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Alternating Quotients of Fuchsian Groups

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It is shown that any finitely generated, non-elementary Fuchsian group has among its homomorphic images all but finitely many of the alternating groups A_n. This settles in the affirmative a long-standing conjecture of Graham Higman.

1. INTRODUCTION

It all started with a theorem of Miller [14]: the classical modular group $\mathrm{PSL}_2(\mathbf{Z})$ has among its homomorphic images every alternating group, except A_6, A_7, and A_8. In the late 1960s Graham Higman conjectured that any (finitely generated non-elementary) Fuchsian group has among its homomorphic images all but finitely many of the alternating groups. This reduces to an investigation of the cocompact (p, q, r)-triangle groups, and in the series of papers [3, 4, 9, 15, 16] the conjecture was verified in the affirmative when $p = 2$. Assuming the Fuchsian group is finitely generated and non-elementary, and taking "almost all" to be synonymous with "all but finitely many," and "surjects" with "has among its homomorphic images," we build on this earlier work to prove

THEOREM. *Any Fuchsian group surjects almost all of the alternating groups.*

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There are several motivations behind the conjecture: Fuchsian groups have an algebraic structure that is somewhat complicated, and to get a firmer grip on this situation, one may be tempted to consider their finite, or even simple, homomorphic images. There is also a geometric incentive, namely, any compact Riemann surface (or complex algebraic curve) of genus > 1 has conformal automorphism group a finite homomorphic image of some Fuchsian group.

Schreier coset diagrams supply the technology used to prove the theorem, and they appear in the literature in various guises (see [2, 11] for alternative formulations as hypermaps or *dessin d'enfants*). Section 3 has the definition and the basic properties. Section 4 contains the proof of the theorem.

2. THE PLAN

Suppose X is the 2-sphere S^2, the Euclidean plane $\mathbf{E}^2$, or the hyperbolic plane $\mathbf{H}^2$. Let G be a finitely generated non-elementary discrete group of orientation preserving isometries of X. By classical work of Fricke and Klein (see, for instance, [21]), G has a presentation of the form

$$\begin{array}{rll} \text{generators}: & a_1, b_1, \ldots, a_g, b_g & \text{(hyperbolic)}, \\ & x_1, \ldots, x_e & \text{(elliptic)}, \\ & y_1, \ldots, y_s & \text{(parabolic)}, \\ & z_1, \ldots, z_t & \text{(hyperbolic boundary elements)}. \\ \text{relations}: & x_1^{m_1} = \cdots = x_e^{m_e} = 1, & \\ & \prod_{i=1}^{e} x_i \prod_{j=1}^{s} y_j \prod_{k=1}^{t} z_k \prod_{l=1}^{g} [a_l, b_l] = 1. & \end{array}$$

When $X = \mathbf{H}^2$, G is called a Fuchsian group. The division into spherical, Euclidean, and Fuchsian is governed by the quantity

$$\mu(G) = 2g - 2 + \sum_{i=1}^{e}\left(1 - \frac{1}{m_i}\right) + s + t, \tag{1}$$

with $\mu(G) < 0, = 0$, or > 0 as $X = S^2, \mathbf{E}^2$, or $\mathbf{H^2}$. The quotient X/G is an orientable 2-orbifold of genus g with e cone points, s punctures, and t boundary components. Its geometry and the algebraic structure of G are intimately connected, so that G is determined up to isomorphism by its signature $(g; m_1, \ldots, m_e; s; t)$, $2 \leq m_1 \leq \cdots \leq m_e$.

To prove the theorem, it suffices to just consider the cocompact Dyck groups—the cases where in the signature we have $g = s = t = 0$. To see why, we make a few elementary observations.

1. A group of signature $(g; m_1, \ldots, m_e; s; t)$ is isomorphic to one of $(g; m_1, \ldots, m_e; s+t; 0)$, and by (1), the former is Fuchsian if and only if the latter is. We may assume then that $t = 0$. Write $(g; m_1, \ldots, m_e; s)$ instead of $(g; m_1, \ldots, m_e; s; 0)$ from now on.

2. We can surject $G = (g; m_1, \ldots, m_e; s)$ onto $G' = (g'; m_1, \ldots, m_i', \ldots, \hat{m}_j, \ldots, m_e; s')$, for any $g' \leq g$, $s' \leq s$, and m_i' a divisor of m_i. The hat denotes omission. Here is how: map the jth elliptic, $s - s'$ of the parabolic, and $g - g'$ hyperbolic pairs of generators of G to the identity of G'; map the ith elliptic generator of G to the corresponding elliptic generator of G' raised to the power m_i/m_i'. All other generators of G map to the corresponding ones in G'. The map then extends to the desired homomorphism.

3. Writing $(m_1, \ldots, m_e)$ when $g = s = 0$, suppose

$$\psi : G = (m_1, \ldots, m_e) \to S_n$$

is a homomorphism with transitive image and let G_j be the subgroup of G consisting of those elements stabilising some fixed point j of $\{1, 2, \ldots, n\}$. By Theorem 1 of [17], G_j has signature $(g'; n_{11}, n_{12}, \ldots, n_{1\rho_1}, \ldots, n_{r1}, n_{r2}, \ldots, n_{r\rho_r})$, where $\psi(x_i)$ has exactly one cycle each of lengths $m_i/n_{i1}, \ldots, m_i/n_{i\rho_i}$, with all other cycles of length m_i, and $\mu(G_j) = n\mu(G)$. Moreover, if G_j is normal in G, and we have the theorem for G, the simplicity of A_n for $n \geq 5$ gives the result for G_1 as well.

4. Finally, any k-cycle $(a_1, \ldots, a_k) \in A_n$ can be written as a product

$$(a_2, a_k)(a_3, a_{k-1}) \cdots (a_{k/2+1/2}, a_{k/2+3/2})(a_1, a_2)(a_3, a_k) \cdots (a_{k/2+1/2}, a_{k/2+5/2})$$

of two involutions in A_n. Similarly any cycle of even length in S_n can be written as a product of a involution in S_n and an involution in A_n. Thus, if we have the result for $(m_1, \ldots, k, \ldots, m_e; s)$ we have it for $(m_1, \ldots, 2, 2, \hat{k}, \ldots, m_e; s)$ too.

LEMMA 2.1. *The theorem is true for every Fuchsian group if it holds for every Dyck group.*

Proof. Proceeding according to the genus, suppose G has signature $(g, m_1, \ldots, m_e; s)$ with $g \geq 2$. Map G onto $\langle x, y | - \rangle$, free of rank 2, by sending $a_1 \mapsto x$, $a_2 \mapsto y$, and all the other generators to the identity. Since A_n is 2-generated for $n \geq 3$ (see [5]), we are done.

A group of genus 1 with $e \geq 1$ can be surjected onto $(1; m_1; 0)$ for $m_1 \geq 2$, by comment 2 above. The map $\psi : (0; 2, 2, 2, 2m_1; 0) \to S_2$ sending all generators to the permutation $(1, 2)$ has kernel isomorphic to $(1; m_1; 0)$ by comment 3 above; hence the result holds for groups of genus 1 with $e \geq 1$ by the assumption of the Lemma. For groups of genus 1 with no elliptic generators or periods, hence signature $(1;-;s)$ for $s \geq 1$, we may surject onto $(1;-;1)$. But this is easily seen to be free of rank 2, so the result holds here also.

A group of genus 0 with no periods must, by (1), have at least three parabolic generators, and hence surject $(0;-;3)$. But this is free of rank 2 also. With a single period we have $s \geq 2$, and the group surjects $(0;m_1;2) \cong \mathbf{Z}_{m_1} * \mathbf{Z}$, the free product of $\mathbf{Z}_{m_1}$ and $\mathbf{Z}$. This surjects $\mathbf{Z}_{m_1} * \mathbf{Z}_3$, which in turn surjects any Fuchsian triangle group of the form $(0;3,m_1,r;0)$ for which the result holds by assumption.

With two periods and one parabolic, we have $(0;m_1,m_2;1) \cong \mathbf{Z}_{m_1} * \mathbf{Z}_{m_2}$, where $m_2 \geq 3$, so we can surject any Fuchsian triangle group like $(0;m_1,m_2,r;0)$. A group with more parabolics, $(0;m_1,m_2;s)$ for $s \geq 2$, surjects $(0;m_1;2)$ done above. Finally, $(0;m_1,\dots,m_e;s)$, $e \geq 3$, surjects either $\mathbf{Z}_2 * \mathbf{Z}_2 * \mathbf{Z}_2$ or $\mathbf{Z}_{m_1} * \mathbf{Z}_{m_2}$ for $m_2 \geq 3$. Surject the former onto a Fuchsian $(0;2,2,2,p;0)$. The latter has already been handled. ∎

LEMMA 2.2. *The theorem holds for every Dyck group if it holds for the following*:

1. *the Fuchsian triangle groups* (p,q,r) *with* $2 \leq p < q < r$ *distinct primes*;
2. *the triangle groups* $(2,4,r)$ *for* $r \geq 5$ *a prime*;
3. *the groups* $(2,3,8)$, $(2,3,9)$, $(2,3,10)$, $(2,3,12)$, $(2,3,15)$, $(2,3,25)$, $(2,4,6)$, $(2,4,8)$, $(2,4,9)$, $(2,5,6)$, $(2,5,9)$ *and* $(3,4,5)$;
4. *the groups* $(2,3,3,3)$ *and* $(3,3,3,3)$.

Proof. The hyperbolic triangle group $(2,m_1,m_2)$ surjects $(2,q,r)$ for q and r some prime divisors of m_1 and m_2. If $(2,q,r)$ is Fuchsian, we have by (1) that $1/q + 1/r < 1/2$. If q and r are distinct, we have a group listed in part 1 of the lemma. If $q = r$, the map $\psi\colon (2,q,4) \to S_2$ that sends the generators of orders 2 and 4 to the permutation $(1,2)$ and the generator of order q to the identity has kernel $(q,q,2) \cong (2,q,q)$. We have $2/q < 1/2$, hence $q \geq 5$, and the theorem holds for $(2,q,q)$ as it holds for $(2,4,q)$, a group listed in part 2 of the lemma.

If $(2,q,r)$ is not Fuchsian, it must be, after a possible reordering, one of $(2,2,r)$ for $r \geq 2$, $(2,3,3)$, or $(2,3,5)$. The first gives that $(2,m_1,m_2)$ must have the form $(2,m_1,2^l)$, for $m_1 \geq 3$ and $l \geq 2$. If $m_1 = 3$ or 4 then $l \geq 3$, as $(2,3,4)$ is spherical and $(2,4,4)$ Euclidean, so the group surjects $(2,3,8)$ or $(2,4,8)$, both of which are listed in the lemma. For $m_1 \geq 5$, $(2,m_1,2^l)$ surjects $(2,m_1,4) \cong (2,4,m_1)$. This in turn surjects $(2,4,r)$, r prime, and we have a group listed in part 2 unless $r = 2$ or 3. In the first case, $m_1 = 2^n \geq 8$, so $(2,4,m_1)$ surjects $(2,4,8)$. In the second, $m_1 = 2^{l_1}3^{n_1}$, and the group surjects $(2,4,9)$ when $l_1 = 0$, or $(2,4,6)$ otherwise. The cases $(2,q,r) = (2,3,3)$ or $(2,3,5)$ are entirely similar.

This accounts for the $(2, m_1, m_2)$ Fuchsian groups, and the case of general triangle groups is much the same. Similarly for the groups with four or five elliptic generators—either they can be surjected directly onto triangle groups or they can be eliminated from consideration using comment 4 at the beginning of the section. The only exceptions are those listed in the lemma. Finally, a group with six or more elliptic generators can always be surjected directly onto a Fuchsian group with five. No doubt the reader can fill in the details. ∎

In [3, 4, 9], the groups $(2, 3, r)$ for all $r \geq 7$ and $(2, 4, r)$ for all $r \geq 5$ were dealt with. Theorems 1–3 of [15] take care of the $(2, q, r)$, $5 \leq q < r$ prime, with the exception of 60 cases. These 60, and those from parts 3 and 4 of Lemma 2.2 can be found in the preprint version of this paper [7, Sect. 6]. This leaves the triangle groups (p, q, r), $3 \leq p < q < r$, to consider, and they can be found in Section 4.

Later we will construct permutation groups as homomorphic images of Fuchsian groups and will identify the images as alternating, using

THEOREM 2.1 ([12]; Refer to [19, Theorem 13.9]). *Let G be a primitive permutation group of degree n containing a prime cycle for some prime $q \leq n - 3$. Then G is either the alternating group A_n or the symmetric group S_n.*

The following lemma, well known to the cognoscenti, allows one to replace primitivity by more easily verifiable criteria. Recall that the support of a permutation $\sigma \in S_n$ consists of those elements of $\{1, 2, \dots, n\}$ not fixed by σ.

LEMMA 2.3. *Let $G = \langle \sigma_1, \sigma_2, \dots, \sigma_k \rangle$ be a transitive permutation group of degree n containing a prime cycle μ. For each σ_i, suppose there is a point in the support of μ whose image under σ_i is also in the support of μ. Then G is primitive.*

Proof. Suppose on the contrary that G is imprimitive with block system $\mathfrak{B}$. For $\sigma \in G$, let $\bar{\sigma}$ be the permutation induced by σ on $\mathfrak{B}$, and let $\overline{G}$ be the group generated by the $\bar{\sigma}_i$. The map $\sigma \mapsto \bar{\sigma}$ is an epimorphism from G onto $\overline{G}$, and $\overline{G}$ acts transitively on $\mathfrak{B}$. All blocks $B \in \mathfrak{B}$ thus have the same size, say $|B|$. If $B \in \mathfrak{B}$ is in $\text{supp}(\bar{\mu})$, the support of $\bar{\mu}$, then B and its image under μ are distinct blocks, and so B is contained in $\text{supp}(\mu)$. Taking the union of all the blocks in $\text{supp}(\bar{\mu})$ thus gives

$$|B|\,|\text{supp}(\bar{\mu})| \leq |\text{supp}(\mu)|. \tag{2}$$

Now μ has order q a prime, and $\bar{\mu}$ is a homomorphic image of μ. Thus, if $\bar{\mu} \neq 1$, then $\bar{\mu}$ has order q, and so $|\text{supp}(\bar{\mu})| \geq q$. Since $\mathfrak{B}$ is non-trivial, we have $|B| > 1$, and hence, by (2), $|\text{supp}(\mu)| > q$. This contradicts the fact that μ is a q-cycle, so we must have $\bar{\mu} = 1$. This means that $\bar{\mu}$ fixes every block, or equivalently, any point and its image under μ lie in the same block. But μ is a single cycle, so there is a block B^* with $\mathrm{supp}(\mu) \subseteq B^*$. By the condition stated in the lemma, B^* and its image under σ_i intersect for all i, and so are equal. Since the σ_i generate G, the whole group must fix B^*, and by transitivity, $B^* = \{1, 2, \ldots, n\}$, so there is just one block. This is the desired contradiction. ∎

3. COSET DIAGRAMS

Suppose G is a group with a finite presentation $\langle X; R \rangle$, and let $K_0 = K_0(X; R)$ be the standard 2-dimensional *CW*-complex with $\pi_1(K_0) \cong G$. The 1-skeleton of K_0 consists of a single vertex incident with oriented loops or *edges* that are in one-to-one correspondence with the generators X. Each edge $x \in X$ is a pair of oppositely oriented arcs, an x-arc and an x^{-1}-arc. The former coincides with the edge under its given orientation and the latter with the edge with the reverse orientation. The faces of K_0 are in one-to-one correspondence with the relators R and are obtained by sewing discs onto the 1-skeleton, each with boundary label a relator word $r \in R$; see [10, Sect. 6.3].

A Schreier coset diagram for G is a cellular (that is, k-cells lift to k-cells) covering of K_0 (see [18, Sects. 2.2.1 and 4.3.2] or [1]). A covering K realises a subgroup $H \cong \pi_1(K)$ of G, with the vertices of K in one-to-one correspondence with the cosets of H in G. Conversely, every subgroup is realisable in this way from some diagram.

Their usefulness for our purposes stems from the fact that any coset diagram K yields a homomorphism $\theta_K \colon G \to \mathrm{Sym}\{\text{vertices of } K\} \cong S_n$. Here n is the sheet number of the covering, hence the number $|K|$ of vertices in K. For any $g \in G$ the image of vertex v under the permutation $\theta_K(g)$ is the terminal vertex of the path starting at v with label g. In particular, $\theta_K(G)$ is transitive if and only if K is path-connected.

All of this is, of course, well known. The *CW*-complexes that form coset diagrams for G are characterised by two simple properties:

1. For each vertex v and generator $x \in X$, there is precisely one x-arc and one x^{-1}-arc having initial vertex v.

2. The faces are in one-to-one correspondence with the paths obtained by starting at some vertex v and traversing a path with label some $r \in R$.

Condition 2 indicates that, in their unrefined form, coset diagrams are a little unwieldy—there will be many faces sharing the same set of boundary edges. To alleviate matters, we use an equivalent construct, suggested by

Higman and used in [3, 4, 8, 9, 15, 16]. It is what results by identifying such multiple faces.

Let $G = (m_1, \ldots, m_e)$ be some fixed but arbitrarily chosen Dyck group. A more convenient presentation than given in the Introduction is

$$\langle x_1, x_2, \ldots, x_{e-1} \mid x_1^{m_1} = x_2^{m_2} = \cdots = x_{e-1}^{m_{e-1}} = (x_1 x_2 \ldots x_{e-1})^{m_e} = 1 \rangle.$$

A G-graph is a directed graph with edges labelled $x_1, \ldots, x_{e-1}$ satisfying property (1) above. Ordering the edges incident with every vertex as shown in Fig. 1 yields a 2-cell embedding of a G-graph into a closed orientable surface (see [20] for more details on graph embeddings). Each face of this complex S will have boundary label some power of x_i or $x_1 x_2 \cdots x_{e-1}$. Call S a *G-diagram* if, for each face, this power divides the order of the appropriate word given in the presentation.

In a G-diagram, a path starting at v with label $x_i^{m_i}$ or $(x_1 \cdots x_{e-1})^{m_e}$ circumnavigates a face an integral number of times. Taking the underlying G-graph and sewing in a 2-cell for each such vertex–relator pair yields a coset diagram for G. Conversely, the 1-skeleton of a coset diagram is a G-graph in which a path from any vertex with label a relator is closed (as it bounds a face). Embedding the graph as above gives a G-diagram. We therefore have

LEMMA 3.1. *A coset diagram for G yields a unique G-diagram, and vice versa.*

Consequently, we use the same terminology for G-diagrams as for coset diagrams. In particular, call a face an x_i-face or $(x_1 \cdots x_{e-1})$-face whenever it has boundary label some power of x_i or $x_1 \cdots x_{e-1}$.

The key property of G-diagrams, as Higman observed, is that they can sometimes be combined to form new ones. For this we use *handles*, that is, pairs of vertices α and β, each incident with x_1-loops, so that the path starting at α with label $x_1 \cdots x_{e-1}$ terminates at β.

Let $K_1, \ldots, K_t$, $t \leq m_1$, be a collection of disjoint G-diagrams, and the $2m_1$ distinct vertices $\alpha_1, \beta_1, \ldots, \alpha_{m_1}, \beta_{m_1}$ a collection of m_1 handles with

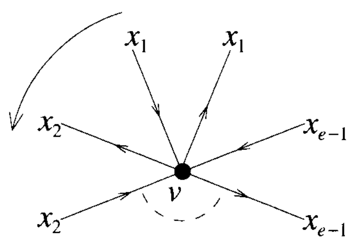

FIGURE 1

at least one in each diagram. Take the disjoint union of all the underlying G-graphs, remove the x_1-loops at the vertices α_j and β_j, and replace them by x_1-edges from α_j to α_{j+1} and β_j to β_{j-1}, subscripts taken modulo m_1. Embed the graph in the usual way, and call the resulting complex the *composition* of $K_1, \dots, K_t$ denoted by $[\![K_1, \dots, K_t]\!]$.

PROPOSITION 3.1. *$[\![K_1, \dots, K_t]\!]$ is also a G-diagram with $\sum |K_i|$ vertices.*

Proof. The underlying graph of $[\![K_1, \dots, K_t]\!]$ is clearly a G-graph, so it remains to show that all faces have boundary labels of the required form. If the boundary of a face does not contain an x_1-edge with initial vertex one of the α_j or β_j, then all edges are contained in a single G-diagram K_i, and we are done.

Otherwise, we obtain the boundary label for the face by starting at an α_j or β_j and traversing a path with label some power of x_1 or some power of $x_1 \cdots x_{e-1}$, until it closes (which it does by repeating an arc). The path obtained by traversing just x_1-edges passes through the vertices $\alpha_{j+1}, \dots, \alpha_{m_1}, \alpha_1, \dots, \alpha_j$ or $\beta_{j-1}, \dots, \beta_1, \beta_{m_1}, \dots, \beta_j$, before closing with label $x_1^{m_1}$, so such faces are as they should be. Observe that *before* composition, the path starting at α_j with label some power of $x_1 \cdots x_{e-1}$ arrived at vertex β_j after $e-1$ directed edges, and proceeded to traverse the x_1-loop at β_j and then an x_2-edge. After composition, the path from α_j with such a label arrives instead at β_{j+1} after $e-1$ directed edges, traverses the new x_1-edge to β_j, and is then identical with the path before composition. So the boundary label behaves as if the composition never happened and is thus of the required form. The number of vertices is obvious. ∎

Now suppose G is the triangle group

$$\langle x, y \mid x^p = y^q = (xy)^r = 1 \rangle, \qquad 3 \le p < q < r,$$

with p, q, and r prime. In practice, we simplify (p, q, r)-diagrams when drawing them: a shaded q-gon indicates a y-face with boundary label y^q, and a shaded wedge a y-face with label y; the orientation on arcs runs anticlockwise around any face they bound unless indicated otherwise; x-faces with boundary x are removed completely, leaving only the incident vertex which will be called *free*. On occasion, we will talk of attaching x-arcs to free vertices, by which we mean attach the arcs to the underlying G-graph and re-embed.

As a consequence, the unshaded faces are precisely the x- and xy-faces, and for an embedded G-graph to be a G-diagram, it is sufficient that the xy-faces have a number of y-arcs dividing r in their boundaries, and the x-faces a number of x-arcs dividing p. These criteria can usually be verified at a glance.

We devote the remainder of this section to diagrams for triangle groups. An x-face is of type $[l_1, \ldots, l_\lambda, \ldots, \bar{l}_\mu, \ldots, l_t]$, $\sum_{i=1}^{t} l_i = p$, if it has boundary label x^p, and in traversing the boundary with the orientation

- vertices $(\sum_{i<\lambda} l_i + 1)$ through $(\sum_{i\le\lambda} l_i)$ are consecutive on some q-gon,
- vertices $(\sum_{i<\mu} l_i + 1)$ through $(\sum_{i\le\mu} l_i)$ are incident with shaded wedges.

Of course the face also has type X for X any cyclic permutation of the l_i, but in practice this ambiguity causes no confusion. We tend to say *type X x-cycle* rather than x-face of type X. Fig. 2 shows a type $[k, \overline{p-k}]$ x-cycle, $1 \le k \le p$, or *type k pendant*.

Suppose we have k consecutive free vertices on a q-gon, all in the boundary of the same xy-face F. Attaching a type k pendant to these vertices increases the number of y-arcs in the boundary of F by $p - 2k + 1$. The modification also produces a new x-face with boundary x^p and some y- and xy-faces with labels y and xy.

Suppose $q = lp + s$ for $l \ge 1$ and $1 \le s \le p - 1$. Take a shaded q-gon, and attach $l - 1$ type p pendants to $p(l-1)$ consecutive vertices. Attach a single type s pendant so that p consecutive vertices are left free. The resulting q-gon together with the attachments will be called a *booster*.

Let $X_i = [l_{i1}, \bar{l}_{i2}, l_{i3}, \bar{l}_{i4}]$, $i = 1, \ldots, m$. Suppose that, for integers $1 \le k_1, \ldots, k_t \le p$, we have $l_{11} + \sum k_i$ consecutive free vertices on a q-gon and in the boundary of an xy-face F. By attaching a type $\{k_1, \ldots, k_t; X_1, \ldots, X_m\}$ array to these free vertices we mean

- attach t pendants of types $k_1, \ldots, k_t$, and
- a collection of m boosters, joined into a chain, with l_{i3} vertices of the ith booster connected to l_{i1} vertices of the $(i-1)$st by an x-cycle of type X_i (taking the 0th booster to be the original q-gon)—see Fig. 3

Write $\{k_1, \ldots, k_i^{\delta_i}, \ldots, k_t;\ X_1, \ldots, X_j^{\delta_j}, \ldots, X_m\}$ when the array includes δ_i type k_i pendants and δ_j x-cycles of type X_j. Notice that a type $\{k; -\}$

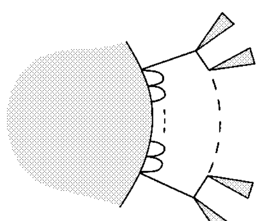

FIG. 2. Type k pendant.

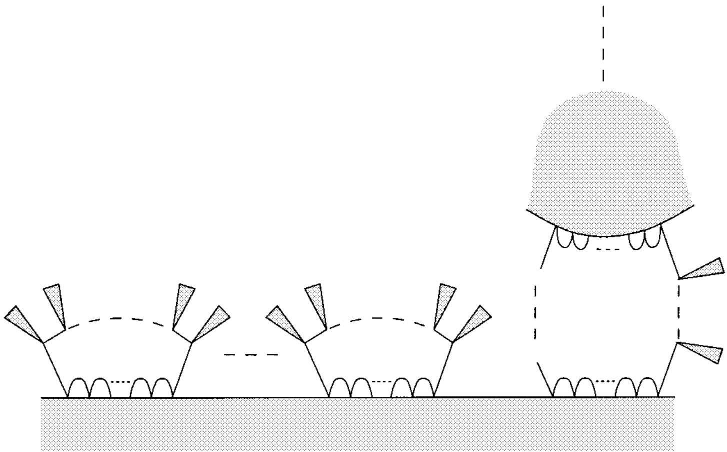

FIG. 3. Type $\{k_1, \ldots, k_t; X_1, \ldots, X_m\}$ array.

array is merely a type k pendant. In attaching an array, the number of y-arcs in the boundary of xy-face F increases by

$$m(p+l+2-s) + \sum_{i=1}^{t}(p-2k_i+1) + 2\sum_{\bar{l}_{ij}\in X_i} l_{ij}, \tag{3}$$

together with the creation of the usual complement of x-, y-, and xy-faces having boundary x, x^p, y, and xy. All other faces are unaffected. To see (3), start with each $X_i = [1, p-1]$, and observe that replacing it by $[1, \bar{1}, p-2]$ increases the y-arc count by 2, while a change to $[2, p-2]$ has no effect.

If K is a (p, q, r)-diagram with $g \in (p, q, r)$, the cycle structure of $\theta_K(g)$ is a function $\mathbf{s}\colon \mathbf{Z}^+ \to \mathbf{Z}^+ \cup \{0\}$, such that $\mathbf{s}(i)$ is the number of cycles of length i when $\theta_K(g)$ is written as a product of disjoint cycles. Given two structures $\mathbf{s}_1$ and $\mathbf{s}_2$, let $\mathbf{s}_1 + \mathbf{s}_2$ be their pointwise sum as functions. In Section 4 we will be interested in the structure of the element $x^{-1}y$.

LEMMA 3.2. *Suppose $K_1, \ldots, K_t$ are (p, q, r)-diagrams with $\mathbf{s}_i$ the cycle structure of $\theta_{K_i}(x^{-1}y)$. If $K = [\![K_1, \ldots, K_t]\!]$, then $\theta_K(x^{-1}y)$ has cycle structure $\sum \mathbf{s}_i$.*

Proof. Only cycles in $\theta_{K_i}(x^{-1}y)$ that pass through handle points are affected by the composition. If α_j and β_j lie in such a cycle, then in $\theta_K(x^{-1}y)$ the cycle is identical, except that β_j is replaced by β_{j-1}. ∎

Consequently, consideration of the cycle structure of $\theta_{[\![K_1,\ldots,K_t]\!]}(x^{-1}y)$ reduces to an investigation of the $\theta_{K_i}(x^{-1}y)$.

We determine the effect on $\theta_K(x^{-1}y)$ of attaching an array by considering the various ingredients. From now on, when we talk of a *cycle* in K, we will mean a cycle of $\theta_K(x^{-1}y)$, and the context should make clear which cycle we mean. Notice first that consecutive free vertices on a q-gon are contained in the same cycle. Attaching a type k pendant to these vertices increases the length of this cycle by $p-k$ when k is odd. When k is even, the length decreases by $k/2$, and a new cycle of length $p-k/2$ is created. Next, the vertices of an isolated booster are organised into a single cycle of length

$$q+\begin{cases} p-s, & s \text{ odd},\\ -\dfrac{s}{2}, & s \text{ even}.\end{cases}$$

When contained as the ith booster of an array, vertices may be gained or lost from this cycle (it may even be fused with cycles from neighbouring boosters) depending on whether l_{i1} and l_{i3} are even or odd. Fig. 4 shows the possible orbits on the vertices, illustrated by small circles and squares.

It will be useful to have at our disposal various maneuvers in which an array is replaced by another. Replacing an array of type $\{k_1,\dots,k_t;X_1,\dots,X_m\}$ by one of type $\{k_1,\dots,k_t,\frac{p+1}{2};X_1,\dots,X_m\}$ is called spoiling. A push–pull substitutes $\{k_1,\dots,k_i-1,\dots,k_j+1,\dots,k_t;X_1,\dots,X_m\}$, while replacing by $\{k_1,\dots,k_t;X_1,\dots,X_i',\dots,X_m\}$, where $X_i'=[l_{i1}\pm 1,\bar{l}_{i2},l_{i3}\mp 1,\bar{l}_{i4}]$, will be known as modifying a chain.

A few brief remarks on each then. Suppose K' is the result of performing such a maneuver on some array in the (p,q,r)-diagram K:

- $K \xrightarrow{\text{spoiling}} K'$: since (3) is unchanged, K' is also a (p,q,r)-diagram.

The modification requires $\frac{p+1}{2}$ free vertices and $|K'|=|K|+\frac{p-1}{2}$. The length of the cycle containing these free vertices changes by a non-trivial amount $<q$.

- $K \xrightarrow{\text{push}-\text{pull}} K'$: again (3) is invariant so K' is a (p,q,r)-diagram. No

free vertices are required and $|K'|=|K|$. The length of the cycle on the q-gon to which the array is attached changes by

$$\sum_{\substack{k\in\{k_i,k_j\}\\ k \text{ even}}}\left(p-\frac{k}{2}\right)-\sum_{\substack{k\in\{k_i,k_j\}\\ k \text{ odd}}}\frac{k}{2}. \tag{4}$$

FIGURE 4

• $K \xrightarrow{\text{modifying chain}} K'$: again K' is a (p, q, r)-diagram, with $|K'| = |K|$. The operation requires a free vertex on either the $(i-1)$st or ith booster, creating one on the other. Use Fig. 4 to monitor the effect on cycles in $\theta_K(x^{-1}y)$.

4. THE PROOF OF THE THEOREM

Higman's construction, forming the basis of [3, 4, 8, 9, 15, 16], is essentially

PROPOSITION 4.1. *Let K_1, K_2, and K_3 be path-connected diagrams for the triangle group (p, q, r) such that*

1. *$|K_1|, |K_2|$ are relatively prime, and $|K_3| \geq q+3$;*
2. *K_1 and K_2 each contain at least two handles and K_3 one;*
3. *if $\mathbf{s}_i$ is the cycle structure of $\theta_{K_i}(x^{-1}y)$, then $\mathbf{s}_1(kq) = \mathbf{s}_2(kq) = 0$, $k \geq 1$, and*

$$\mathbf{s}_3(kq) = \begin{cases} 1, & k = 1, \\ 0, & k > 1; \end{cases}$$

4. *if μ is the q-cycle in $\theta_{K_3}(x^{-1}y)$ there are $i, j \in \mu$, not contained in the handle, with $i^x, j^y \in \mu$.*

Then $G = (p, q, r)$ surjects almost all of the alternating groups.

Proof. Let $p_1, p_2 > p$ be distinct primes not dividing $|K_1|$ and $|K_2|$. For k_1 and k_2 arbitrary non-negative integers we construct a sequence of diagrams $C_0, C_1, \ldots, C_{k_1}, \ldots, C_{k_1+k_2} := K$ as follows: for the 0th step, if either k_1 or $k_2 = 0$, take $C_0 = K_3$; otherwise, $C_0 = K_2$. At step i, $1 \leq i \leq k_1$, take p_1 identical copies of K_1 and let C_i be the composition

$$[\![[\![\cdots [\![[\![C_{i-1}, \underbrace{K_1, \ldots, K_1}_{p-1}]\!], \ldots]\!], \underbrace{K_1, \ldots, K_1}_{p-1}]\!], \underbrace{K_1, \ldots, K_1}_{\leq p-2}]\!]. \tag{5}$$

In particular, the two handles on each K_1 allow us to perform the composition, which is a (p, q, r)-diagram by Proposition 3.1. Observe that each C_i has at least two handles. For $k_1 + 1 \leq i \leq k_1 + k_2 - 1$, take p_2 identical copies of K_2 and let C_i be a composite diagram of the form (5) but with p_2 copies of K_2 instead of p_1 copies of K_1. Finally, at step $k_1 + k_2$, if k_1 or $k_2 = 0$, let $C_{k_1+k_2}$ be as in the previous step. Otherwise, take a diagram of the form (5) but replace one of the K_2's by a K_3 (using its sole handle).

A quick sketch may help the reader to see what is going on. Now $|K| = k_1 p_1 |K_1| + k_2 p_2 |K_2| + |K_3|$, and since $|K_1|$ and $|K_2|$ are relatively prime, so too are $p_1|K_1|$ and $p_2|K_2|$. By choosing k_1 and k_2 suitably, $|K|$ can thus be made to equal any integer greater than $(p_1|K_1|-1)(p_2|K_2|-1)+|K_3|$. So, if $\theta_K: (p,q,r) \to S_{|K|}$ is the homomorphism arising from K, we have permutation representations of (p,q,r) for all but finitely many degrees. By Lemma 3.2 the permutation $\theta_K(x^{-1}y)$ contains the q-cycle μ and no other cycles of length divisible by q, so some power of $\theta_K(x^{-1}y)$ is just μ. Path-connectedness, Lemma 2.3, and Theorem 2.1 give $\theta_K(G) = A_{|K|}$ or $S_{|K|}$, but the generators of G have odd order, so in fact $\theta_K(G) = A_{|K|}$. ∎

It remains then to give the details. For each of the following cases, the diagrams K_1, K_2, and K_3 are given and parts 1, 2, and 4 of the proposition are easily established. Part 3 will prove to be somewhat messier.

(1) The *Case* $p \geq 7$ and $q \geq p+6$. Consider Fig. 5. We have q-gons, $Q_1, \ldots, Q_{p-1}$, with Q_1 at the top and the ordering going clockwise. They are connected by two type $[2, 1, \ldots, 1]$ x-cycles, the number of 1's being $p-2$. The connections are such that Q_i contributes one y-arc to the boundary of region F_{i-1}, subscripts taken modulo $p-1$. The usual embedding places Fig. 5 on the 2-sphere, as depicted in the picture in fact. The face F_{p-1} has $q-2$ y-arcs in its boundary, faces $F_1, \ldots, F_{p-2}$ have q, and there are four other unshaded faces, two each with labels xy and x^p.

Similarly for Fig. 6, we have q-gons, $Q_1, \ldots, Q_p$, connected by two type $[1, \ldots, 1]$ x-cycles, the number of 1's being p. The connections are meant to allow Q_i to contribute $\frac{q-1}{2}$ y-arcs to the boundary of region F_{i-1}, subscripts taken modulo p. The usual embedding places the figure on the 2-sphere also.

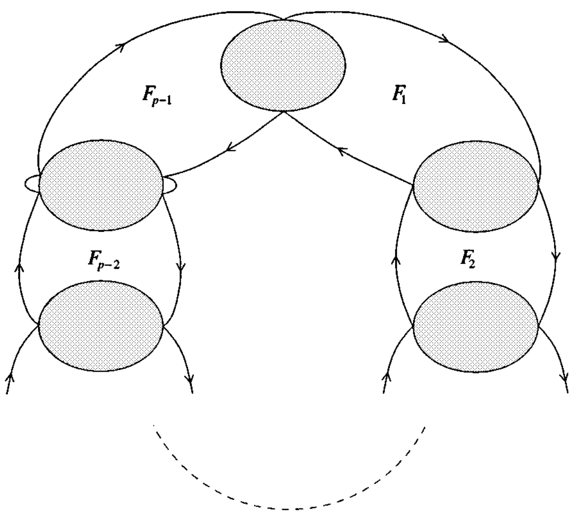

FIGURE 5

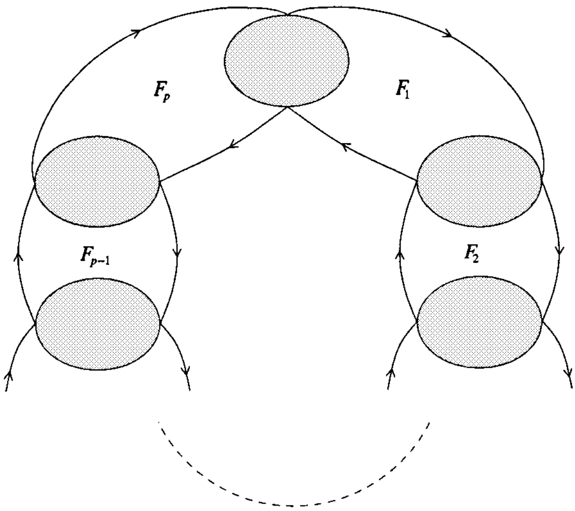

FIGURE 6

Recalling that $q = lp + s$, and $r \geq q + 2$ is prime, let m, δ, and k be positive integers such that

- m is largest with $(q + 2) + m(p + l + 2 - s) \leq r$;
- δ is largest with $(q + 2) + m(p + l + 2 - s) + \delta(p - 3) \leq r$;
- k is determined by $p - 2k + 1 = r - q - m(p + l + 2 - s) - \delta(p - 3)$.

Notice that $2 \leq k \leq \frac{p-1}{2}$. Each q-gon Q_i of Fig. 5 has a number of consecutive free vertices laying in the boundary of face F_i. Assuming for now that this number is sufficient to do so, attach to $Q_1, \dots, Q_{p-2}$ arrays of type $\{2^\delta, k; [2, p-2]^m\}$, and one of type $\{2^\delta, k-1; [2, p-2]^m\}$ to Q_{p-1}. By (3) and the definitions of m, δ, and k, each face F_i now has r y-arcs in its boundary. We thus have a spherical (p, q, r)-diagram, K_1^r. The actual value of r is usually irrelevant, so we just call this diagram K_1.

Take a single q-gon, attach to it a type $\{2^\delta, k; [2, p-2]^m\}$ array and embed. The resulting spherical (p, q, r)-diagram will be our $K_2 := K_2^r$. Our third diagram is slightly more complicated. In Fig. 6 attach type $\{k; -\}$ arrays to $Q_1, \dots, Q_{p-3}$ and Q_{p-1}, using free vertices in the boundary of $F_1, \dots, F_{p-3}$ and F_{p-1}. To $Q_2, \dots, Q_{p-2}$ and Q_p, attach type $\{2^\delta; [2, p-2]^m\}$'s, adjacent to $F_1, \dots, F_{p-3}$ and F_{p-1}, while to Q_1 and Q_{p-2}, connect $\{2^\delta, k; [2, p-2]^m\}$'s adjacent to F_p and F_{p-2} (the reader should sketch the positions of the various attachments as a guide). Again assume for now that there is sufficient space to do all these things. Each F_i receives $r-q$ new y-arcs. The resulting (p, q, r)-diagram is $K_3 := K_3^r$.

Let N be the number of new vertices introduced by an array of type $\{2^\delta, k; [2, p-2]^m\}$. We have $|K_1| = (p-1)q + (p-2)N + N + 1$ and $|K_2| = q + N$. Thus, any common divisor of $|K_1|$ and $|K_2|$ also divides

$$|K_1| - (p-1)|K_2| = 1, \tag{6}$$

so that $|K_1|$ and $|K_2|$ are relatively prime. Clearly $|K_3| \geq q+3$ and the K_i are path-connected.

Let $\mathbf{s}_i$ be as in the proposition, and observe that, in K_3, the $\frac{q+1}{2}$ free vertices of Q_{p-1} adjacent to F_{p-2}, and the $\frac{q-1}{2}$ free vertices of Q_p adjacent to F_p, form a q-cycle in $\mathbf{s}_3$. It is easy to check that this cycle satisfies part 4 of the proposition. Call any other cycle in $\mathbf{s}_1$, $\mathbf{s}_2$, or $\mathbf{s}_3$ with length divisible by q a *bad* cycle.

We can always arrange things so that bad cycles dissappear and part 3 of the proposition is thus satisfied. The vertices of Figs. 5 and 6 and the q-gon that forms the nucleus of K_2 (that is, before any arrays are attached to them) are organised into various cycles. In fact, there are $p-3$ q-cycles, a $(q-2)$-cycle, and a $(q+2)$-cycle in Fig. 5; p q-cycles in Fig. 6; and a q-cycle in the q-gon of K_2. A crucial observation is that in K_1 and K_2, each of these cycles has exactly one array attached. Things are more complicated with K_3—one q-cycle has $\{k;-\}$ and $\{2^\delta, k; [2, p-2]^m\}$ arrays attached, another has $\{2^\delta; [2, p-2]^m\}$ and $\{2^\delta, k; [2, p-2]^m\}$ arrays, while $p-3$ of them have $\{k;-\}$ and $\{2^\delta; [2, p-2]^m\}$. The single q-cycle not mentioned is our precious prime cycle.

We monitor the effect on these cycles of the attached arrays. First, using the observations following Lemma 3.2, one can check that the boosters in a type $\{k_1, \ldots, k_t; [2, p-2]^m\}$ array contribute bad cycles only when $s = 2$. In this case, the mth booster contains a q-cycle. No problem; just modify the chain, and replace X_m by $X'_m = [1, p-1]$.

Next we examine the effect of the pendants in an array. Consider one of the q-cycles in K_1 or K_2. If $m = 0$, so that a $\{2^\delta, k; -\}$ array is attached to the cycle, its length becomes

$$q - \delta + \begin{cases} p-k, & k \text{ odd}, \\ -\dfrac{k}{2}, & k \text{ even}. \end{cases}$$

Since $k \leq \frac{p-1}{2}$, we have $p - k \geq \frac{p+1}{2}$, and so the cycle is bad only if $\delta \geq \frac{p+1}{2}$. The definitions of m, δ, and k give $\delta(p-3) + p - 3 \leq p + l + 2 - s$, so the cycle is bad only if $l - s \geq 7$, that is, $q \geq 8p+1$ (in fact, $q \geq 2p+1$ will do). By an identical argument, the $(q-2)$-cycle in K_1 becomes bad only if $q \geq 2p+1$, and the $(q+2)$-cycle suffers the same fate under the addition of a type $\{4;-\}$ array, or only if $q \geq 2p+1$. Similarly for the q-cycles in K_3, when $m=0$, we must have $q \geq 3p$ before any turn bad, and when $m=1$, we must have $q \geq 2p+1$.

What do we do with these bad cycles? When $m \geq 1$ it is simple. Take one of the $\{k_1, \ldots, k_t; [2, p-2]^m\}$ arrays attached to the cycle and perform a simultaneous volley of chain modifications: either replace all $X_i = [2, p-2]$ by $X_i' = [1, p-1]$, or all X_i by $X_i' = [3, p-3]$, whichever does not create a bad cycle on the mth booster (they both cannot). When $s=2$ and $m \geq 2$, change all X_i to $[1, p-1]$. If $s=2$ and $m=1$, change $X_1 = [1, p-1]$ to $X_1' = [3, p-3]$. In any case the bad cycle is obliterated and no new bad cycles are created. Remember that when $X_1' = [3, p-3]$, we are assuming there are two free vertices where the array is attached, but more on this later.

If $m=0$ and a bad cycle arises in K_3, spoil one of the attached arrays, assuming for now that there is enough room to do so. If the bad cycle is in K_1 or K_2, it would be nice to be rid of it by spoiling the attached array. Unfortunately, spoiling changes the number of vertices, and (6) would no longer be valid. So, except for when a $\{4;-\}$ is attached to the $(q+2)$-cycle, spoil *every* array in these two diagrams (again assuming there is enough room). This certainly removes the bad cycle. The danger is that it may have created a new one elsewhere. If so, remove it by performing a push–pull on the attached array: replace $\{2^\delta, k \text{ or } k-1, \frac{p+1}{2}; [2, p-2]^m\}$ by $\{2^\delta, k-1 \text{ or } k-2, \frac{p+3}{2}; [2, p-2]^m\}$, or $\{2^\delta, 1, \frac{p+1}{2}; [2, p-2]^m\}$ by $\{2^\delta, 2, \frac{p-1}{2}; [2, p-2]^m\}$. In all the cases that bad cycles arise, $q \geq 2p+1$, so the effect (4) of these push–pulls is both non-trivial and $< q$, so the new bad cycle is removed.

The bad cycle arising when a $\{4;-\}$ array is attached to the $(q+2)$-cycle in K_1 is removed by similarly spoiling every array in K_1 and K_2. It can be checked that this creates no new bad cycles elsewhere. This accounts for all situations where bad cycles arise and establishes part 3 of the proposition.

Our final task is to see that there are sufficient free vertices in the appropriate places for all the above to happen. Fix p, and for a given q, let Δ be the maximum value obtained by δ. When $m=0$ the largest number of consecutive free vertices needed anywhere is $2\delta + k + \frac{p+1}{2}$: room for a type $\{2^\delta, k; -\}$ array and a possible spoil. Similarly, when $m \geq 1$ we need $2(\delta+1)+k+1$: room for a $\{2^\delta, k; [2, p-2]^m\}$ array and a potential volley of chain modifications. The $m \geq 1$ requirements are less than the $m=0$, and since $k \leq \frac{p-1}{2}$, these in turn are less than $2\Delta + p$.

Take four consecutive vertices on the q-gon of K_2 and two on each of $Q_1, \ldots, Q_{p-2}$ of K_1. These are the handles for K_1 and K_2. Thus, before any arrays are added, the q-gons of K_1 and K_2 are left with $q-4$ consecutive free vertices. When $p+6 \le q \le 2p+1$, we have $\Delta = 1$, so $2\Delta + p \le q-4$, and we are happy.

Now Δ is the largest multiple of $p-3$ less than $p+l+2-s$. Thus for a fixed l, Δ and hence $2\Delta + p$, is biggest, and $q-4$ smallest, when $s=1$. It therefore suffices to show that $2\Delta + p \le q-4$ for $q = lp+1$. We already have this for $l=2$. If the inequality is valid for a given l, and we increase l by 1, then $p+l+2-s$, and hence Δ, increases by at most 1, and so $2\Delta + p$ by at most 2. But $q-4$ increases by $p \ge 7$, and we are home.

In K_3 the vertex requirements are greatest and the availability is the least, on the side of Q_{p-2} adjacent to F_{p-2}. By considering the possible values of Δ for q in the range $p+6 \le q \le 4p-1$, one can show, using the discussion of when bad cycles arise, that the $\frac{q-3}{2}$ consecutive free vertices that are available suffice. For $q \ge 4p+1$, argue as for K_1 and K_2.

Finally, place a handle on K_3 using two vertices of the precious q-cycle.

(2) Case $p \ge 7$ and $q = p+2$ or $p+4$. Diagrams K_1 and K_2 are the same as in the previous case. That there is sufficient room on K_1 and K_2 is a slightly more delicate matter, but the argument is essentially the same. These diagrams can be of no help to $(11, 13, 17)$, however, which can be found in [7, Sect. 6].

Unfortunately, there are not enough free vertices on the K_3 from case 1 once q is this close to p. Instead, consider Fig. 7. When $q = p+2$ the large xy-face has $q_0 = p+10$ y-arcs in its boundary, while the minimum r of interest is $r_0 = p+6$. For $r \ge r_0$ prime, let m and k be positive integers such that m is largest with $r_0 + m(p+1) \le r$, and k is determined by $p - 2k + 1 = r - q_0 - m(p+1)$. Add a type $\{k; [1, p-1]^m\}$ array to the top q-gon. The resulting (p, q, r)-diagram will be our K_3 for $q = p+2$.

Since $k \le \frac{p+5}{2}$, there is sufficient room on the top q-gon for the array with at least three vertices to spare. Put a handle on the bottom q-gon, which also has at least three vertices to spare. The middle q-gon supplies us with a q-cycle. Bad cycles can only arise on the q-gon to which the array is attached. In such a situation, change the two $[1, p-1]$ cycles in Fig. 7 to type $[2, p-2]$'s. This removes the bad cycle.

With $q = p+4$, a K_3 diagram for $(7, 11, 13)$ is in [7, Sect. 6]. Otherwise the argument is identical with $q_0 = p+16$, and $r_0 = p+6$ when $p \ge 13$, or $r_0 = 17$ when $p = 7$.

(3) The Case $p = 5$ and $q \ge 17$. Except for the arrays, diagrams K_1, K_2, and K_3 are the same as in case 1. For $r \ge q+2$ prime, let m be largest with $(q+2) + m(5+l+2-s) \le r$; δ_1 largest with $(q+2) + m(5+l+2-s) + 4\delta_1 \le r$; δ_2 largest with $(q+2) + m(5+l+2-s) + 4\delta_1 + 2\delta_2 \le r$; and k determined by $p + 2k - 1 = r - q - m(5+l+2-s) - 4\delta_1 - 2\delta_2$. Add arrays in the same places as in case 1, except replace each 2^δ in an array by $1^{\delta_1}, 2^{\delta_2}$. The remainder of the argument is the same.

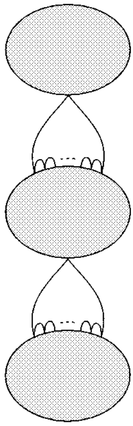

FIGURE 7

(4) The Case $p = 5$ and $q = 11; 13$. Diagrams K_1 and K_2 are as in case 3. For K_3, let $r_0 = 13$ and $q_0 = 15$ when $q = 11$, or $r_0 = 17$ and $q_0 = 21$ when $q = 13$. Given $r \geq r_0$ prime, take m largest with $r_0 + m(9 - s) \leq r$, and k determined by $p - 2k + 1 = r - q_0 - m(9 - s)$. Add a type $\{5, k; [1, p - 1]^m\}$ array to the top q-gon of Fig. 7, and type $\{5; -\}$'s to the other two. The resulting $(5, q, r)$-diagram is our K_3. Proceed as in case 2.

(5) The Case $p = 5$ and $q = 7$. We do $(5, 7, 11)$ and $(5, 7, 13)$ in [7, Sect. 6]. Diagrams K_1 and K_2 are the same as in case 1, except for the arrays. For $r \geq 17$ prime, take m largest with $9 + 6m \leq r$, δ largest with $9 + 6m + 2\delta \leq r$, and k given by $5 + 2k - 1 = r - 7 - 6m - 2\delta$. Somewhat unusually, add type $\{k; [1, 4]^{m-1}, [1, \bar{\delta}, 4 - \delta]\}$'s and a single type $\{k - 1; [1, 4]^{m-1}, [1, \bar{\delta}, 4 - \delta]\}$ in all the usual places. When $m \geq 2$ and $\delta = 2$, a bad cycle arises in the chain of boosters. Remove it by modifying, X'_m being $[2, \bar{2}, 1]$, and by replacing the type 2 pendant on each of the last two boosters by types 1 and 3. For K_3, follow the construction of case 2.

(6) The Case $p = 3$ and $q \geq 17$. Most of the $p = 3$ case was handled in [8]. The methods we use to cope with what remains will just as easily do the whole lot—for completness we do this. Use Fig. 6, and allow Q_i to contribute a single y-arc to region F_{i-1}. For $r > q$ prime, take $m \geq 0$ largest with $q + m(5 + l - s) \leq r$ and $\delta \geq 0$ largest with $q + m(5 + l - s) + 2\delta \leq r$. Add type $\{1^\delta; [2, 1]^m\}$ arrays to each Q_i, using the free vertices adjacent to region F_i. The resulting $(3, q, r)$-diagram is our K_1.

Spoil the array on Q_1, that is, replace by one of type $\{1^\delta, 2; [2, 1]^m\}$. This gives another $(3, q, r)$-diagram, K_2. Notice that $|K_1| - |K_2| = 1$, so $|K_1|$ and $|K_2|$ are relatively prime. Place a handle on Q_2 and Q_3 in each diagram. We can remove bad cycles from the chains of boosters by the methods of case 1. It is easy to show that none arise elsewhere in K_1. A bad cycle will arise on Q_1 in K_2 precisely when $m \geq 1$ and $\delta = 1$, but the replacement

$$\{1^\delta, 2; [2, 1]^m\} \to \{1^\delta, 3; [1, 1, \bar{1}], [2, 1]^{m-1}\}$$

removes it. The argument of case 1 shows that there are sufficient free vertices for all the arrays and subsequent modifications.

Take Fig. 6 with the connecting type $[1, 1, 1]$ x-cycles allowing Q_i to contribute $\frac{q-1}{2}$ to F_{i-1}. Attach type $\{1^\delta; [2, 1]^m\}$ arrays to Q_1 adjacent to F_1 and F_3, and also to Q_2 adjacent to F_2. The result is K_3. By the usual argument, there is sufficient room for the arrays as well as to spoil any array incident with a bad cycle. A q-cycle occupies the untouched vertices of Q_3 adjacent to F_3 and Q_2 adjacent to F_1, and a handle for K_3 can be safely placed here.

(7) The Case $p = 3$ and $q = 13$. You can find $(3, 13, 17)$ and $(3, 13, 19)$ in [7, Sect. 6]. For $r \geq 23$ prime, use the K_1 and K_2 of case 6. For K_3 attach $\{3^3; -\}$ arrays to the bottom two q-gons of Fig. 7, and place a handle on the bottom one as well. Place a type $\{3; -\}$ on the top q-gon. In addition, we need a type $\{1^\delta; [2, 1]^m\}$ on the top q-gon, with δ and m chosen as in case 6, and this can be spoiled if necessary to remove bad cycles.

(8) The Case $p = 3$ and $q = 11$. We do $(3, 11, 13)$ in [7, Sect. 6]. For $r \geq 17$ prime, diagrams K_1 and K_2 are as in case 6. For K_3 attach type $\{3^2; -\}$ arrays to the top two q-gons in Fig. 7, and a $\{3^3; -\}$ array to the bottom. Place a handle on the middle q-gon (which contains our q-cycle) and a type $\{1^\delta; [2, 1]^m\}$ array on the top one. Choose m and δ according to the usual scheme. Spoil the array to remove any bad cycles.

(9) The Case $p = 3$ and $q = 7$. Look in [7, Sect. 6] for $(3, 7, 11)$. For $r \geq 13$ prime, variations on Fig. 7 yield all three diagrams. For consider just the top two q-gons and the type $[1, 2]$ x-cycle connecting them. Place a type $\{1^\delta; [2, 1]^m\}$ array on the top one as usual and a handle on each of the top two. The resulting $(3, 7, r)$-diagram is K_1. In addition, attach a type $\{2; -\}$ array to the bottom q-gon. The result is K_2. For K_3, start from scratch with Fig. 7, and attach to the bottom two q-gons arrays of type $\{3; -\}$, while to the top, attach a type $\{1^\delta, 3; [1, 2]^m\}$. Place a handle on the bottom q-gon.

(10) The Case $p = 3$ and $q = 5$. You can find $(3, 5, 7)$ and $(3, 5, 11)$ in [7, Sect. 6]. Otherwise, for K_1 take Fig. 7 with type $\{-; [2, 1]^m\}$ and $\{1^\delta; -\}$ arrays attached to the second and third q-gons, respectively, and with two handles on the top. For K_2, place type $\{2; -\}$ and $\{1^\delta; [1, 2]^m\}$ arrays on the second and third q-gons instead. To get K_3, attach a $\{1^\delta; [2, 1]^m\}$ to the top q-gon and a handle on the bottom one.

This completes the proof of the theorem.

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