(ii) $\text{pd}_F \mathcal{X}(T) \leq 1$.

(iii) If $Y \in \mathcal{Y}(T)$ is indecomposable nonprojective, then $\text{id}_T Y \leq 1$.

(iv) $\text{Ext}^2(\mathcal{Y}(T), \mathcal{X}(T)) \neq 0$.

**Proof.** We first show (ii). For this let $X \in \mathcal{Y}(T)$. Thus, there is $X' \in \mathcal{Y}(T)$ such that $X = \text{Ext}^1_A(T, X')$. By 4.2 we have that $X' \in \text{Sub} T$ and an exact sequence $0 \to X' \to T_0 \to T_1 \to 0$, with $T_0, T_1 \in \text{add} T$. Applying $\text{Hom}_A(T, -)$ yields the following exact sequence of $T$-modules:

$$0 \to \text{Hom}_A(T, T_0) \to \text{Hom}_A(T, T_1) \to X \to 0.$$}

Since $\text{Hom}_A(T, T_0)$ and $\text{Hom}_A(T, T_1)$ are both projective $T$-modules this shows that $\text{pd}_F X \leq 1$.

Next we show (iii). For this let $Y \in \mathcal{Y}(T)$ be an indecomposable nonprojective module. Thus there is $Y' \in \mathcal{Y}(T)$ such that $Y = \text{Hom}_A(T, Y')$. Since $Y$ is not projective we have that $Y'$ is not a direct summand of $T$. So by assumption (c) we have that $\text{id}_A Y' \leq 1$. Thus, $\text{Ext}^2_A(\mathcal{Y}(T), Y') = 0$. By 1.1(i), we see that then $\text{Ext}^1_A(\mathcal{Y}(T), Y) = 0$. By (ii) we have that $\text{Ext}^1_T(\mathcal{Y}(T), Y) = 0$. Since $(\mathcal{Y}(T), \mathcal{X}(T))$ is a torsion pair in $\text{mod} \Gamma$, we infer that $\text{id}_T Y \leq 1$. Also note that it follows from 2.6 that $\text{id}_A \mathcal{Y}(T) \leq 2$.

Next we show the first part of (i). The second always holds. For this it is enough to show while using (ii) that each $Y \in \mathcal{Y}(T)$ satisfies $\text{pd}_F Y \leq 2$. Let $Y' \in \mathcal{Y}(T)$ such that $Y = \text{Hom}_A(T, Y')$. By tilting theory there exists an exact sequence

$$0 \to T_s \to \cdots \to T_0 \to Y' \to 0$$

with $T_0, \ldots, T_s \in \text{add} T$. By 2.6 we have that $\text{id}_A T \leq 2$. But then it follows that we may use $s = 2$ (cf. Lemma 1.2). Applying $\text{Hom}_A(T, -)$ to this sequence then shows the assertion.

Finally we show (iv). Since $\text{gl.dim} A = 3$ there exists an injective $A$-module $I$ and a projective $A$-module $P$ such that $\text{Ext}^3_A(I, P) \neq 0$. Now $I \in \mathcal{Y}(T)$. Since $\text{id}_A P = 3$, we infer that $P$ is not a direct summand of $T$. We consider the torsion exact sequence

$$0 \to t(P) \to P \to P/t(P) \to 0.$$}

Then $P/t(P) \neq 0$. By the previous considerations we have that $\text{id}_A t(P) \leq 2$. Applying $\text{Hom}_A(I, -)$ to this sequence shows that $\text{Ext}^3_A(I, P/t(P)) \neq 0$. But then it follows from Lemma 1.1(iii) that $\text{Ext}^1_T(\mathcal{Y}(T), \mathcal{X}(T)) \neq 0$. □

Note that the canonical tilting module for a strict shod algebra satisfies the assumptions of the last proposition. In fact every proper successor of $J'$ is contained in $\mathcal{R}_A$, hence any indecomposable torsion module $X$ which is not a direct summand of $J'$ satisfies $\text{id}_A X \leq 1$. The second example given in Section 2 shows that there may exist more tilting modules satisfying these properties.
Corollary 4.4. Let $\Lambda$ be a strict shod algebra. Then the Hochschild cohomology groups $H^i(\Lambda) = 0$ for all $i \geq 3$.

Proof. Let $T$ and $\Gamma$ be as in 4.3. Then it follows from [12] that $H^i(\Lambda) \simeq H^i(\Gamma)$ and $H^i(\Gamma) = 0$ for $i \geq 3$, since $\text{gl.dim} \, \Gamma = 2$.

Note that it is shown in [7] that actually $H^i(\Lambda) = 0$ for all $i \geq 2$ for a strict shod algebra. The corollary gives an alternative proof for $i = 3$.

Next we will show the converse of Proposition 4.3. Since we want to deal with tilting modules we show the dual version. We will need some additional notation. If $M$ is a cotilting module for a strict shod algebra $\Lambda$ we have a dual canonical decomposition $M = M_1 \bigoplus M_r$, where $M_1 \in \text{add} (\mathcal{L}_A \setminus \mathcal{R}_A)$ and $M_r \in \mathcal{R}_A$. Again it is easy to see that $M_1 \neq 0 \neq M_r$. □

Proposition 4.5. Let $\Gamma$ be an artin algebra of global dimension two. Let $\Gamma T$ be a tilting module with $\text{id}_{\Gamma} \mathcal{F}(T) \leq 1$ and every indecomposable noninjective $X \in \mathcal{F}(T)$ satisfies $\text{pd}_{\Gamma} X \leq 1$. Then $\Lambda = \text{End}_{\Gamma} T$ is a shod algebra. If $\text{Ext}^2_{\mathcal{F}(T)}(\mathcal{F}(T), T) \neq 0$, then $\Lambda$ is strict shod. In this case the cotilting module $D(T) = M = M_1 \bigoplus M_r$ satisfies

(i) $M_1$ is Ext-injective in $\text{add} (\mathcal{L}_A \setminus \mathcal{R}_A)$,
(ii) the torsion pair $\langle \mathcal{X}(T), \mathcal{Y}(T) \rangle$ splits, and
(iii) $\text{pd}_{\mathcal{R}} X \leq 1$ for each indecomposable $X \in \mathcal{Y}(T)$ which is not a direct summand of $M$.

Proof. Since $\text{id}_{\Gamma} \mathcal{F}(T) \leq 1$ we see that the induced torsion pair $\langle \mathcal{X}(T), \mathcal{Y}(T) \rangle$ splits. Thus, $\text{Ext}^1_A(\mathcal{Y}(T), \mathcal{X}(T)) = 0$. Since $\mathcal{Y}(T)$ is a torsion free class containing $\mathcal{L}_A$, we infer that then also $\text{Ext}^2_A(\mathcal{Y}(T), \mathcal{X}(T)) = 0$. Since $\text{id}_{\Gamma} \mathcal{F}(T) \leq 1$ we have $\text{Ext}^2_{\mathcal{F}(T)}(\mathcal{F}(T), \mathcal{F}(T)) = 0$, hence $\text{Ext}^2_{\mathcal{F}(T)}(\mathcal{F}(T), \mathcal{F}(T)) = 0$. But then $\text{id}_{\mathcal{A}} \mathcal{X}(T) \leq 1$. Let $X \in \mathcal{Y}(T)$ be an indecomposable $\mathcal{A}$-module. Then there is $X' \in \mathcal{F}(T)$ such that $X = \text{Hom}_{\mathcal{F}}(T, X')$. If $X'$ is $\mathcal{A}$-injective, then we claim that $\text{id}_{\mathcal{A}} X \leq 1$. Assume to the contrary that $\text{id}_{\mathcal{A}} X \geq 2$. Then there exists an indecomposable $\mathcal{A}$-module $Y$ such that $\text{Ext}^2_{\mathcal{A}}(Y, X) \neq 0$. If $Y \in \mathcal{Y}(T)$ with $Y = \text{Hom}_{\mathcal{F}}(T, Y')$ for some $Y' \in \mathcal{F}(T)$, we infer by 1.1(v) that then also $\text{Ext}^2_{\mathcal{F}}(Y', X') \neq 0$ in contrast to $X'$ being injective. If $Y \in \mathcal{X}(T)$ with $Y = \text{Ext}^1_{\mathcal{F}}(T, Y')$ for some $Y' \in \mathcal{F}(T)$, we infer by 1.1(iv) that then $\text{Ext}^1_{\mathcal{F}}(Y', X') \neq 0$ in contrast to $X'$ being injective. So it remains to consider the case that $X'$ is not injective. Then we have by assumption that $\text{pd}_{\mathcal{F}} X' \leq 1$. Since $X' \in \mathcal{F}(T)$ and $\text{pd}_{\mathcal{F}} X' \leq 1$ there exists an exact sequence

$$0 \to T_1 \to T_0 \to X' \to 0$$

with $T_0, T_1 \in \mathcal{F}$. Apply $\text{Hom}_{\mathcal{F}}(T, -)$ to this sequence. This then shows that $\text{pd}_{\mathcal{F}} X \leq 1$. In particular, we see that $\mathcal{A}$ is a shod algebra. Note that this also shows (iii) by observing that $M = \text{Hom}_{\mathcal{F}}(T, D(T))$.

From Lemma 1.1(iv) we know that $\text{Ext}^1_{\mathcal{A}}(\mathcal{X}(T), \mathcal{Y}(T)) \simeq \text{Ext}^1_{\mathcal{F}}(\mathcal{F}(T), \mathcal{F}(T))$. If the last term is nonzero, then $\text{gl.dim} \, \mathcal{A} = 3$. Since $\mathcal{A}$ is a shod algebra this just means that $\mathcal{A}$ is strict shod.

Let $D(T) = M = M_1 \bigoplus M_r$ be the induced cotilting module. We know that $M = \text{Hom}_{\mathcal{F}}(T, D(T))$. We have observed before that the torsion pair $\langle \mathcal{X}(T), \mathcal{Y}(T) \rangle$ splits.
Moreover we have seen that \( \text{id}_A \mathcal{T}(T) \leq 1 \). This implies that \( \mathcal{X}(T) \subseteq \mathcal{Y}(T) \). Since \( \text{Ext}^1_A(\mathcal{Y}(T), M) = 0 \), we infer that \( M_i \) is Ext-injective in \( \text{add}(L_A \setminus R_A) \). 

It is easy to construct examples of algebras of global dimension two with a tilting module \( T \) satisfying the assumptions of Proposition 4.5 but \( \text{Ext}^2(B, \mathcal{T}(T), \mathcal{T}(T)) = 0 \). For example consider the following algebra given as quiver with relations.

Let \( T = \tau^{-} S \bigoplus P \), where \( S \) is the unique simple projective module and \( P \) is the direct sum of the remaining indecomposable projectives. Then \( \text{gl dim } \text{End } T = 2 \).

Proposition 4.5 and the dual of Proposition 4.3 now give a bijective correspondence between

\[
\mathcal{S} = \{(A, M) \mid A \text{ strict shod, } M \text{ cotilting module, } M \text{ Ext-injective in } \text{add}(L_A \setminus R_A), \text{ the torsion pair } (\mathcal{X}(T), \mathcal{Y}(T)) \text{ splits, and } \text{pd}_A X \leq 1 \text{ for each indecomposable } X \in \mathcal{Y}(T) \text{ not a direct summand of } T \}
\]

and

\[
\mathcal{G} = \{(\Gamma, M) \mid \text{gl dim } \Gamma = 2, \text{M tilting module, } \text{id}_\Gamma X \leq 1 \text{ for } X \in \mathcal{F}(T), \text{ and } \text{pd}_\Gamma Y \leq 1 \text{ for all noninjective } Y \in \text{ind } \mathcal{F}(T) \text{ and } \text{Ext}^2_{\Gamma}(\mathcal{F}(T), \mathcal{F}(T)) \neq 0 \}. \]

We know from Sections 3 and 4 that \( \mathcal{S} \) is not empty. The opposite algebra of the second example in Section 4 shows that for a fixed strict shod algebra \( A \) there may exist more than one cotilting module \( M \) such that \( (A, M) \in \mathcal{S} \).

Also note that there exists a representation-finite algebra \( \Gamma \) and a tilting module \( \Gamma \) such that \( (\Gamma, M) \in \mathcal{G} \) and \( \Gamma \) is not representation directed. For this consider the first example \( A \) in Section 4 and let \( A M \) be a cotilting module such that \( (A, M) \in \mathcal{S} \).

5. Properties of algebras in \( \mathcal{G} \)

In this section, we will investigate in more detail the algebras contained in the set \( \mathcal{G} \) which we defined at the end of the last section.

We start with an elementary observation.

**Lemma 5.1.** Let \( (\Gamma, M) \in \mathcal{G} \). Let \( S \) be a simple \( \Gamma \)-module. Then \( \text{pd}_\Gamma S \leq 1 \text{ or } \text{id}_\Gamma S \leq 1 \).

**Proof.** Let \( (\mathcal{F}(T), \mathcal{F}(T)) \) be the torsion pair induced by \( T \). Let \( S \) be a simple \( \Gamma \)-module. Then \( S \in (T(T) \cup \mathcal{F}(T)) \). If \( S \in \mathcal{F}(T) \), then \( \text{id}_\Gamma S \leq 1 \). If \( S \in \mathcal{F}(T) \) is not injective, then \( \text{pd}_\Gamma S \leq 1 \). 

**Lemma 5.2.** Let \( (\Gamma, M) \in \mathcal{G} \). Then \( \Gamma T = P \bigoplus \Gamma C \), where \( 0 \neq P \) is projective and \( 0 \neq \Gamma C \) has no indecomposable projective direct summand and \( \text{Hom}_\Gamma(\Gamma C, \Gamma) = 0 \).
Proposition 5.3. With the notation above we have that $\Gamma'$ is a hereditary artin algebra and $\tau_F C$ is a $\Gamma'$-tilting module. In particular we have that $\text{End}_I C$ is a tilted algebra.

Proof. We identify $\text{mod} \Gamma'$ with the full subcategory of $\text{mod} \Gamma$ containing those $\Gamma$-modules $\tau_F X$ such that $\text{Hom}_I(P, X) = 0$. The following properties are straightforward. For $X, Y \in \text{mod} \Gamma'$ we have that $\text{Ext}_I^1(\tau_F C, \tau_F C) = \text{Ext}_I^1(X, Y).$ Also we infer that $\mathcal{F}(T) \subset \text{mod} \Gamma'$, thus $\tau_F C \in \text{mod} \Gamma'$. We claim that $\text{Ext}_I^1(\tau_F C, \tau_F C) = 0$ and that $\text{End}_I \tau_F C \simeq \text{End}_I C$.

Indeed, by the first property we that $\text{Ext}_I^1(\tau_F C, \tau_F C) \simeq \text{Ext}_I^1(\tau_F C, \tau_F C).$ By the Auslander–Reiten formula [3] we infer that the second term is isomorphic to $D\text{Hom}_I(C, \tau C).$ By Lemma 5.2, we have that $D\text{Hom}_I(C, \tau C) = D\text{Hom}_I(C, \tau C).$ Since $\text{pd}_I C = 1$, we infer that $D\text{Hom}_I(C, \tau C) = \text{Ext}_I^1(C, C) = 0$, since $C$ is a direct summand of a tilting module. Clearly we have that $\text{End}_I \tau_F C \simeq \text{End}_I \tau_F C$. Since $\text{pd}_I C = 1$ we see that $\text{End}_I \tau_F C \simeq \text{End}_I \tau_F C$, which in turn is isomorphic to $\text{End}_I C$. Using again 5.2 we see that $\text{End}_I(C) = \text{End}_I C$.

Let $P' = \Gamma(1 - e)$. We consider the minimal add $P$-approximation $f : \tilde{P} \rightarrow P'$ of $P'$. By construction we infer that $Q' = \text{cok} f \in \text{mod} \Gamma'$. Indeed, let $B = \text{im} f$. Then $B \in \mathcal{F}(T)$ and we have an exact sequence

$$(*) \quad 0 \rightarrow B \xrightarrow{n} P' \xrightarrow{\pi} Q' \rightarrow 0.$$ 

Applying $\text{Hom}_I(P, -)$ to $(*)$ yields the following exact sequence

$$0 \rightarrow \text{Hom}_I(P, B) \rightarrow \text{Hom}_I(P, P') \rightarrow \text{Hom}_I(P, Q') \rightarrow \text{Ext}_I^1(P, B).$$

Since $B \in \mathcal{F}(T)$ we see that the last term vanishes. By construction we have that the first map is surjective, hence $Q' \in \text{mod} \Gamma'$.

Next we show that $Q'$ is a projective generator for $\text{mod} \Gamma'$. Let $Z \in \text{mod} \Gamma'$. Applying $\text{Hom}_I(-, Z)$ to $(*)$ yields the exact sequence

$$\text{Hom}_I(B, Z) \rightarrow \text{Ext}_I^1(Q', Z) \rightarrow \text{Ext}_I^1(P', Z).$$

Since $P'$ is projective the last term vanishes. The surjection $\tilde{P} \rightarrow B$ induces an injective map from $\text{Hom}_I(B, Z)$ to $\text{Hom}_I(\tilde{P}, Z)$. Since $Z \in \text{mod} \Gamma'$ and $\tilde{P} \in \text{add} P$ we infer
Thus \( \text{Hom}_F(B, Z) = 0 \). Hence \( 0 = \text{Ext}^1_F(Q', Z) \simeq \text{Ext}^1_F(Q', Z) \). Thus \( Q' \) is a projective \( \Gamma' \)-module. Let \( S \) be a simple \( \Gamma' \)-module. Then \( S \) is a simple \( \Gamma \)-module with projective cover in \( \text{add} P' \) and \( \text{Hom}_F(P, S) = 0 \). Applying \( \text{Hom}_F(-, S) \) to (**) shows that the surjection \( \pi : P' \to S \) induces a map \( \beta : Q' \to S \) such that \( \pi = \pi \beta \), hence \( \beta \) is surjective. Thus \( Q' \) is a projective generator for \( \text{mod} \Gamma' \).

Next we claim that \( Q' \in \mathcal{F}(T) \). For this apply \( \text{Hom}_F(C, -) \) to the exact sequence (***). Since \( \text{Hom}_F(C, P') = 0 \) and \( \text{Ext}^1_F(C, B) = 0 \), since \( B \in \mathcal{F}(T) \), we infer that \( \text{Hom}_F(C, Q') = 0 \). By assumption we have that then \( \text{id}_F Q' \leq 1 \), and so \( \text{Ext}^1_F(Q', Q') = 0 \) for all \( i > 0 \).

Next, we claim that for \( X, Y \in \text{mod} \Gamma' \) we have that \( \text{Ext}^2_F(X, Y) \simeq \text{Ext}^2_F(X, Y) \). Indeed, consider the exact sequence of \( \Gamma' \)-modules

\[
(**) \quad 0 \to X' \to Q'_0 \to X \to 0
\]

with \( Q'_0 \in \text{add} Q' \). Then \( \text{Ext}^2_F(X, Y) \simeq \text{Ext}^1_F(X', Y) \simeq \text{Ext}^1_F(X', Y) \). Also consider the exact sequence of \( \Gamma' \)-modules

\[
(***) \quad 0 \to Y' \to Q'_1 \to Y \to 0
\]

with \( Q'_1 \in \text{add} Q' \). Applying \( \text{Hom}_F(Q', -) \) to (*** ) yields the following exact sequence

\[
\text{Ext}^1_F(Q', Q'_1) \to \text{Ext}^2_F(Q', Y) \to \text{Ext}^2_F(Q', Y).
\]

As noticed before the first term vanishes. The last term vanishes since \( \text{gl} \dim \Gamma = 2 \), hence \( \text{Ext}^2_F(Q', Y) = 0 \). We consider (**) as a sequence of \( \Gamma \)-modules. Applying \( \text{Hom}_F(-, Y) \) to (**) then yields the following exact sequence:

\[
\text{Ext}^1_F(Q'_0, Y) \to \text{Ext}^1_F(X', Y) \to \text{Ext}^2_F(X, Y) \to \text{Ext}^2_F(Q'_0, Y).
\]

By the previous considerations we see that the first and the last term vanish, hence we have that \( \text{Ext}^1_F(X', Y) \simeq \text{Ext}^2_F(X, Y) \), which shows the claim.

As an immediate consequence we see that a \( \Gamma' \)-module \( Z \) with \( \text{id}_F Z \leq 1 \) will satisfy \( \text{id}_F Z \leq 1 \). Indeed, let \( Y \in \text{mod} \Gamma' \). Then \( 0 = \text{Ext}^2_F(Y, Z) \simeq \text{Ext}^2_F(Y, Z) \). In particular we see that each \( Z \in \mathcal{F}(T) \) satisfies \( \text{id}_F Z \leq 1 \).

Next let \( Z \in \text{mod} \Gamma' \). We claim that \( \text{id}_F Z \leq 1 \). For this consider

\[
0 \to Z' \to Q'_0 \to Z \to 0
\]

exact with \( Q'_0 \in \text{add} Q' \). Since \( Q' \in \mathcal{F}(T) \) and \( \mathcal{F}(T) \) is a torsion free class, we infer that \( Z' \in \mathcal{F}(T) \). But then \( \text{id}_F Z' \leq 1 \), hence \( \text{id}_F Z \leq 1 \). This shows that \( \Gamma' \) is a hereditary artin algebra.

Since \( \tau_F C \in \text{mod} \Gamma' \) we have that \( \text{pd}_F \tau_F C \leq 1 \). Since \( \text{Ext}^1_F(\tau_F C, \tau_F C) = 0 \) and \( \tau_F C \) has the correct number of indecomposable direct summands we infer that \( \tau_F C \) is a \( \Gamma' \)-tilting module, hence \( \text{End}_F(\tau_F C) \) is a tilted algebra. By the first part of the proof we then also have that \( \text{End}_F C \) is a tilted algebra. \( \square \)

It is easy to construct examples satisfying the conditions of the last proposition such that \( \text{End}_F C \) is not hereditary.

**Proposition 5.4.** With the notation above let \( \Gamma'' = \text{End}_F P \). Then \( \Gamma'' \) is a tilted algebra such that an indecomposable noninjective \( \Gamma'' \)-module \( Y \) satisfies \( \text{pd}_F Y \leq 1 \).
Proof. We consider the perpendicular category
\[ C^\perp = \{ Z \in \mod \Gamma \mid \hom_{\Gamma}(C, Z) = \ext^1_{\Gamma}(C, Z) = 0 \}. \]
Then clearly \( C^\perp \subseteq \mathcal{F}(T) \). Also \( P \in C^\perp \) by 5.2. Let \( X \in C^\perp \). Since by assumption we have that \( \text{gl} \ \dim \ BNUL = 2 \) there exists by 1.2 an exact sequence
\[
0 \to T_2 \to T_1 \to T_0 \to X \to 0
\]
with \( T_0, T_1, T_2 \in \add T \). It is easy to see that in fact \( T_0, T_1, T_2 \in \add P \). So \( P \) is a projective generator in \( C^\perp \). Thus \( C^\perp \simeq \mod \ \text{End}_P \mod \ = \mod \ I'' \). Let \( X \in C^\perp \). Then the above sequence is a projective resolution of \( X \) when considered as \( I'' \)-module; but also when considered as \( I \)-module. Thus \( \text{pd}_{I''} X = \text{pd}_{I''} X \). But if \( X \in C^\perp \) is an injective \( I'' \)-module, then clearly \( X \) is an indecomposable module which is not injective as \( I'' \)-module, then \( X \) is not injective as \( I \)-module, hence \( \text{pd}_{I} X \leq 1 \). The assertion now follows from Lemma 1.4. □

Note that in Proposition 1.6 we have given a characterization of these algebras.

We will now consider classes of algebras \( I \) of global dimension two and show that they do not admit a tilting module \( \Gamma T \) such that \( (\Gamma, \Gamma T) \in \mathcal{G} \).

For simplicity we will consider finite dimensional algebras over an algebraically closed field \( k \). We refer for the details on the category of coherent sheaves on a weighted projective line to [9].

Proposition 5.5. Let \( I \) be a quasitilted algebra such that \( I = \text{End}_M \) for a tilting object \( M \in \mathcal{H} \) and \( \mathcal{H} \) is derived equivalent to \( \text{coh}_X \) for a weighted projective line with at least four nontrivial weights. Then for all \( I \)-tilting modules \( \Gamma T \) the pair \( (\Gamma, \Gamma T) \notin \mathcal{G} \).

Proof. Let \( t \) be the number of nontrivial weights. Then it was shown in [14] that \( \dim_k H^2(\Gamma) = t - 3 \), so by assumption we have that \( H^2(\Gamma) \neq 0 \). By the result in [7] we have that a strict shod algebra \( A \) satisfies \( H^2(A) = 0 \). So if \( I \) would admit a tilting module \( \Gamma T \) such that the pair \( (\Gamma, \Gamma T) \in \mathcal{G} \) we would have a strict shod algebra \( A = \text{End}_{\Gamma} T \) with \( H^2(A) \neq 0 \), using the tilting invariance of Hochschild cohomology from [13]. □

The following easy lemma also excludes certain algebras of global dimension two.

Lemma 5.6. Let \( I \) be an algebra of global dimension two. If there is a simple injective \( I \)-module \( S \) such that \( \text{pd}_{I} S = 2 \) and that \( \text{pd}_{\Gamma T} S = 2 \), then for all \( I \)-tilting modules \( \Gamma T \) the pair \( (\Gamma, \Gamma T) \notin \mathcal{G} \).

Proof. Assume to the contrary that \( \Gamma T \) is a tilting module such that the pair \( (\Gamma, \Gamma T) \in \mathcal{G} \). Since \( \tau T S \) is not injective and \( \text{pd}_{I} \tau T S = 2 \) we see that \( \tau T S \notin \mathcal{F}(T) \), hence \( \ext^1(T, \tau T S) \neq 0 \). Since \( \text{id}_{I} \tau T S = 1 \), we infer by using the Auslander–Reiten formula that \( \ext^1(T, \tau T S) \neq 0 \) and \( \text{id}_{I} \tau T S \neq 1 \).
As an application of this lemma we will show that a canonical algebra \( C \) of global dimension two (cf. [21]) will not admit a tilting module \( \mathcal{T} \) such that the pair \((C, \mathcal{T}) \in \mathcal{G}\). It is straightforward to see that a canonical algebra which does not satisfy the assumptions of 5.6 is a concealed algebra of type \( \tilde{D}_4 \). But then the simple injective \( S \) is the only indecomposable module such that \( \text{pd}\_B S = 2 \). But then \( S \in \mathcal{T}(T) \) and so \( \text{pd}\_B \mathcal{T}(T) \leq 1 \). Thus \( \text{Ext}^2_B(\mathcal{T}(T), \mathcal{T}(T)) = 0 \).

Finally, we consider the case of Auslander algebras. For details, we refer to [3]. For this let \( A \) be a basic representation finite algebra over an algebraically closed field \( k \) and let \( X_1, \ldots, X_m \) be a complete set of representatives from the isomorphism classes of the indecomposable \( A \)-modules. Set \( \mathcal{I} = \text{End}_A(\bigoplus_{i=1}^m X_i) \). Then \( \mathcal{I} \) is an Auslander algebra and it is well known that \( \text{gl dim}\_A \mathcal{I} \leq 2 \). If \( X \) is an indecomposable \( \mathcal{I} \)-module from the above list we denote by \( S_X \) the corresponding simple \( \mathcal{I} \)-module. So \( S_X \) is the top of the indecomposable projective \( \mathcal{I} \)-module \( \text{Hom}_A(\bigoplus_{i=1}^m X_i, X) \). It is well known that \( \text{pd}\_B S_X \leq 1 \) iff \( X \) is projective as \( A \)-module and that \( \text{id}\_B S_X \leq 1 \) iff \( X \) is injective as \( A \)-module. This will be used in the following proposition.

**Proposition 5.7.** Let \( \mathcal{I} \) be an Auslander algebra. Then there is no \( \mathcal{I} \)-tilting module \( rT \) such that \((r\mathcal{I}, rT) \in \mathcal{G}\).

**Proof.** Let \( \mathcal{I} \) be an Auslander algebra with \( \text{gl dim}\_A \mathcal{I} = 2 \). Assume that there exists a \( \mathcal{I} \)-tilting module \( rT \) such that \((r\mathcal{I}, rT) \in \mathcal{G}\). Let \( A \) be the representation finite algebra associated with \( \mathcal{I} \). Let \( S = S_X \) be a simple \( \mathcal{I} \)-module for some indecomposable \( A \)-module \( X \). So from Lemma 5.1 we infer that \( \text{pd}\_B S \leq 1 \) or \( \text{id}\_B S \leq 1 \). By the previous remarks we infer that \( X \) is either a projective \( A \)-module or an injective \( A \)-module. So each indecomposable \( A \)-module is either projective or injective. In particular each simple \( A \)-module is either projective or injective. But then \( \text{rad}^2 A = 0 \) and \( A \) is a hereditary algebra with the property that each indecomposable \( A \)-module is either projective or injective. It is straightforward to see that the only possibilities are the path algebras over \( k \) of the following quivers:

```
 o---o---o---o---o
```

If \( rT \) is a tilting module, then any indecomposable projective-injective is a direct summand of \( T \). In all three cases the simple projective \( \mathcal{I} \)-modules \( P \) satisfy \( \text{id}\_B P = 2 \), hence cannot be torsion free and so are direct summands of \( T \). This shows that in the first case \( rT = r\mathcal{I} \), but this contradicts the fact that the torsion pair is nontrivial. In the second case it is easily seen that the indecomposable projective which is neither simple nor injective is an extension of a simple \( rS \) with \( \text{id}\_B S = 2 \) by the two simple projectives. So \( S \in \mathcal{T}(T) \) by assumption and so also in this case \( rT = r\mathcal{I} \), but this again contradicts the fact that the torsion pair is nontrivial. The last case is similar.
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