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Additive functions on quivers

Helmut Lenzing*, Liane Hasenberg

Fachbereich Mathematik-Informatik, Universität Paderborn, D-33095 Paderborn, Germany

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Dedicated to Daniel Simson on the occasion of his 60th birthday

Abstract

An integral function on the set of vertices of a graph is additive if twice its value at any vertex v equals the sum of its values at all adjacent vertices, counting multiple edges. It is well known that among finite connected graphs exactly the extended Dynkin graphs admit a positive additive function, whereas the Dynkin diagrams themselves only allow almost-additive functions, violating additivity in a single vertex.

In the present paper we study—usually non-positive—additive or non-additive functions on finite quivers, and relate the concept of additivity to the radical of the homological Euler form. Our main results concern the existence and construction of such functions for wild quivers. Our results are most specific in case the underlying graph is a tree, possibly with multiple edges.

Keywords: Additive function; Euler form; Quiver; Graph; Coxeter polynomial; Cartan matrix

0. Introduction

Let Δ be a finite graph without loops, possibly with multiple edges, whose set of vertices is denoted Δ_0. For vertices p, q of Δ let $a(p, q)$ denote the number of edges between p and q. An integral function f on Δ_0 is called *additive* if twice its value at any vertex v equals the sum of values at all vertices adjacent to v counting multiple edges, that is,

$$2f(v) = \sum_{z \in \Delta_0} a(v, z) f(z). \tag{1}$$

* Corresponding author.
E-mail address: helmut@math.uni-paderborn.de (H. Lenzing).

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Expressed in terms of the adjacency matrix $A = (a(p, q))$ of Δ, additive functions are just integral solutions of the matrix equation $(2Id - A)x = 0$. This paper deals with the problem to determine the set of solutions and its structure.

Among connected graphs, only the extended Dynkin graphs $\widetilde{\mathbb{A}}_n$, $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$ and $\widetilde{\mathbb{E}}_8$ admit a (strictly) *positive* additive function f. If then f additionally is *normalized*, that is, the subgroup generated by the values of f equals $\mathbb{Z}$, then f is unique, and each additive function is an integral multiple of f. For instance

$$\begin{array}{ccccccccccccccc} & & & & 3 & & & & & & & & & & \\ & & & & | & & & & & & & & & & \\ 2 & — & 4 & — & 6 & — & 5 & — & 4 & — & 3 & — & 2 & — & 1 \end{array}$$

displays this normalized function for the extended Dynkin quiver $\widetilde{\mathbb{E}}_8$. For a survey on these and related facts we refer to [8], see also [6]. By the above, a Dynkin diagram Δ will not admit an additive function. However, Δ has a vertex e such that (Δ, e) has an *almost-additive* function, that is, a function f satisfying additivity (1) for all vertices v with the exception of the vertex e. In the present note we investigate the existence of additive, respectively almost-additive, functions on more general graphs, not restricting to positive functions.

As in [6] we relate additive functions to the Euler form attached to a quiver $\vec{\Delta}$ with underlying graph Δ and to the radical of the quadratic form q_Δ of the graph. Our main results concern trees, possibly with multiple edges. We show that integral-valued almost-additive functions defined on trees and not having zeros are unique up to rational multiples. Further the homological Euler form for a directed tree $\vec{\Delta}$ with underlying graph Δ is trivial on the radical of the quadratic form q_Δ. We show that for bipartite quivers the corank of q_Δ can be derived from the Coxeter polynomial. Finally, we show how to extend arbitrary functions to additive functions on larger graphs. As a result, very different from the case of positive additive functions, a full classification of additive functions for graphs, and more particularly for trees, is impossible. The contents of this paper are basically contained in [4]. The paper was written during a stay of the first-named author at the Mathematical Institute of UNAM, Mexico-City. He (HL) wants to express his thanks to the institution and, more particularly, to J.A. de la Peña and M. Barot for hospitality and helpful comments.

1. Almost-additive and additive functions

Let Δ be a finite graph and let e be a vertex of Δ. We say that a function $f : \Delta_0 \to \mathbb{Z}$ is *additive* for Δ (*almost-additive for* (Δ, e)) if condition (1) is satisfied for all vertices v of Δ (respectively for all vertices v different from e).

This concept is related to other concepts extensively studied in the representation theory of finite dimensional algebras. Let $\vec{\Delta}$ be a quiver, that is, an oriented graph without oriented cycles, and with underlying graph Δ. Let $\vec{a}(p, q)$ denote the number of arrows from p to q such that the number of edges $a(p,q)$ between p and q equals $\vec{a}(p,q)+\vec{a}(q,p)$. The $\Delta_0 \times \Delta_0$-matrices $\vec{A}=(\vec{a}(p,q))$ and its symmetrization $A=(a(p,q))$ are the *adjacency matrices* of $\vec{\Delta}$ and Δ, respectively. Further, $C = Id - \vec{A}$ is called the *Cartan matrix* of $\vec{\Delta}$. The (p,q)-entry of its inverse C^{-1} is the number of paths from p to q in the quiver $\vec{\Delta}$.

Of particular importance is the (non-symmetric) bilinear form

$$\langle -,- \rangle : \mathbb{Z}^{\Delta_0} \times \mathbb{Z}^{\Delta_0} \to \mathbb{Z}, \quad (x,y) \mapsto x^t C y,$$

called the *Euler form*. The associated quadratic form

$$q_\Delta(x) = \langle x, x \rangle$$

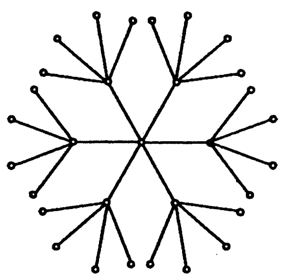
is an invariant of the graph Δ, depending only on the *symmetric bilinear form* $(x \mid y) = \langle x, y\rangle + \langle y, x \rangle = x^t(2Id - A)y$. The *radical* of q_Δ is the direct summand of $\mathbb{Z}^{\Delta_0}$ consisting of all x with $(x \mid -) = 0$; its rank is called the *corank* of q_Δ. The next assertion relates radical and additive functions.

Lemma 1.1. *View $u \in \mathbb{Z}^{\Delta_0}$ as an integral function on the vertices of Δ. Then u is an additive function if and only if u belongs to the radical of q_Δ.*

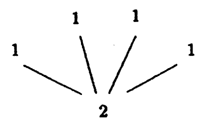
Proof. By the preceding remarks u belongs to the radical of q_Δ if and only if $(C + C^t)u = 0$, that is, if and only if $(2Id - A)u = 0$, where A is the adjacency matrix of Δ. This in turn means that $2u(p) = \sum_q a(p,q)u(q)$ holds for each vertex p, hence that u is an additive function. $\square$

Therefore the radical q_Δ and the group of additive functions on Δ agree. As already recalled, each extended Dynkin graph has a radical of rank one. As the following example shows, the rank of the radical can get arbitrarily large even for trees.

Example 1.2. (i) Consider the family of snowflake trees $S_n(\Delta)$ having n leaves ($n \geqslant 2$), all agreeing with some extended Dynkin diagram Δ. We display below the snowflake $S_6(\widetilde{\mathbb{D}}_4)$:



Note that $S_n = S_n(\widetilde{\mathbb{D}}_4)$ has $5n + 1$ points. It is easily checked that the radical R of the quadratic form for S_n has rank $n - 1$, where a basis of R is formed by the additive functions f_k, $k = 2, \ldots, n$ taking value zero on the central point, agreeing with the normalized additive function for $\widetilde{\mathbb{D}}_4$



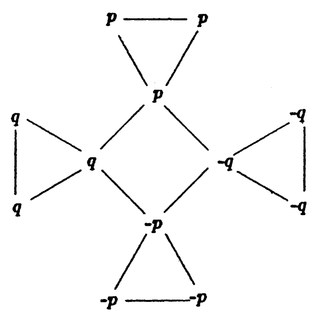
on the first leaf, taking the same values also on the kth leaf, but with negative sign, and being zero on all the remaining vertices.

(ii) The additive function

$$1 = 1 — 0 — -1 = -1$$

on $S_2(\widetilde{\mathbb{A}}_1)$ is a generator for the radical.

(iii) Let p, q be integers, then



displays an additive function f, where the radical has rank two. If p and q are coprime, f is normalized.

For the rest of the section we assume that Δ is a tree, possibly with multiple edges. We thus deal with a graph Δ without loops, allowing multiple edges but no cycles involving three or more vertices. As Example 1.2(iii) shows, the following proposition does not extend to graphs in general.

Proposition 1.3. *Assume that Δ is connected and e is a vertex of Δ. Let f be an almost-additive function for (Δ, e) not having any zero. Then each almost-additive function g for (Δ, e) is a rational multiple of f.*

Proof. We argue by induction on the number n of vertices of Δ. For $n = 1$ the assertion is evident. For $n > 1$ the complement of e in Δ decomposes into connected components $\Delta^{(1)}, \Delta^{(2)}, \ldots, \Delta^{(t)}$. Since Δ has no cycles involving at least three vertices, each $\Delta^{(i)}$ has a unique neighbor e_i of e. Let g be an almost-additive function for (Δ, e), then its restriction to $\Delta_0^{(i)}$ yields an almost-additive function for $(\Delta^{(i)}, e_i)$. By induction there are rational numbers $q_1, q_2, \dots, q_t$ such that

$$g(x) = q_i f(x) \text{ for each vertex } x \text{ from } \Delta^{(i)}. \tag{2}$$

Evaluating additivity (1) of g at vertex e_i yields

$$a(e_i, e)g(e) = 2g(e_i) - \sum_{z \in \Delta_0^{(i)}} a(e_i, z)g(z).$$

Invoking almost-additivity of f now yields in view of (2)

$$a(e_i, e)g(e) = q_i \left(2f(e_i) - \sum_{z \in \Delta_0^{(i)}} a(e_i, z)f(z) \right) = q_i a(e_i, e)f(e).$$

Since $f(e)$ and the $a(e_i, e)$ are non-zero, it follows that all the q_i have the same value q, and $g = qf$ follows. □

The proposition has the following immediate consequence.

Corollary 1.4. *Assume that Δ admits an additive function without zeros hence, in different terminology, a sincere radical vector. Then the quadratic form q_Δ has corank one.*

Call a vertex z from Δ a *zero-vertex* if Δ admits a non-zero additive function, and if all such functions vanish on z. By $Z(\Delta)$ we denote the set of all zero-vertices of Δ. We will repeatedly use the following simple observation.

Lemma 1.5. *Let f and g be additive functions on Δ with zero sets Z_f and Z_g respectively. Then a suitable linear combination h of f and g has zero set $Z_h = Z_f \cap Z_g$.*

Proof. Let $h = af + bg$. The assumption $a_x f(x) + b_x g(x) = 0$ for some x not in $Z_f \cap Z_g$ determines the slope b_x/a_x. Avoiding the finitely many slopes thus arising, proves the claim. □

In the following situation the set of zero vertices is particularly easy to determine.

Proposition 1.6. *Let f be a non-zero additive function on Δ whose set of zeros Z_f does not contain a pair of adjacent vertices. Then $Z(\Delta) = Z_f$.*

Proof. We assume that g is additive on Δ such that $g(z) \neq 0$ for some $z \in Z_f$. Consider the full subgraph Γ of Δ obtained by removing $Z_f \cap Z_g$. Let Γ' be the connected component of Γ containing z. Note that the restrictions f' and g' of f and g to Γ' stay additive; moreover, $f' \neq 0$ by the assumption on f. By Lemma 1.5 there is linear combination $h' = af' + bg'$ without zeros on Γ'. In view of Proposition 1.3, f' is a non-zero rational multiple of h'. Hence $f(z)$ is non-zero, contradiction. $\square$

We illustrate this phenomenon by an example, where the values attached to the vertices display an additive function f. Here, the support set of f is a disjoint union of extended Dynkin diagrams.

$$\begin{array}{ccccccccccccc}
-1 & & & & & & 1 & & & & & & -1 \\
\| & & & & & & | & & & & & & \| \\
-1 & — & 0 & — & 1 & — & 2 & — & 1 & — & 0 & — & -1 \\
 & & & & & & | & & & & & & \\
 & & & & & & 1 & & & & & &
\end{array}$$

Also in the general situation, the set of zero vertices agrees with the zero set of some additive function.

Proposition 1.7. *Assume that Δ is connected and admits a non-zero additive function. Then there exists an additive function f with $Z(\Delta) = Z_f$. In particular, if $Z(\Delta)$ is empty the corank of q_Δ equals one.*

Proof. We choose an additive function f on Δ such that its zero set Z_f has minimal cardinality. By Lemma 1.5 each additive function vanishes on Z_f showing that Z_f equals the set of zero-vertices. This proves the first claim; the second then follows from Corollary 1.4. $\square$

As Example 1.2(ii) shows, zero-vertices may also occur for corank one. The next statement reduces the study of graphs with a non-zero additive function basically to the graphs of corank one.

Corollary 1.8. *Assume that Δ admits an additive function. The full subgraph of Δ, obtained by removing the set $Z(\Delta)$ of zero vertices, decomposes into connected graphs $\Delta^{(i)}$, $i = 1, \ldots, t$, each having an additive function without zeros. In particular each q_{Δ_i} has corank one.*

Proof. Let f be an additive function with $Z_f = Z(\Delta)$. Clearly, restriction of f to $\Delta^{(i)}$ yields an additive function f_i without zeros on $\Delta^{(i)}$, and the claim follows. $\square$

Proposition 1.9. *Let $\vec{\Delta}$ be a quiver which is a tree, multiple arrows allowed. Then the restriction of the Euler form to the radical of q_Δ is zero.*

Proof. We identify members of the radical with additive functions, and assume the radical to be non-zero. As before, let $Z(\Delta)$ be the set of zero vertices of Δ, and $\coprod_{i=1}^{t} \vec{\Delta}^{(i)}$ be the decomposition of the full subquiver of $\vec{\Delta}$, obtained by removing $Z(\Delta)$, into connected components. Observe that for any additive function f on Δ, its restriction to $\Delta^{(i)}$ is again additive. For any two additive functions f, g on Δ we hence get

$$\langle f, g\rangle_{\vec{\Delta}} = \sum_{i=1}^{t} \langle f_i, g_i\rangle_{\vec{\Delta}^{(i)}}, \tag{3}$$

where $\langle -, -\rangle_{\vec{\Delta}}$ and $\langle -, -\rangle_{\vec{\Delta}^{(i)}}$ denote the Euler forms for $\vec{\Delta}$ and $\vec{\Delta}^{(i)}$, respectively.

We have proved before (Corollary 1.8) that each $\Delta^{(i)}$ has corank one. Hence all the terms on the right hand side of (3) vanish, thus proving the claim. □

2. Additive functions and Coxeter polynomials

Let $\vec{\Delta}$ be a finite quiver without oriented cycles, and let C be its Cartan matrix. Then the *Coxeter transformation*, or Coxeter matrix, $\Phi = -C^{-1}C^t$ satisfies

$$\langle y, x\rangle = -\langle x, \Phi y\rangle \quad \text{for all } x, y \in \mathbb{Z}^{\Delta_0}.$$

The characteristic polynomial $\chi_{\vec{\Delta}} = |T\,Id - \Phi|$ of Φ is called the *Coxeter polynomial* of $\vec{\Delta}$. Unlike the quadratic form, Coxeter matrix and polynomial depend on the orientation of $\vec{\Delta}$, not just on the underlying graph. More information on these topics can be found for instance in [5,7,9].

Since $\Phi x = x$ if and only if $(C + C^t)x = 0$, the radical of q_Δ equals the fixed point set of Φ. Hence Δ admits a non-zero additive function if and only if 1 is a root of the Coxeter polynomial $\chi_{\vec{\Delta}}$. Moreover, it is possible to determine the corank of q_Δ from the Coxeter polynomial, in case $\vec{\Delta}$—as in the last section—is a tree, possibly with multiple arrows.

Proposition 2.1. *Assume the quiver $\vec{\Delta}$ has a bipartite orientation or is a tree, possibly with multiple arrows. Then the multiplicity of 1 as a root of the Coxeter polynomial $\chi_{\vec{\Delta}}$ equals twice the corank of q_Δ.*

Proof. For a tree $\vec{\Delta}$ it follows from tilting theory or by a more direct argument, see [2], that reversing the direction of arrows in a sink (respectively, source) does not change the Coxeter polynomial. Accordingly, we may assume that $\vec{\Delta}$ has a bipartite orientation. From now on let therefore $\vec{\Delta}$ be any bipartite quiver. Then the Coxeter polynomial $\chi_{\vec{\Delta}}$ and the characteristic polynomial φ_Δ of the adjacency matrix A of Δ are related by the formula

$$\chi_{\vec{\Delta}}(T^2) = T^{|\Delta_0|}\varphi_\Delta\left(T + \frac{1}{T}\right),$$

see [1]. It follows that the multiplicity of 1 as a root of $\chi_{\vec{\Delta}}$ equals twice the multiplicity of 2 as a root of φ_Δ. Since $A = 2Id - (C + C^t)$ is symmetric, the latter multiplicity equals the rank of the group of solutions of $(C + C^t)x = 0$, hence the corank of q_Δ. $\square$

For instance the Coxeter polynomial of the snowflake graph $S_n(\widetilde{\mathbb{D}}_n)$ from Example 1.2 is given as

$$(T - 1)^{2(n-1)}(T^2 - 2(n - 2)T + 1)(T + 1)^{3n+1},$$

which may either be derived by direct calculation or using a recursive algorithm calculating Coxeter polynomials for trees due to Boldt [3].

3. Construction of additive functions

In this section we deal with arbitrary graphs (quivers) without loops (without oriented cycles, respectively). For a function $f : \Delta_0 \to \mathbb{Z}$ and a vertex v of Δ we form

$$\delta_f(v) = 2f(v) - \sum_{z\in\Delta_0} a(v, z)f(z),$$

the *deviation from additivity* in v.

We start to derive restrictions for additive and almost-additive functions.

Proposition 3.1. *Let f be an integral additive function for a graph Δ. Assume that x and y are neighbors in Δ such that $f(x)$, $f(y)$ and $a(x, y)$ are odd. Then Δ has a cycle containing x and y as neighbors and at least one further vertex.*

Proof. Consider the full subgraph Γ of Δ, formed by all vertices x such that there exists another vertex y with $f(x)$, $f(y)$ and $a(x, y)$ all being odd. Let x be from Γ, then in view of additivity $2f(x) = \sum_{y\in\Delta_0} a(x, y)f(y)$, the cardinality of all y in Γ being neighbors of x is even. Passing to an Euler cycle for Γ proves the claim. $\square$

Corollary 3.2. *Let Δ be a graph and v be any vertex from Δ. There does not exist an almost-additive function f for (Δ, v) such that $f(v)$ is odd and $\delta_v(f) = f(v)$.*

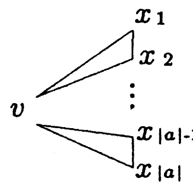
Proof. We take two copies Δ' and Δ'' of Δ, and denote for each vertex x from Δ by x' and x'' the corresponding vertices from Δ' and Δ'', respectively. Using a similar notation, f yields functions f' and f'' for Δ' and Δ''.

Assuming $f(v)$ odd and $\delta_v(f) = f(v)$ for some vertex v of Δ, we form a graph Γ consisting of Δ', Δ'' and a new vertex e joined each with v' and v'' by a single edge. The setting yields an additive function φ on Γ extending f' and f'' and satisfying $\varphi(e) = f(v)$. By construction, e is not lying on any cycle of Γ, contradicting the proposition. $\square$

Proposition 3.3. *Let Δ be a graph and $f : \Delta_0 \to \mathbb{Z}$ be an arbitrary function. Then it is possible to realize Δ as a full subgraph of a graph $\bar{\Delta}$, adding only vertices and single edges, and to extend f to an additive function $\bar{f} : \bar{\Delta} \to \mathbb{Z}$ in such a way that $\bar{f}(x) = \pm 1$ for each vertex $x \in \bar{\Delta}_0 \backslash \Delta_0$.*

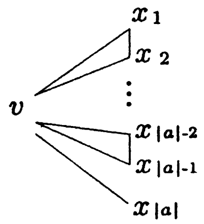
Proof. To each vertex v of Δ we join $|a|$ new vertices $x_1, \ldots, x_{|a|}$, where $a = \delta_f(v)$, and extend f to a function $\bar{f}$ by putting $\bar{f}(x_i) = \mathrm{sgn}(a)$ for $i = 1, \ldots, |a|$. This function $\bar{f}$ then is additive for v. Invoking this process, we assume from now on that f deviates from additivity only in vertices attaining value ± 1.

Let v be such a vertex. We consider first the case that $a = \delta_f(v)$ is *even*. We then adjoin $|a|/2$ triangles to v as follows:

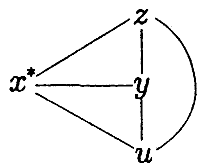


where the new vertices x_i get value $\bar{f}(x_i) = \mathrm{sgn}(a)$. Then $\bar{f}$ is additive at v, and $\delta_{\bar{f}}(x_i) = \mathrm{sgn}(a) - f(v)$. Recall that $f(v) = \pm 1$. If $f(v) = \mathrm{sgn}(a)$, then $\bar{f}$ is additive on each x_i, and we are done. Otherwise $\delta_{\bar{f}}(x_i) = 2\mathrm{sgn}(a)$ holds for each i. Then to each x_i we attach a further triangle (vertices x_i, y_i and z_i) and extend $\bar{f}$ to the new vertices by $\bar{f}(y_i) = \mathrm{sgn}(a) = \bar{f}(z_i)$ such that $\bar{f}$ now is additive on x_i and on the new vertices y_i and z_i.

We therefore assume from now on that f deviates from additivity only in vertices v with $f(v) = \pm 1$, where $a = \delta_f(v)$ is odd. In this case we adjoin $(|a| - 1)/2$ triangles to v as follows:



and a further edge with terminal vertex $x^* = x_{|a|}$, and extend f putting $\bar{f}(x_i) = \mathrm{sgn}(a)$ for $i = 1, \ldots, |a|$. Clearly $\bar{f}$ is additive on v, and $\delta_{\bar{f}}(x_i)$ is even for $i = 1, \ldots, |a| - 1$. Continuing with these vertices as in the first part, it remains to deal with additivity in $x^* = x_{|a|}$. Since $f(v) = \pm 1$, we know that $\delta_{\bar{f}}(x^*) = 2\,\mathrm{sgn}(a) - f(v)$ is either equal to $3\,\mathrm{sgn}(a)$ or to $\mathrm{sgn}(a) = f(x^*)$. In the case $\delta_{\bar{f}}(x^*) = 3\,\mathrm{sgn}(a)$ we adjoin vertices as follows:



Putting $\bar{f}(y) = \bar{f}(z) = \bar{f}(u) = \mathrm{sgn}(a)$ we obtain additivity for x^* and the new vertices.

Finally, we may therefore assume that f deviates from additivity only in 'exceptional' vertices v of order one where $f(v) = \delta_f(v) = -1$. If we join two such vertices v_1, v_2 by a new edge, then f becomes additive also in v_1 and v_2. Continuing, we either obtain an additive function $\bar{f}$ or else one of the 'exceptional' vertices v remains. In this case the function $\bar{f}$ is almost-additive for $(\bar{\Delta}, v)$ contradicting Corollary 3.2, since v does not belong to any cycle. This finishes the proof. □

Proposition 3.4. *Let Δ be a tree, possibly with multiple arrows. A function $f : \Delta_0 \to \mathbb{Z}$ can be extended to an additive function $\bar{f} : \bar{\Delta} \to \mathbb{Z}$ on a tree $\bar{\Delta}$ having Δ as a full subtree if and only if $f(v)\delta_f(v)$ is even for each vertex v of Δ. Moreover, $\bar{\Delta}$ can be chosen to arise from Δ by adjunction of simple edges.*

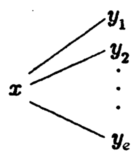
In particular, any function $f : \Delta_0 \to \mathbb{Z}$ with even values can be extended to an additive function on a (possibly) larger tree.

Proof. Since $f(v)\delta_f(v) = 2f(v)^2 - \sum_z a(v, z) f(z) f(v)$, the necessity of the condition follows from Proposition 3.1. We are going to show that the condition is also sufficient.

Let v be a vertex of Δ, we put $a = f(v)$ and $b = \delta_f(v)$. Assume first that b is *odd*. With $d = (|b| - 1)/2$ we join new vertices $x_1, \ldots, x_d$ to v, all getting value $2\,\mathrm{sgn}(b)$, and a further vertex x^* getting value $\mathrm{sgn}(b)$. Since by assumption a is even, the extended function then is additive for v and has even deviation from additivity for all new vertices.

We thus can assume that b is *even*. This time we join $d = |b|/2$ new vertices $x_1, \ldots, x_d$ to v, each getting value $2\,\mathrm{sgn}(b)$. The arising function is additive for v, and deviates from additivity by $c = 4\,\mathrm{sgn}(b) - a$ for each x_i. There are two cases to consider.

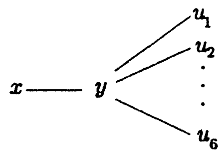
$\mathrm{sgn}(c) = \mathrm{sgn}(b)$: We join to each x_i $(1 \leqslant i \leqslant d)$ a copy of the branch



where $e = |c|$, identifying x with x_i. Attaching value $\mathrm{sgn}(c)$ to the copies of the y_j $(1 \leqslant j \leqslant e)$ then yields a function that is additive for each x_i and for all the new vertices.

$\mathrm{sgn}(c) = -\mathrm{sgn}(b)$: If c is *even*, we join to each x_i, $1 \leqslant i \leqslant d$, $e = |c|/2$ copies of the branch.

(*)



identifying x with x_i. Attaching values $2\,\mathrm{sgn}(c)$ to each copy of y and $\mathrm{sgn}(c)$ to each copy of u_k, $1 \leqslant k \leqslant 6$, we achieve additivity for all the new vertices.

If c is *odd*, we put $e = (|c| + 1)/2$ and proceed as before, joining e copies of the branch $(*)$ to each vertex x_i, $1 \leqslant i \leqslant d$. Additionally, we join a further new vertex y_i^* to x_i. Attaching values $2\,\mathrm{sgn}(c)$ to each copy of y, $\mathrm{sgn}(c)$ to each copy of u_k, and $\mathrm{sgn}(b)$ to each y_i^* finally yields additivity for all the new vertices. $\square$

It follows from the proposition that the function, depicted below,

$$\begin{array}{ccccc} & & 1 & & \\ & & | & & \\ 1 & — & 1 & — & 1 \end{array}$$

cannot be extended to an additive function on any larger tree.

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