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Generalized Cheeger–Gromoll metrics and the Hopf map

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ABSTRACT

We show that there exists a family of Riemannian metrics on the tangent bundle of a two-sphere, which induces metrics of constant curvature on its unit tangent bundle. In other words, given such a metric on the tangent bundle of a two-sphere, the Hopf map is identified with a Riemannian submersion from the universal covering space of the unit tangent bundle, equipped with the induced metric, onto the two-sphere. A hyperbolic counterpart dealing with the tangent bundle of a hyperbolic plane is also presented.

1. Introduction

One of the most studied maps in Differential Geometry is the Hopf map $H : \mathbb{S}^3 \to \mathbb{C}P^1$ from the unit three-sphere $\mathbb{S}^3 \subset \mathbb{C}^2$ onto the complex projective line $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$, defined for $z = (z_1, z_2) \in \mathbb{S}^3$ by

$$H(z) = \begin{cases} z_1/z_2 & \text{if } z_2 \neq 0, \\ \infty & \text{if } z_2 = 0. \end{cases}$$

Composed with the inverse stereographic projection $p^{-1} : \mathbb{C} \to \mathbb{S}^2 \setminus \{(0,0,1)\} \subset \mathbb{R}^3$ given by

$$p^{-1}(\zeta) = \left(\frac{2\operatorname{Re}\zeta}{|\zeta|^2+1}, \frac{2\operatorname{Im}\zeta}{|\zeta|^2+1}, \frac{|\zeta|^2-1}{|\zeta|^2+1} \right), \quad \zeta \in \mathbb{C},$$

it can be regarded as a map $H : \mathbb{S}^3 \to \mathbb{S}^2$ sending

$$z = (z_1, z_2) \mapsto \left(2\operatorname{Re} z_1\bar{z}_2, 2\operatorname{Im} z_1\bar{z}_2, |z_1|^2 - |z_2|^2\right), \tag{1.1}$$

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which, if we choose the two-sphere $\mathbb{S}^2$ to be of radius $1/2$, becomes a Riemannian submersion, relative to the canonical metric on each sphere.

As is well known, the Hopf map is closely linked to the unit tangent bundle $T^1\mathbb{S}^2 \to \mathbb{S}^2$ of the two-sphere. Indeed, the total space $T^1\mathbb{S}^2$ is diffeomorphic to the real projective three-space $\mathbb{R}P^3$, and the Hopf map $H : \mathbb{S}^3 \to \mathbb{S}^2$ is nothing else than the canonical projection from the universal covering space of $T^1\mathbb{S}^2$ onto $\mathbb{S}^2$. This shows that a Riemannian metric of constant positive curvature exists on $T^1\mathbb{S}^2$, inherited from the canonical metric on $\mathbb{S}^3$.

Then it is a pertinent question whether this constant curvature metric on $T^1\mathbb{S}^2$ is induced from some "natural" Riemannian metric defined on the "ambient" total space $T\mathbb{S}^2$ of the tangent bundle $T\mathbb{S}^2 \to \mathbb{S}^2$ of $\mathbb{S}^2$, when one regards the total space of the unit tangent bundle $T^1\mathbb{S}^2$ as a hypersurface of $T\mathbb{S}^2$. This question also arises when the three-sphere $\mathbb{S}^3$ is equipped with one of the Berger metrics, that is, when a homothety is applied on the fibres.

The aim of this paper is to give affirmative answers, using generalized Cheeger–Gromoll metrics $h_{m,r}$ defined in [1] (see Section 3.3 for the precise definition of $h_{m,r}$), that there is a two-parameter family of Riemannian metrics on the tangent bundle of $\mathbb{S}^2$, which induces desired metrics for both questions. Namely, we prove the following

Theorem 1.1. *Let $\mathbb{S}^n(c)$ be the n-sphere of constant curvature $c > 0$, and denote by $T\mathbb{S}^n(c)$ (resp. $T^1\mathbb{S}^n(c)$) its tangent (resp. unit tangent) bundle. Let $F : \mathbb{S}^3(c/4) \to T^1\mathbb{S}^2(c)$ be the covering map defined by (2.8).*

(1) *Then F induces an isometry from the projective three-space $(\mathbb{R}P^3(c/4), g_{\mathrm{can}})$ of constant curvature $c/4$ to $T^1\mathbb{S}^2(c)$, equipped with the metric induced from the generalized Cheeger–Gromoll metric $h_{m,r}$ on $T\mathbb{S}^2(c)$, where $m = \log_2 c$ and $r \geqslant 0$.*
(2) *Similarly, when $\mathbb{S}^3$ is equipped with a Berger metric g_ϵ defined by (3.10), F induces an isometry from $(\mathbb{R}P^3, g_\epsilon)$ to $(T^1\mathbb{S}^2(4), h_{m,r})$, for $m = \log_2 \epsilon^2 + 2$ and $r \geqslant 0$.*

In particular, we see from Theorem 1.1(1) that any three-sphere of constant positive curvature is isometrically immersed into the total space of the tangent bundle of a two-sphere, equipped with a generalized Cheeger–Gromoll metric. A hyperbolic counterpart of this is also true. Namely, any anti-de Sitter three-space of constant negative curvature is isometrically immersed into the total space of the tangent bundle of a hyperbolic plane, equipped with an indefinite generalized Cheeger–Gromoll metric. More precisely, we prove

Theorem 1.2. *Let $H^3_1(c)$ be the anti-de Sitter three-space of constant curvature $-c < 0$. Let $T\mathbb{H}^2(c)$ (resp. $T^1\mathbb{H}^2(c)$) be the tangent (resp. unit tangent) bundle of the hyperbolic plane $\mathbb{H}^2(c)$ of constant curvature $-c < 0$, and endow $T\mathbb{H}^2(c)$ with the indefinite generalized Cheeger–Gromoll metric $h_{m,r}$ defined by (4.14). Then the covering map $F : H^3_1(c/4) \to T^1\mathbb{H}^2(c)$ defined by (4.8) is an isometric immersion from $H^3_1(c/4)$ to $T^1\mathbb{H}^2(c)$, equipped with the metric induced from $h_{m,r}$, where $m = \log_2 c$ and $r \geqslant 0$.*

The paper is organized as follows. In Section 2 we describe the Hopf map $\mathbb{S}^3(c/4) \to \mathbb{S}^2(c)$ in terms of the natural identification of the three-sphere $\mathbb{S}^3(c/4)$ and the unit tangent bundle $T^1\mathbb{S}^2(c)$ with Lie groups $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$, respectively. Then, using these descriptions, we prove Theorem 1.1 in Section 3. For this end, we compute the differential of the covering map $F : \mathbb{S}^3(c/4) \to T^1\mathbb{S}^2(c)$ and find explicitly a suitable induced metric on $T^1\mathbb{S}^2(c)$ making F to be isometric. An alternative proof of Theorem 1.1, based on our previous knowledge of the curvature of generalized Cheeger–Gromoll metrics, is presented in Remark 3.3.

In Section 4 we prove a hyperbolic counterpart of Theorem 1.1(1). Namely, we define the hyperbolic Hopf map $H^3_1(c/4) \to \mathbb{H}^2(c)$ for the hyperbolic plane, and extend the notion of generalized Cheeger–Gromoll metrics to admit indefinite ones. Then we prove Theorem 1.2 by the same method as in Section 3, namely, by identifying the anti-de Sitter three-space $H^3_1(c/4)$ and the unit tangent bundle $T^1\mathbb{H}^2(c)$ with Lie groups $\mathrm{SU}(1,1)$ and $\mathrm{SO}^+(1,2)$, respectively.

2. Hopf map

To fix our notation and conventions, we first review how one can identify the Hopf map $H : \mathbb{S}^3 \to \mathbb{S}^2$ with the canonical projection from the universal covering space of the unit tangent bundle $T^1\mathbb{S}^2$ onto the 2-sphere $\mathbb{S}^2$.

To begin with, recall that the unit 3-sphere

$$\mathbb{S}^3 = \left\{ \left(x^1, x^2, x^3, x^4\right) \in \mathbb{R}^4 \;\middle|\; \left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2 + \left(x^4\right)^2 = 1 \right\}$$

is diffeomorphic to the special unitary group

$$\begin{aligned} \mathrm{SU}(2) &= \left\{ A \in \mathrm{GL}(2, \mathbb{C}) \;\middle|\; {}^t\bar{A}A = \mathrm{Id},\ \det A = 1 \right\} \\ &= \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \;\middle|\; a, b \in \mathbb{C},\ |a|^2 + |b|^2 = 1 \right\} \end{aligned}$$

under the map

$$\begin{aligned}&\psi:\mathbb{S}^3\to \mathrm{SU}(2),\\ &x=\big(x^1,x^2,x^3,x^4\big)\mapsto A_x=\begin{pmatrix} z_1 & -\bar z_2\\ z_2 & \bar z_1\end{pmatrix},\end{aligned}\tag{2.1}$$

where $z_1=x^1+\sqrt{-1}x^2$ and $z_2=x^3+\sqrt{-1}x^4$.

Moreover, SU(2) is the universal covering space of the special orthogonal group SO(3) with the covering map

$$\rho:\mathrm{SU}(2)\to \mathrm{SO}(3),\qquad A_x\mapsto \rho(A_x)$$

described as follows. First, we regard SO(3) as SO($\mathfrak{su}(2)$), where the Lie algebra of SU(2),

$$\begin{aligned}\mathfrak{su}(2)&=\big\{X\in \mathfrak{gl}(2,\mathbb{C})\;\big|\;{}^tX+\bar X=0,\ \mathrm{Tr}\,X=0\big\}\\ &=\left\{\begin{pmatrix}\sqrt{-1}x^3 & -x^2+\sqrt{-1}x^1\\ x^2+\sqrt{-1}x^1 & -\sqrt{-1}x^3\end{pmatrix}\;\middle|\;x^1,x^2,x^3\in\mathbb{R}\right\},\end{aligned}$$

is identified with $\mathbb{R}^3$, equipped with the scalar product $\langle X,Y\rangle=-(1/2)\,\mathrm{Tr}(XY)$, so that

$$e_1=\begin{pmatrix}0 & \sqrt{-1}\\ \sqrt{-1} & 0\end{pmatrix},\qquad e_2=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\qquad e_3=\begin{pmatrix}\sqrt{-1} & 0\\ 0 & -\sqrt{-1}\end{pmatrix}\tag{2.2}$$

form an orthonormal basis of $(\mathfrak{su}(2),\langle,\rangle)$. Then $\rho(A_x)$ is defined by the adjoint representation of SU(2) as

$$\rho(A_x):\mathfrak{su}(2)\to\mathfrak{su}(2),\qquad Y\mapsto \mathrm{Ad}(A_x)Y=A_xYA_x^{-1},\tag{2.3}$$

and so $\rho(A_x)\in \mathrm{SO}(3)\cong \mathrm{SO}(\mathfrak{su}(2),\langle,\rangle)$.

The matrix representation of $\rho(A_x)$, with respect to the orthonormal basis (2.2) of $\mathfrak{su}(2)$, is given by

$$\begin{aligned}\rho(A_x)&=\begin{pmatrix}\mathrm{Re}(z_1^2-\bar z_2^2) & \mathrm{Im}(\bar z_1^2+z_2^2) & 2\,\mathrm{Re}(z_1\bar z_2)\\ \mathrm{Im}(z_1^2-\bar z_2^2) & \mathrm{Re}(\bar z_1^2+z_2^2) & 2\,\mathrm{Im}(z_1\bar z_2)\\ -2\,\mathrm{Re}(z_1z_2) & 2\,\mathrm{Im}(z_1z_2) & |z_1|^2-|z_2|^2\end{pmatrix}\\ &=\begin{pmatrix}A_xe_1A_x^{-1} & A_xe_2A_x^{-1} & A_xe_3A_x^{-1}\end{pmatrix}.\end{aligned}\tag{2.4}$$

Note that $\rho:\mathrm{SU}(2)\to\mathrm{SO}(3)$ is a homomorphism with kernel $\{\pm\mathrm{Id}\}$, and hence SO(3) is diffeomorphic to the real projective three-space $\mathbb{R}\mathrm{P}^3$.

Given $c>0$, let $\mathbb{S}^n(c)\subset\mathbb{R}^{n+1}$ denote the n-sphere of radius $1/\sqrt{c}$ with center at the origin of $\mathbb{R}^{n+1}$. We also denote the unit n-sphere $\mathbb{S}^n(1)$ simply by $\mathbb{S}^n$. Recall that the unit vectors tangent to $\mathbb{S}^2(c)$ form the *unit tangent bundle*

$$\begin{aligned}T^1\mathbb{S}^2(c)&=\big\{(x,v)\in\mathbb{R}^3\times\mathbb{R}^3\;\big|\;x\in\mathbb{S}^2(c),\ v\in T_x\mathbb{S}^2(c),\ |v|=1\big\}\\ &=\big\{(x,v)\in\mathbb{R}^3\times\mathbb{R}^3\;\big|\;|x|=1/\sqrt{c},\ |v|=1,\ \langle x,v\rangle=0\big\}\end{aligned}\tag{2.5}$$

of $\mathbb{S}^2(c)$ with the canonical projection $\pi:T^1\mathbb{S}^2(c)\to\mathbb{S}^2(c)$ given by $\pi(x,v)=x$. Since $T^1\mathbb{S}^2(c)$ is composed of orthogonal vectors of $\mathbb{R}^3$, one can define the diffeomorphism

$$\phi:\mathrm{SO}(3)\to T^1\mathbb{S}^2(c),\quad (c_1\ c_2\ c_3)\mapsto (c_3/\sqrt{c},c_1).\tag{2.6}$$

Finally, let ι be the homothety defined by

$$\iota:\mathbb{S}^3(c/4)\to\mathbb{S}^3(1),\quad 2x/\sqrt{c}\mapsto x.\tag{2.7}$$

Then we have the following

Proposition 2.1. *The composition of the covering map*

$$F=\phi\circ\rho\circ\psi\circ\iota:\mathbb{S}^3(c/4)\to T^1\mathbb{S}^2(c)\tag{2.8}$$

with the canonical projection $\pi:T^1\mathbb{S}^2(c)\to\mathbb{S}^2(c)$ is identical with the Hopf map $H:\mathbb{S}^3(c/4)\to\mathbb{S}^2(c)$.

Indeed, from (2.1) through (2.7), we see that the composition $\pi\circ F$ is a map sending

$$(2/\sqrt{c})(z_1,z_2)\mapsto (1/\sqrt{c})\big(2z_1\bar z_2,|z_1|^2-|z_2|^2\big),$$

which is nothing but the *Hopf map* H of (1.1) normalized in our context.

3. Differential of the covering map

The most direct path to an answer to our problem is to compute the differential of the covering map $F : \mathbb{S}^3(c/4) \to T^1\mathbb{S}^2(c)$, determine the image of an orthonormal frame of $T\mathbb{S}^3(c/4)$, and then find explicitly a suitable induced metric on $T^1\mathbb{S}^2(c)$ making F to be isometric. This can be carried out as follows.

3.1. Differentials of maps

(1) The map $\psi : \mathbb{S}^3 \to \mathrm{SU}(2)$ in (2.1) gives rise to a linear map from $\mathbb{R}^4$ into the space of complex 2×2 matrices of the form $\left(\begin{smallmatrix} a & -\bar{b} \\ b & \bar{a} \end{smallmatrix}\right)$, so that $d\psi_x = \psi$ for all $x \in \mathbb{R}^4$.

Noting that the fibres of the Hopf map (1.1) are described as the orbits of the $\mathbb{S}^1$-action $\mathbb{S}^1 \times \mathbb{S}^3 \to \mathbb{S}^3$ on $\mathbb{S}^3$ defined by

$$\big(e^{\sqrt{-1}t}, (z_1, z_2)\big) \mapsto e^{\sqrt{-1}t}(z_1, z_2) = \big(e^{\sqrt{-1}t} z_1, e^{\sqrt{-1}t} z_2\big),$$

we see that if $x = (x^1, x^2, x^3, x^4) \in \mathbb{S}^3$, then

$$X_3(x) = (\sqrt{-1}z_1, \sqrt{-1}z_2) = \big(-x^2, x^1, -x^4, x^3\big)$$

is a vector tangent to a fibre of the Hopf map, and

$$X_3(x), \qquad X_2(x) = \big(-x^3, x^4, x^1, -x^2\big), \qquad X_1(x) = \big(-x^4, -x^3, x^2, x^1\big)$$

form a global orthonormal frame of $T\mathbb{S}^3$. Since $\psi(x) = A_x = \left(\begin{smallmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{smallmatrix}\right)$, it follows that

$$d\psi_x = \psi : T_x\mathbb{S}^3 \to T_{\psi(x)}\big(\mathrm{SU}(2)\big) = A_x \cdot \mathfrak{su}(2)$$

and

$$d\psi_x\big(X_3(x)\big) = \begin{pmatrix} -x^2 + \sqrt{-1}x^1 & x^4 + \sqrt{-1}x^3 \\ -x^4 + \sqrt{-1}x^3 & -x^2 - \sqrt{-1}x^1 \end{pmatrix} = A_x e_3. \tag{3.1}$$

Similarly, we have $d\psi_x(X_2(x)) = A_x e_2$ and $d\psi_x(X_1(x)) = A_x e_1$.

(2) The differential of the covering map

$$\rho : \mathrm{SU}(2) \to \mathrm{SO}(3), \qquad A_x \mapsto \rho(A_x),$$

given by (2.3), is a linear map

$$d\rho_{A_x} : T_{A_x}\big(\mathrm{SU}(2)\big) = A_x \cdot \mathfrak{su}(2) \to T_{\rho(A_x)}\mathrm{SO}(3) = \rho(A_x) \cdot \mathfrak{so}(3)$$

defined by

$$A_x Y \mapsto d\rho_{A_x}(A_x Y) = \rho(A_x) \circ \mathrm{ad}(Y), \tag{3.2}$$

where

$$\mathrm{ad}(Y) : \mathfrak{su}(2) \to \mathfrak{su}(2), \qquad Z \mapsto \mathrm{ad}(Y)(Z) = [Y, Z].$$

Consequently, for the orthonormal basis (2.2) of $\mathfrak{su}(2)$, we obtain, for instance,

$$d\rho_{A_x} : A_x \cdot \mathfrak{su}(2) \to \rho(A_x) \cdot \mathfrak{so}(3), \qquad A_x e_3 \mapsto \rho(A_x) \circ \mathrm{ad}(e_3),$$

and $\mathrm{ad}(e_3)(e_3) = 0$, $\mathrm{ad}(e_3)(e_2) = -2e_1$, $\mathrm{ad}(e_3)(e_1) = 2e_2$. Therefore, as a matrix,

$$\mathrm{ad}(e_3) = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$\rho(A_x) \circ \mathrm{ad}(e_3) = \big(2A_x e_2 A_x^{-1} \quad -2A_x e_1 A_x^{-1} \quad 0\big).$$

Similarly, since $\mathrm{ad}(e_2)(e_1) = -2e_3$, we obtain

$$\rho(A_x) \circ \mathrm{ad}(e_2) = \big(-2A_x e_3 A_x^{-1} \quad 0 \quad 2A_x e_1 A_x^{-1}\big),$$
$$\rho(A_x) \circ \mathrm{ad}(e_1) = \big(0 \quad 2A_x e_3 A_x^{-1} \quad -2A_x e_2 A_x^{-1}\big).$$

(3) Finally, we note that the diffeomorphism ϕ defined by (2.6) is linear, so $d\phi_g = \phi$ and, for $\rho(A_x) \in \mathrm{SO}(3)$

$$d\phi_{\rho(A_x)} = \phi : T_{\rho(A_x)}\mathrm{SO}(3) = \rho(A_x) \cdot \mathfrak{so}(3) \to T_{\phi(\rho(A_x))}\big(T^1\mathbb{S}^2(c)\big)$$

is given by

$$(\alpha_1\ \alpha_2\ \alpha_3) \mapsto (\alpha_3/\sqrt{c}, \alpha_1).$$

Therefore we obtain

$$\begin{aligned} d\phi_{\rho(A_x)}\big(\rho(A_x) \circ \mathrm{ad}(e_3)\big) &= \big(0, 2A_x e_2 A_x^{-1}\big) = \tilde{e}_3, \\ d\phi_{\rho(A_x)}\big(\rho(A_x) \circ \mathrm{ad}(e_2)\big) &= \big(2A_x e_1 A_x^{-1}/\sqrt{c}, -2A_x e_3 A_x^{-1}\big) = \tilde{e}_2, \\ d\phi_{\rho(A_x)}\big(\rho(A_x) \circ \mathrm{ad}(e_1)\big) &= \big({-2}A_x e_2 A_x^{-1}/\sqrt{c}, 0\big) = \tilde{e}_1. \end{aligned} \tag{3.3}$$

In conclusion, combining (2.8) together with (3.1) through (3.3) yields

$$dF_x\big(2X_3(x)/\sqrt{c}\,\big) = \tilde{e}_3, \qquad dF_x\big(2X_2(x)/\sqrt{c}\,\big) = \tilde{e}_2, \qquad dF_x\big(2X_1(x)/\sqrt{c}\,\big) = \tilde{e}_1. \tag{3.4}$$

3.2. Lifts to the unit tangent bundle

In general, each tangent space of the tangent bundle TM of a Riemannian manifold (M, g) admits a canonical decomposition into its vertical and horizontal subspaces. Indeed, given a point $(x, e) \in TM$, the kernel of the differential of the canonical projection $\pi : TM \to M$ defines the vertical space $\mathcal{V}_{(x,e)} = \ker d\pi_{(x,e)}$, while the horizontal space $\mathcal{H}_{(x,e)}$ is given by the kernel of the connection map

$$K_{(x,e)} = K : T_{(x,e)}TM \to T_xM, \qquad K(Z) = d(\exp_x \circ R_{-e} \circ \tau)(Z).$$

Here $\tau : U \subset TM \to T_xM$ is the map, defined on an open neighbourhood U of $(x, e) \in TM$, sending a vector $v \in T_yM$, with $(y, v) \in U$, to a vector in T_xM by parallel transport along the unique geodesic arc from y to x. The map $R_{-e} : T_xM \to T_xM$ is the translation given by $R_{-e}(X) = X - e$ for $X \in T_xM$.

One can see that $\mathcal{H}_{(x,e)} \cap \mathcal{V}_{(x,e)} = \{0\}$ and $\mathcal{H}_{(x,e)} \oplus \mathcal{V}_{(x,e)} = T_{(x,e)}TM$, and define the *horizontal lift* $X^h \in \mathcal{H}_{(x,e)}$ and the *vertical lift* $X^v \in \mathcal{V}_{(x,e)}$ of $X \in T_xM$ by

$$K_{(x,e)}\big(X^v\big) = X, \qquad d\pi_{(x,e)}\big(X^h\big) = X.$$

An alternative description of the horizontal lift X^h is given as follows. Let $X \in T_xM$ and choose $e \in T_xM$. Take a curve $\gamma : I \to M$ such that $\gamma(0) = x$ and $\dot{\gamma}(0) = X$. (Since the result is independent of the curve chosen, we can take it to be a geodesic.) Let $\Gamma : I \to TM$ be the unique curve in TM such that $\Gamma(0) = (x, e)$ and $\Gamma(t)$ is parallel to $\dot{\gamma}(t)$ in the sense that $\nabla_{\dot{\gamma}(t)}\Gamma(t) = 0$ for all $t \in I$. Namely, $\Gamma(t) = (\gamma(t), v(t))$, where $v(t) \in T_{\gamma(t)}M$ and $\nabla_{\dot{\gamma}(t)}v(t) = 0$ for all $t \in I$, so that $v(t)$ is the parallel transport of the vector e along the curve γ. Then $\dot{\Gamma}(0) = X^h \in T_{(x,e)}TM$. We will use this approach below.

Now, recall that the unit tangent bundle $T^1\mathbb{S}^2(c)$ is a 3-dimensional hypersurface of $T\mathbb{S}^2(c)$. Then we note that at $(x, e) \in T^1\mathbb{S}^2(c)$ the tangent space of the tangent bundle $T\mathbb{S}^2(c)$ is written as

$$T_{(x,e)}\big(T\mathbb{S}^2(c)\big) = \big\{X^h + Y^v \bigm| X, Y \in T_x\mathbb{S}^2(c)\big\},$$

where X^h (resp. Y^v) is the horizontal (resp. vertical) lift of X (resp. Y). Also, that of the unit tangent bundle $T^1\mathbb{S}^2(c)$ is given by

$$T_{(x,e)}\big(T^1\mathbb{S}^2(c)\big) = \big\{X^h + Y^v \bigm| X, Y \in T_x\mathbb{S}^2(c),\ \langle Y, e\rangle = 0\big\}, \tag{3.5}$$

since the tangent vector at (x, e) of any vertical curve on $T^1\mathbb{S}^2(c)$ must be orthogonal to e.

We know the differential of the covering map $F : \mathbb{S}^3(c/4) \to T^1\mathbb{S}^2(c)$ from (3.4) and recall that

$$F(2x/\sqrt{c}\,) = (\tilde{x}, e) \in T^1\mathbb{S}^2(c)$$

for each $2x/\sqrt{c} \in \mathbb{S}^3(c/4)$, where $\tilde{x} = (1/\sqrt{c}\,)A_x e_3 A_x^{-1}$ and $e = A_x e_1 A_x^{-1}$. We set

$$f = -A_x e_2 A_x^{-1}.$$

Then $(\tilde{x}, f) \in T^1\mathbb{S}^2(c)$ and $\langle f, e\rangle = 0$, so that, by virtue of (3.5),

$$T_{(\tilde{x},e)}\big(T^1\mathbb{S}^2(c)\big) = \mathrm{Span}\big\{e^h, f^h, f^v\big\}.$$

Now, we are going to show

Proposition 3.1. *Let $\tilde{x}$, e and f be as above. Then*

$$(\sqrt{c}/2)\tilde{e}_2 = e^h, \qquad (\sqrt{c}/2)\tilde{e}_1 = f^h, \qquad \tilde{e}_3 = -2f^v. \tag{3.6}$$

Proof. To construct the horizontal lift $e^h \in T_{(\tilde{x},e)}(T^1\mathbb{S}^2(c))$, we take the great circle γ in $\mathbb{S}^2(c)$ such that $\gamma(0) = \tilde{x}$ and $\dot{\gamma}(0) = e$, that is,

$$\gamma(t) = \cos(\sqrt{c}t)\tilde{x} + \sin(\sqrt{c}t)(e/\sqrt{c}).$$

Then the curve $\Gamma : I \to T^1\mathbb{S}^2(c)$ given by $\Gamma(t) = (\gamma(t), \dot{\gamma}(t))$ is parallel to $\dot{\gamma}(t)$, so that $e^h = \dot{\Gamma}(0) = (\dot{\gamma}(0), \ddot{\gamma}(0))$. Namely,

$$e^h = \left(A_x e_1 A_x^{-1}, -\sqrt{c}A_x e_3 A_x^{-1}\right) = (\sqrt{c}/2)\tilde{e}_2.$$

Similarly, to construct $f^h \in T_{(\tilde{x},e)}(T^1\mathbb{S}^2(c))$ for $f = -A_x e_2 A_x^{-1}$, we take the great circle $\gamma(t) = \cos(\sqrt{c}t)\tilde{x} + \sin(\sqrt{c}t) \times (f/\sqrt{c})$, so that $\gamma(0) = \tilde{x}$ and $\dot{\gamma}(0) = f$. Then the curve $\Gamma : I \to T^1\mathbb{S}^2(c)$ given by $\Gamma(t) = (\gamma(t), v(t) = e)$ satisfies $\nabla_{\dot{\gamma}(t)} v(t) = 0$ for all $t \in I$. Hence

$$f^h = \dot{\Gamma}(0) = (f, 0) = \left(-A_x e_2 A_x^{-1}, 0\right) = (\sqrt{c}/2)\tilde{e}_1.$$

Finally, since $d\pi(\tilde{e}_3) = 0$, to show that $\tilde{e}_3 = -2f^v$ we compute $K(\tilde{e}_3)$. Since $\tilde{e}_3 = dF_x(2X_3/\sqrt{c})$ and $X_3 = \dot{\gamma}(0)$ for $\gamma(t) = e^{\sqrt{-1}t}x$, which is indeed a geodesic of $\mathbb{S}^3$ along a fibre of the Hopf map, we can write $\tilde{e}_3$ as a vector tangent to a curve $\tilde{\gamma}(t) = F \circ (2/\sqrt{c})\gamma(t)$ in $T^1\mathbb{S}^2(c)$ and then

$$K(\tilde{e}_3) = \frac{d}{dt}\bigg|_{t=0} (\exp_{\tilde{x}} \circ R_{-e} \circ \tau)\big(\tilde{\gamma}(t)\big). \tag{3.7}$$

Also, it is immediate from (2.4) and (2.6) that

$$\tilde{\gamma}(t) = \left((1/\sqrt{c})A_x e_3 A_x^{-1},\ A_{\gamma(t)} e_1 A_{\gamma(t)}^{-1}\right) \in T^1\mathbb{S}^2(c)$$

and $\pi(\tilde{\gamma}(t)) = \tilde{x}$, so that $\tilde{\gamma}(t)$ is a curve along the fibre over $\tilde{x}$. Consequently, the parallel transport τ in (3.7) is the identity map, and

$$K(\tilde{e}_3) = \frac{d}{dt}\bigg|_{t=0} \exp_{\tilde{x}}\left(\frac{1}{\sqrt{c}}A_x e_3 A_x^{-1}, A_{\gamma(t)} e_1 A_{\gamma(t)}^{-1} - A_x e_1 A_x^{-1}\right),$$

since $e = A_x e_1 A_x^{-1}$.

Put $W(t) = A_{\gamma(t)} e_1 A_{\gamma(t)}^{-1} - A_x e_1 A_x^{-1}$. Then the geodesic of $\mathbb{S}^2(c)$ starting at $\tilde{x}$ with initial vector $W(t)$ is given by

$$\delta_t(s) = \frac{1}{\sqrt{c}} A_x e_1 A_x^{-1} \cos\left(\sqrt{c}\left|W(t)\right|s\right) + \frac{1}{\sqrt{c}} \frac{W(t)}{|W(t)|} \sin\left(\sqrt{c}\left|W(t)\right|s\right),$$

and $K(\tilde{e}_3) = (d/dt)|_{t=0}\delta_t(1)$. On the other hand, since

$$\gamma(t) = \left(x^1\cos t - x^2\sin t, x^2\cos t + x^1\sin t, x^3\cos t - x^4\sin t, x^4\cos t + x^3\sin t\right),$$

we have

$$\begin{aligned} W(t) &= A_{\gamma(t)} e_1 A_{\gamma(t)}^{-1} - A_x e_1 A_x^{-1} \\ &= \begin{pmatrix} -4(-x^1x^3 + x^2x^4)\sin^2 t + 2(x^1x^4 + x^2x^3)\sin 2t \\ -4(x^1x^2 + x^3x^4)\sin^2 t + ((x^1)^2 - (x^2)^2 + (x^3)^2 - (x^4)^2)\sin 2t \\ -2((x^1)^2 - (x^2)^2 - (x^3)^2 + (x^4)^2)\sin^2 t - 2(x^1x^2 - x^3x^4)\sin 2t \end{pmatrix} \end{aligned}$$

and $|W(t)| = 2\sin t$.

Therefore we obtain

$$\begin{aligned} K(\tilde{e}_3) &= \frac{d}{dt}\bigg|_{t=0}\left(\frac{1}{\sqrt{c}}A_x e_3 A_x^{-1}\cos(2\sqrt{c}\sin t) + \frac{W(t)}{2\sqrt{c}\sin t}\sin(2\sqrt{c}\sin t)\right), \\ &= \left(\frac{W(t)}{2\sqrt{c}\sin t}\right)(0)\frac{d}{dt}\bigg|_{t=0}\sin(2\sqrt{c}\sin t) \\ &= \begin{pmatrix} 4(x^1x^4 + x^2x^3) \\ 2((x^1)^2 - (x^2)^2 + (x^3)^2 - (x^4)^2) \\ -4(x^1x^2 - x^3x^4) \end{pmatrix} = 2A_x e_2 A_x^{-1} \\ &= -2f, \end{aligned}$$

which shows that $\tilde{e}_3 = -2f^v$. □

3.3. Generalized Cheeger–Gromoll metrics

For the tangent bundle TM of a Riemannian manifold (M,g), a *natural* Riemannian metric on TM, in the sense that the vertical and horizontal subspaces of each tangent space of TM are orthogonal and the canonical projection $\pi : TM \to M$ becomes a Riemannian submersion, was first defined by Sasaki [7]. This metric, now called the Sasaki metric, appears as having the simplest possible form, but its geometry is known to be rather rigid (cf. [1,5]). Later on, a more general metric, called the Cheeger–Gromoll metric, was given on TM by Musso and Tricerri [5], which has been further generalized in [1] toward the discovery of new harmonic sections of Riemannian vector bundles.

To be precise, given the two-sphere $\mathbb{S}^2(c)$, for $m \in \mathbb{R}$ and $r \geqslant 0$, the *generalized Cheeger–Gromoll metric* $h_{m,r}$ on the tangent bundle $T\mathbb{S}^2(c)$ is defined, on each tangent space $T_{(x,e)}(T\mathbb{S}^2(c))$ at $(x,e) \in T\mathbb{S}^2(c)$, by

$$\begin{aligned} &h_{m,r}\big(X^h, Y^h\big) = \langle X, Y\rangle, \qquad h_{m,r}\big(X^h, Y^v\big) = 0, \\ &h_{m,r}\big(X^v, Y^v\big) = \omega^m\big(\langle X, Y\rangle + r\langle X, e\rangle\langle Y, e\rangle\big), \end{aligned} \tag{3.8}$$

where $X, Y \in T_x\mathbb{S}^2(c)$ and $\omega = 1/(1+|e|^2)$. In particular, when $(x,e) \in T^1\mathbb{S}^2(c)$, this metric restricts on $T_{(x,e)}(T^1\mathbb{S}^2(c))$ to

$$\begin{aligned} &h_{m,r}\big(X^h, Y^h\big) = \langle X, Y\rangle, \quad h_{m,r}\big(X^h, Y^v\big) = 0, \\ &h_{m,r}\big(X^v, Y^v\big) = \frac{1}{2^m}\langle X, Y\rangle, \end{aligned} \tag{3.9}$$

since $\langle Y, e\rangle = 0$ by virtue of (3.5). Namely, the parameter r disappears if $h_{m,r}$ is restricted to the unit tangent bundle $T^1\mathbb{S}^2(c)$. It should be noted that the original Cheeger–Gromoll metric corresponds to $m = r = 1$ and the Sasaki metric to $m = r = 0$.

Now, our Theorem 1.1 can be proved as follows. If we choose $m = \log_2 c$, then, noting (3.4) and (3.6), we obtain from (3.9) that

$$\begin{aligned} &h_{m,r}\big((\sqrt{c}/2)\tilde{e}_1, (\sqrt{c}/2)\tilde{e}_1\big) = h_{m,r}\big(f^h, f^h\big) = \langle f, f\rangle = 1, \\ &h_{m,r}\big((\sqrt{c}/2)\tilde{e}_2, (\sqrt{c}/2)\tilde{e}_2\big) = h_{m,r}\big(e^h, e^h\big) = \langle e, e\rangle = 1, \\ &h_{m,r}\big((\sqrt{c}/2)\tilde{e}_1, (\sqrt{c}/2)\tilde{e}_2\big) = h_{m,r}\big(f^h, e^h\big) = \langle f, e\rangle = 0, \\ &h_{m,r}\big((\sqrt{c}/2)\tilde{e}_2, (\sqrt{c}/2)\tilde{e}_3\big) = -h_{m,r}\big(e^h, \sqrt{c}f^v\big) = 0, \\ &h_{m,r}\big((\sqrt{c}/2)\tilde{e}_1, (\sqrt{c}/2)\tilde{e}_3\big) = -h_{m,r}\big(f^h, \sqrt{c}f^v\big) = 0, \end{aligned}$$

and

$$h_{m,r}\big((\sqrt{c}/2)\tilde{e}_3, (\sqrt{c}/2)\tilde{e}_3\big) = h_{m,r}\big(-\sqrt{c}f^v, -\sqrt{c}f^v\big) = \frac{c}{2^m}\langle f, f\rangle = 1.$$

This shows that $F : \mathbb{S}^3(c/4) \to T^1\mathbb{S}^2(c)$ defined by (2.8) induces an isometry from $(\mathbb{R}\mathrm{P}^3(c/4), g_{\mathrm{can}})$ to $(T^1\mathbb{S}^2(c), h_{m,r})$ for $m = \log_2 c$ and any $r \geqslant 0$.

Moreover, if we equip the unit three-sphere $\mathbb{S}^3$ with a *Berger metric* g_ϵ in [3] such that

$$\{X_1, X_2, \epsilon X_3\} \text{ is an orthonormal frame of } T\mathbb{S}^3, \tag{3.10}$$

then we see from (3.4) that $dF_x(\epsilon X_3) = \epsilon\tilde{e}_3$ and

$$h_{m,r}(\epsilon\tilde{e}_3, \epsilon\tilde{e}_3) = h_{m,r}\big(-2\epsilon f^v, -2\epsilon f^v\big) = \frac{1}{2^m}\langle 2\epsilon f, 2\epsilon f\rangle = \frac{4\epsilon^2}{2^m}.$$

Therefore, for $m = \log_2 \epsilon^2 + 2$, the map $F : \mathbb{S}^3 \to T^1\mathbb{S}^2(4)$ yields an isometry from $(\mathbb{R}\mathrm{P}^3, g_\epsilon)$ to $(T^1\mathbb{S}^2(4), h_{m,r})$ for any $r \geqslant 0$.

Remark 3.2. In Theorem 1.1(1), if we choose $c = 1$, then $m = 0$. Thus, for $r = 0$ the generalized Cheeger–Gromoll metric $h_{0,0}$ defined by (3.8) is nothing but the Sasaki metric defined on $T\mathbb{S}^2(1)$. In this case, Theorem 1.1(1) is proved in [4].

Remark 3.3 *(Curvature approach).* An alternative method would be to compute that $(T^1\mathbb{S}^2(c), h_{m,r})$ with $m = \log_2 c$ has constant sectional curvature $c/4$, looking at $T^1\mathbb{S}^2(c)$ as a hypersurface of $T\mathbb{S}^2(c)$ and use previous knowledge of the curvature of $(T\mathbb{S}^2(c), h_{m,r})$ (cf. [2]). Fairly simple computations show that the second fundamental form B of $T^1\mathbb{S}^2(c)$ in $T\mathbb{S}^2(c)$ is given by

$$\begin{aligned} &B\big(X^h, Y^h\big) = B\big(X^h, Y^v\big) = 0, \\ &B\big(X^v, Y^v\big) = \sqrt{2^m/(1+r)}\,\frac{m/2 + r}{1 + r}\langle X, Y\rangle\boldsymbol{n}, \end{aligned}$$

and from the Gauss formula, we see that the sectional curvature $\hat{K}$ of $(T^1\mathbb{S}^2(c), h_{m,r})$ at the point $(x,e) \in T^1\mathbb{S}^2(c)$ is given by

$$\hat{K}\big(e^h \wedge f^h\big) = c - \frac{3c^2}{2^{m+2}},$$
$$\hat{K}\big(e^h \wedge f^v\big) = \hat{K}\big(f^h \wedge f^v\big) = \frac{c^2}{2^{m+2}}, \tag{3.11}$$

where $f \in T^1_x\mathbb{S}^2(c)$ with $\langle e, f\rangle = 0$ and $T_{(x,e)}(T^1\mathbb{S}^2(c)) = \mathrm{Span}\{e^h, f^h, f^v\}$. Clearly, the sectional curvatures are equal to $c/4$ if $m = \log_2 c$ (for any $r \geqslant 0$), whilst for the Berger metric g_ϵ, we need to choose

$$m = \log_2 \epsilon^2 + 2.$$

4. Hyperbolic counterpart

In what follows, we denote by $\mathbb{R}^n_\nu$ the pseudo-Euclidean n-space of index ν, that is, $\mathbb{R}^n$ equipped with the indefinite metric

$$\langle x, y\rangle = \sum_{i=1}^{n-\nu} x^i y^i - \sum_{j=n-\nu+1}^{n} x^j y^j.$$

4.1. Hyperbolic Hopf map

Let $H^3_1(c)$ be the *anti-de Sitter* 3*-space* of constant negative curvature $-c<0$ (cf. [6]), which is, by definition, a hypersurface in $\mathbb{R}^4_2$ defined by $\langle x,x\rangle = -1/c$, that is,

$$H^3_1(c) = \big\{\big(x^1,x^2,x^3,x^4\big) \in \mathbb{R}^4_2 \;\big|\; \big(x^1\big)^2 + \big(x^2\big)^2 - \big(x^3\big)^2 - \big(x^4\big)^2 = -1/c\big\}.$$

Note that $H^3_1(c)$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^2$. If we introduce complex coordinates $z_1 = x^1 + \sqrt{-1}x^2$ and $z_2 = x^3 + \sqrt{-1}x^4$, then $H^3_1(c)$ is represented as

$$H^3_1(c) = \big\{(z_1,z_2) \in \mathbb{C}^2 \;\big|\; |z_1|^2 - |z_2|^2 = -1/c\big\}.$$

To define the hyperbolic Hopf map, let $\varpi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}\mathrm{P}^1$ be the canonical projection defining the complex projective line $\mathbb{C}\mathrm{P}^1$. Restricting ϖ to $H^3_1(c) \subset \mathbb{C}^2 \setminus \{0\}$, we have a mapping

$$\varpi : H^3_1(c) \to \mathbb{C}, \qquad z = (z_1,z_2) \mapsto \varpi(z) = z_1/z_2,$$

which maps $H^3_1(c)$ diffeomorphically onto the unit ball $B^2 = \{\zeta \in \mathbb{C} \mid |\zeta| < 1\}$ in $\mathbb{C}$. Let

$$\mathbb{H}^2(c) = \big\{\big(x^1,x^2,x^3\big) \in \mathbb{R}^3_1 \;\big|\; \big(x^1\big)^1 + \big(x^2\big)^2 - \big(x^3\big)^2 = -1/c, x^3 > 0\big\}$$

be the hyperbolic plane of constant curvature $-c<0$ embedded in $\mathbb{R}^3_1$. Denote by

$$p^{-1}(\zeta) = \left(\frac{2\,\mathrm{Re}\,\zeta}{1-|\zeta|^2}, \frac{2\,\mathrm{Im}\,\zeta}{1-|\zeta|^2}, \frac{1+|\zeta|^2}{1-|\zeta|^2}\right), \quad \zeta \in B^2 \subset \mathbb{C},$$

the inverse stereographic projection $p^{-1} : B \to \mathbb{H}^2(1)$ from the south pole $(0,0,-1) \in \mathbb{H}^2(1)$, and let η be the homothety defined by

$$\eta : \mathbb{H}^2(1) \to \mathbb{H}^2(c), \quad x \mapsto x/\sqrt{c}.$$

Then, composing ϖ with $\eta \circ p^{-1}$, we obtain the *hyperbolic Hopf map*

$$H = \eta \circ p^{-1} \circ \varpi : H^3_1(c/4) \to \mathbb{H}^2(c), \tag{4.1}$$

given by

$$H(z) = (1/\sqrt{c})\big(2z_1\bar{z}_2, |z_1|^2 + |z_2|^2\big) \in \mathbb{C} \times \mathbb{R}. \tag{4.2}$$

Note that the hyperbolic Hopf map H is a submersion from a pseudo-Riemannian manifold $H^3_1(c/4)$ with geodesic fibres, which can be described as the orbits of the $\mathbb{S}^1$-action $\mathbb{S}^1 \times H^3_1(c/4) \to H^3_1(c/4)$ on $H^3_1(c/4)$ defined by

$$\big(e^{\sqrt{-1}t}, (z_1,z_2)\big) \mapsto e^{\sqrt{-1}t}(z_1,z_2) = \big(e^{\sqrt{-1}t}z_1, e^{\sqrt{-1}t}z_2\big).$$

In particular, if $x=(x^1,x^2,x^3,x^4)\in H_1^3(1)$, then

$$X_3(x)=(\sqrt{-1}z_1,\sqrt{-1}z_2)=\left(-x^2,x^1,-x^4,x^3\right)$$

is a vector tangent to a fibre of the hyperbolic Hopf map with $\langle X_3,X_3\rangle=-1$, and

$$X_3(x),\qquad X_2(x)=\left(x^3,-x^4,x^1,-x^2\right),\qquad X_1(x)=\left(x^4,x^3,x^2,x^1\right)$$

form a global pseudo-orthonormal frame of TH_1^3 such that $\langle X_2,X_2\rangle=\langle X_1,X_1\rangle=1$ and $\langle X_1,X_2\rangle=\langle X_1,X_3\rangle=\langle X_2,X_3\rangle=0$.

Now, recall that the Lie group

$$\begin{aligned}\mathrm{SU}(1,1)&=\left\{A\in\mathrm{GL}(2,\mathbb{C})\;\middle|\;{}^tAI_1\bar A=I_1,\ \det A=1\right\}\\&=\left\{\begin{pmatrix}a&\bar b\\ b&\bar a\end{pmatrix}\;\middle|\;a,b\in\mathbb{C},\ |a|^2-|b|^2=1\right\},\end{aligned}$$

where $I_1=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, has the Lie algebra

$$\begin{aligned}\mathfrak{su}(1,1)&=\left\{X\in\mathfrak{gl}(2,\mathbb{C})\;\middle|\;{}^tXI_1+I_1\bar X=0,\ \operatorname{Tr}X=0\right\}\\&=\left\{\begin{pmatrix}\sqrt{-1}x^3&x^2-\sqrt{-1}x^1\\ x^2+\sqrt{-1}x^1&-\sqrt{-1}x^3\end{pmatrix}\;\middle|\;x^1,x^2,x^3\in\mathbb{R}\right\},\end{aligned}$$

which is identified with $\mathbb{R}_1^3$, equipped with the scalar product $\langle X,Y\rangle=(1/2)\operatorname{Tr}(XY)$, so that

$$e_1=\begin{pmatrix}0&-\sqrt{-1}\\ \sqrt{-1}&0\end{pmatrix},\qquad e_2=\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad e_3=\begin{pmatrix}\sqrt{-1}&0\\0&-\sqrt{-1}\end{pmatrix}\tag{4.3}$$

form a pseudo-orthonormal basis of $(\mathfrak{su}(1,1),\langle,\rangle)$.

Note that the anti-de Sitter 3-space $H_1^3(1)$ is identified with SU(1,1) under the map

$$\begin{aligned}&\psi:H_1^3(1)\to\mathrm{SU}(1,1),\\&x=\left(x^1,x^2,x^3,x^4\right)\quad\mapsto\quad A_x=\sqrt{-1}\begin{pmatrix}\bar z_2&-z_1\\ \bar z_1&-z_2\end{pmatrix}.\end{aligned}\tag{4.4}$$

Moreover, the adjoint representation of SU(1,1) induces a covering homomorphism

$$\rho:\mathrm{SU}(1,1)\to\mathrm{SO}^+(1,2),\tag{4.5}$$

where $\mathrm{SO}^+(1,2)$ is the restricted Lorentz group with signature (1,2), that is, the identity component of the group of linear isometries O(1,2) of $\mathbb{R}_1^3$. Indeed, $\rho(A_x)$ is defined as

$$\rho(A_x):\mathfrak{su}(1,1)\to\mathfrak{su}(1,1),\qquad Y\mapsto\operatorname{Ad}(A_x)Y=A_xYA_x^{-1},$$

and, with respect to the pseudo-orthonormal basis (4.3) of $\mathfrak{su}(1,1)$, the matrix representation of $\rho(A_x)$ is given by

$$\begin{aligned}\rho(A_x)&=\begin{pmatrix}-\operatorname{Re}(z_1^2+\bar z_2^2)&-\operatorname{Im}(z_1^2-\bar z_2^2)&2\operatorname{Re}(z_1\bar z_2)\\-\operatorname{Im}(z_1^2+\bar z_2^2)&\operatorname{Re}(z_1^2-\bar z_2^2)&2\operatorname{Im}(z_1\bar z_2)\\-2\operatorname{Re}(z_1z_2)&-2\operatorname{Im}(z_1z_2)&|z_1|^2+|z_2|^2\end{pmatrix}\\&=\left(A_xe_1A_x^{-1}\quad A_xe_2A_x^{-1}\quad A_xe_3A_x^{-1}\right),\end{aligned}\tag{4.6}$$

from which we easily see that the kernel of ρ is $\{\pm\mathrm{Id}\}$.

The unit tangent bundle $\pi:T^1\mathbb{H}^2(c)\to\mathbb{H}^2(c)$ of the hyperbolic plane $\mathbb{H}^2(c)$ is defined to be

$$\begin{aligned}T^1\mathbb{H}^2(c)&=\left\{(x,v)\in\mathbb{R}_1^3\times\mathbb{R}_1^3\;\middle|\;x\in\mathbb{H}^2(c),\ v\in T_x\mathbb{H}^2(c),\ |v|=1\right\}\\&=\left\{(x,v)\in\mathbb{R}_1^3\times\mathbb{R}_1^3\;\middle|\;\langle x,x\rangle=-1/c,\ \langle v,v\rangle=1,\ \langle x,v\rangle=0\right\}\end{aligned}$$

with the canonical projection $\pi(x,v)=x$. As in the spherical case in Section 2, we may identify $T^1\mathbb{H}^2(c)$ with $\mathrm{SO}^+(1,2)$ by the diffeomorphism

$$\phi:\mathrm{SO}^+(1,2)\to T^1\mathbb{H}^2(c),\quad(c_1\ c_2\ c_3)\mapsto(c_3/\sqrt{c},c_1).\tag{4.7}$$

Finally, let ι be the homothety defined by

$$\iota:H_1^3(c/4)\to H_1^3(1),\quad 2x/\sqrt{c}\mapsto x.$$

Then, it is immediate from (4.1) through (4.7) that the composition of the covering map

$$F = \phi \circ \rho \circ \psi \circ \iota : H_1^3(c/4) \to T^1\mathbb{H}^2(c) \tag{4.8}$$

with the canonical projection $\pi : T^1\mathbb{H}^2(c) \to \mathbb{H}^2(c)$ yields the hyperbolic Hopf map $H : H_1^3(c/4) \to \mathbb{H}^2(c)$ of (4.1). Indeed, for each $2x/\sqrt{c} \in H_1^3(c/4)$ we have

$$F(2x/\sqrt{c}) = (\tilde{x}, e) \in T^1\mathbb{H}^2(c), \tag{4.9}$$

where $\tilde{x} = (1/\sqrt{c})A_x e_3 A_x^{-1}$ and $e = A_x e_1 A_x^{-1}$, so that

$$\pi \circ F(2x/\sqrt{c}) = (1/\sqrt{c})\big(2z_1\bar{z}_2, |z_1|^2 + |z_2|^2\big) = H(z).$$

4.2. Differentials of maps

The differentials of maps involved in (4.8) can be computed in the same way as in Section 3.1, so we only remark on the following.

(1) Given $x \in H_1^3(1)$, the differential of ψ in (4.4)

$$d\psi_x : T_x H_1^3(1) \to T_{\psi(x)}\big(\mathrm{SU}(1,1)\big) = A_x \cdot \mathfrak{su}(1,1)$$

is given by

$$d\psi_x\big(X_3(x)\big) = A_x e_3, \qquad d\psi_x\big(X_2(x)\big) = A_x e_2, \qquad d\psi_x\big(X_1(x)\big) = A_x e_1. \tag{4.10}$$

(2) The differential of ρ in (4.5)

$$d\rho_{A_x} : T_{A_x}\big(\mathrm{SU}(1,1)\big) = A_x \cdot \mathfrak{su}(1,1) \to T_{\rho(A_x)}\mathrm{SO}^+(1,2) = \rho(A_x) \cdot \mathfrak{so}(1,2)$$

is a linear map sending

$$A_x Y \mapsto d\rho_{A_x}(A_x Y) = \rho(A_x) \circ \mathrm{ad}(Y),$$

so that we have

$$\begin{aligned} d\rho_{A_x}(A_x e_3) &= \big(2A_x e_2 A_x^{-1} \quad -2A_x e_1 A_x^{-1} \quad 0\big), \\ d\rho_{A_x}(A_x e_2) &= \big(2A_x e_3 A_x^{-1} \quad 0 \quad 2A_x e_1 A_x^{-1}\big), \\ d\rho_{A_x}(A_x e_1) &= \big(0 \quad -2A_x e_3 A_x^{-1} \quad -2A_x e_2 A_x^{-1}\big), \end{aligned} \tag{4.11}$$

since $\mathrm{ad}(e_1)(e_1) = 0$, $\mathrm{ad}(e_1)(e_2) = -2e_3$, $\mathrm{ad}(e_1)(e_3) = -2e_2$, $\mathrm{ad}(e_2)(e_3) = 2e_1$ for the pseudo-orthonormal basis (4.3) of $\mathfrak{su}(1,1)$.

(3) Combining (4.10) with (4.11) and taking into account the differentials of the diffeomorphism ϕ and the homothety ι, we find that the differential of F in (4.8)

$$dF_x : T_x H_1^3(c/4) \to T_{F(x)}\big(T^1\mathbb{H}^2(c)\big)$$

is determined by

$$\begin{aligned} dF_x\big(2X_3(x)/\sqrt{c}\big) &= \big(0, 2A_x e_2 A_x^{-1}\big) = \tilde{e}_3, \\ dF_x\big(2X_2(x)/\sqrt{c}\big) &= \big(2A_x e_1 A_x^{-1}/\sqrt{c},\ 2A_x e_3 A_x^{-1}\big) = \tilde{e}_2, \\ dF_x\big(2X_1(x)/\sqrt{c}\big) &= \big(-2A_x e_2 A_x^{-1}/\sqrt{c},\ 0\big) = \tilde{e}_1 \end{aligned} \tag{4.12}$$

for each $x \in H_1^3(c/4)$.

4.3. Lifts to the unit tangent bundle

Recall that the unit tangent bundle $T^1\mathbb{H}^2(c)$ is a 3-dimensional hypersurface of $T\mathbb{H}^2(c)$. As in the spherical case in Section 3.2, denoting by X^h (resp. Y^v) the horizontal (resp. vertical) lift of X (resp. Y), we see that at $(x,e) \in T^1\mathbb{H}^2(c)$ the tangent space of the tangent bundle $T\mathbb{H}^2(c)$ is written as

$$T_{(x,e)}\big(T\mathbb{H}^2(c)\big) = \big\{X^h + Y^v \;\big|\; X, Y \in T_x\mathbb{H}^2(c)\big\},$$

whereas that of the unit tangent bundle $T^1\mathbb{H}^2(c)$ is given by

$$T_{(x,e)}\big(T^1\mathbb{H}^2(c)\big) = \big\{X^h + Y^v \;\big|\; X, Y \in T_x\mathbb{H}^2(c),\ \langle Y, e\rangle = 0\big\}.$$

Recalling (4.9), we set

$$e = A_x e_1 A_x^{-1}, \qquad f = -A_x e_2 A_x^{-1},$$

and $\tilde{x} = (1/\sqrt{c})A_x e_3 A_x^{-1}$. Then $(\tilde{x}, f) \in T^1\mathbb{H}^2(c)$ and $\langle f, e\rangle = 0$, so that

$$T_{(\tilde{x},e)}\big(T^1\mathbb{H}^2(c)\big) = \mathrm{Span}\big\{e^h, f^h, f^v\big\}.$$

Furthermore, we have the following

Proposition 4.1. *Let $\tilde{x}$, e and f be as above. Then*

$$(\sqrt{c}/2)\tilde{e}_2 = e^h, \qquad (\sqrt{c}/2)\tilde{e}_1 = f^h, \qquad \tilde{e}_3 = -2f^v. \tag{4.13}$$

Proof. This can be seen in the same manner as in the proof of Proposition 3.1, so we only remark on the following for the sake of completeness.

For the horizontal lift e^h, we consider a geodesic $\gamma : I \to \mathbb{H}^2(c)$ starting from $\tilde{x} \in \mathbb{H}^2(c)$ with initial vector $e \in T^1_{\tilde{x}}\mathbb{H}^2(c)$. Then the curve $\Gamma : I \to T\mathbb{H}^2(c)$ given by $\Gamma(t) = (\gamma(t), v(t) = \dot{\gamma}(t))$ satisfies that $\Gamma(0) = (\tilde{x}, e)$ and $\nabla_{\dot{\gamma}(t)} v(t) = 0$ for all $t \in I$. Since

$$\gamma(t) = \cosh(\sqrt{c}t)\tilde{x} + \sinh(\sqrt{c}t)(e/\sqrt{c}),$$

we deduce that

$$e^h = \dot{\Gamma}(0) = (e, c\tilde{x}) = (\sqrt{c}/2)\tilde{e}_2.$$

Similarly, for f^h, we take a geodesic $\gamma : I \to \mathbb{H}^2(c)$ defined by

$$\gamma(t) = \cosh(\sqrt{c}t)\tilde{x} + \sinh(\sqrt{c}t)(f/\sqrt{c}),$$

starting from $\tilde{x} \in \mathbb{H}^2(c)$ with initial vector $f \in T^1_{\tilde{x}}\mathbb{H}^2(c)$. Then the curve $\Gamma : I \to T\mathbb{H}^2(c)$ given by $\Gamma(t) = (\gamma(t), v(t) = e)$ satisfies that $\Gamma(0) = (\tilde{x}, e)$ and $\nabla_{\dot{\gamma}(t)} v(t) = 0$ for all $t \in I$. Hence

$$f^h = \dot{\Gamma}(0) = (f, 0) = (\sqrt{c}/2)\tilde{e}_1.$$

To construct the vertical lift f^v, we now consider a curve $\gamma : I \to T\mathbb{H}^2(c)$ defined by $\gamma(t) = (\tilde{x}, (\cos t)e + (\sin t)f)$. Then $\gamma(t)$ is a curve along the fibre over $\tilde{x}$ and satisfies $\gamma(0) = (\tilde{x}, e)$ and $\dot{\gamma}(0) = (0, f)$. Hence $\tilde{e}_3 = (0, -2f) \in \mathcal{V}_{(\tilde{x},e)} \subset T_{(\tilde{x},e)}(T^1\mathbb{H}^2(c)) \subset T_{(\tilde{x},e)}(T\mathbb{H}^2(c))$. Moreover, for the connection map we have

$$\begin{aligned} K_{(\tilde{x},e)}(-\tilde{e}_3/2) &= \frac{d}{dt}\bigg|_{t=0} (\exp_{\tilde{x}} \circ R_{-e} \circ \tau)\big(\gamma(t)\big) \\ &= \frac{d}{dt}\bigg|_{t=0} \exp_{\tilde{x}}\big((\cos t - 1)e + (\sin t)f\big). \end{aligned}$$

Noting that the geodesic of $\mathbb{H}^2(c)$ starting from $\tilde{x}$ with unit initial vector v is given by $\delta_{(\tilde{x},v)}(s) = \cosh(\sqrt{c}s)\tilde{x} + \sinh(\sqrt{c}s)(v/\sqrt{c})$, we then see

$$\begin{aligned} &\exp_{\tilde{x}}\big((\cos t - 1)e + (\sin t)f\big) \\ &\quad = \cosh\big(\sqrt{c}\,\theta(t)\big)\tilde{x} + \frac{\sinh(\sqrt{c}\,\theta(t))}{\sqrt{c}}\left(\frac{(\cos t - 1)e + (\sin t)f}{\theta(t)}\right), \end{aligned}$$

where

$$\theta(t) = \big|(\cos t - 1)e + (\sin t)f\big|_{\mathbb{R}^3_1} = \sqrt{2(1 - \cos t)}.$$

Therefore we obtain

$$K_{(\tilde{x},e)}(-\tilde{e}_3/2) = \frac{d}{dt}\bigg|_{t=0} \exp_{\tilde{x}}\big((\cos t - 1)e + (\sin t)f\big) = f,$$

which shows that $\tilde{e}_3 = -2f^v$. □

4.4. Indefinite generalized Cheeger–Gromoll metrics

We extend the notion of the generalized Cheeger–Gromoll metric $h_{m,r}$ defined in Section 3.3 to admit indefinite ones.

More specifically, for the hyperbolic plane $\mathbb{H}^2(c)$, we define on its tangent bundle $T\mathbb{H}^2(c)$ the *indefinite generalized Cheeger–Gromoll metric* $h_{m,r}$ as follows. Given $m \in \mathbb{R}$ and $r \geqslant 0$, we set on each tangent space $T_{(x,e)}(T\mathbb{H}^2(c))$

$$\begin{aligned} &h_{m,r}\big(X^h, Y^h\big) = \langle X, Y\rangle, \qquad h_{m,r}\big(X^h, Y^v\big) = 0, \\ &h_{m,r}\big(X^v, Y^v\big) = -\omega^m\big(\langle X, Y\rangle + r\langle X, e\rangle\langle Y, e\rangle\big), \end{aligned} \tag{4.14}$$

where $X, Y \in T_x\mathbb{H}^2(c)$ and $\omega = 1/(1+|e|^2)$. It should be noted that, equipped with $h_{m,r}$ on $T\mathbb{H}^2(c)$ and the canonical metric $\langle,\rangle$ on $\mathbb{H}^2(c)$, the canonical projection $\pi : T\mathbb{H}^2(c) \to \mathbb{H}^2(c)$ yields a submersion which is isometric on horizontal directions. Moreover, when $(x, e) \in T^1\mathbb{H}^2(c)$, this metric restricts on $T_{(x,e)}(T^1\mathbb{H}^2(c))$ to

$$\begin{aligned} &h_{m,r}\big(X^h, Y^h\big) = \langle X, Y\rangle, \qquad h_{m,r}\big(X^h, Y^v\big) = 0, \\ &h_{m,r}\big(X^v, Y^v\big) = -\frac{1}{2^m}\langle X, Y\rangle. \end{aligned} \tag{4.15}$$

Note that the parameter r disappears when restricted to the unit tangent bundle, and $h_{m,r}$ has a negative signature on vertical directions.

With these understood, the proof of Theorem 1.2 is immediate. Indeed, if we choose $m = \log_2 c$, then, it follows from (4.12) and (4.13) together with (4.15) that

$$\begin{aligned} &h_{m,r}\big((\sqrt{c}/2)\tilde{e}_1, (\sqrt{c}/2)\tilde{e}_1\big) = h_{m,r}\big(f^h, f^h\big) = \langle f, f\rangle = 1, \\ &h_{m,r}\big((\sqrt{c}/2)\tilde{e}_2, (\sqrt{c}/2)\tilde{e}_2\big) = h_{m,r}\big(e^h, e^h\big) = \langle e, e\rangle = 1, \\ &h_{m,r}\big((\sqrt{c}/2)\tilde{e}_1, (\sqrt{c}/2)\tilde{e}_2\big) = h_{m,r}\big(f^h, e^h\big) = \langle f, e\rangle = 0, \\ &h_{m,r}\big((\sqrt{c}/2)\tilde{e}_2, (\sqrt{c}/2)\tilde{e}_3\big) = -h_{m,r}\big(e^h, \sqrt{c}f^v\big) = 0, \\ &h_{m,r}\big((\sqrt{c}/2)\tilde{e}_1, (\sqrt{c}/2)\tilde{e}_3\big) = -h_{m,r}\big(f^h, \sqrt{c}f^v\big) = 0, \end{aligned}$$

and

$$h_{m,r}\big((\sqrt{c}/2)\tilde{e}_3, (\sqrt{c}/2)\tilde{e}_3\big) = h_{m,r}\big(-\sqrt{c}f^v, -\sqrt{c}f^v\big) = -\frac{c}{2^m}\langle f, f\rangle = -1.$$

Consequently, the covering map $F : H_1^3(c/4) \to T^1\mathbb{H}^2(c)$ defined by (4.8) gives rise to an isometric immersion from $(H_1^3(c/4), g_{\text{can}})$ to $(T^1\mathbb{H}^2(c), h_{m,r})$ for $m = \log_2 c$ and $r \geqslant 0$.

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