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An extended *GS* method for dense linear systems

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ABSTRACT

Davey and Rosindale [K. Davey, I. Rosindale, An iterative solution scheme for systems of boundary element equations, Internat. J. Numer. Methods Engrg. 37 (1994) 1399–1411] derived the GSOR method, which uses an upper triangular matrix Ω in order to solve dense linear systems. By applying functional analysis, the authors presented an expression for the optimum Ω. Moreover, Davey and Bounds [K. Davey, S. Bounds, A generalized SOR method for dense linear systems of boundary element equations, SIAM J. Comput. 19 (1998) 953–967] also introduced further interesting results. In this note, we employ a matrix analysis approach to investigate these schemes, and derive theorems that compare these schemes with existing preconditioners for dense linear systems. We show that the convergence rate of the Gauss–Seidel method with preconditioner P_G is superior to that of the GSOR method. Moreover, we define some splittings associated with the iterative schemes. Some numerical examples are reported to confirm the theoretical analysis. We show that the EGS method with preconditioner $P_G(\gamma_{\text{opt}})$ produces an extremely small spectral radius in comparison with the other schemes considered.

1. Introduction

In this study, we consider the solution of the dense linear system

$$A\boldsymbol{x} = \boldsymbol{b},$$

where $A = (a_{ij}) \in R^{n\times n}$ is an M-matrix, and $\boldsymbol{x}, \boldsymbol{b} \in R^n$ are vectors. Without loss of generality, we assume that A has a splitting of the form $A = I - L - U$, where I denotes the $n \times n$ identity, and $-L$ and $-U$ are the strictly lower, and strictly upper triangular parts of A, respectively. The SOR method is defined by

$$\boldsymbol{x}^{(k+1)} = (I - \omega L)^{-1}((1-\omega)I + \omega U)\boldsymbol{x}^{(k)} + \omega(I - \omega L)^{-1}\boldsymbol{b}, \tag{1.1}$$

and is derived from the following equations:

$$\tilde{\boldsymbol{x}}^{(k+1)} = \boldsymbol{b} + L\boldsymbol{x}^{(k+1)} + U\boldsymbol{x}^{(k)}, \tag{1.2}$$

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} + \omega(\tilde{\boldsymbol{x}}^{(k+1)} - \boldsymbol{x}^{(k)}), \tag{1.3}$$

where ω is an over-relaxation parameter (an acceleration parameter) $(1 \le \omega < 2)$ and $\tilde{\boldsymbol{x}}^{(k+1)}$ is the auxiliary vector. It is well known that the SOR method with the optimum parameter ω is an effective iterative method. However, the optimum parameter cannot be obtained *a priori* except for special cases. In 1973, James [6] proposed the generalized SOR (GSOR) method with the accelerated matrix $\Omega = (\omega_{ij})$ constructed by $\omega_{ii} > 0$ and $\omega_{ij} = 0$ for $j \ne i$ as

$$\boldsymbol{x}^{(k+1)} = (I - \Omega L)^{-1}(I - \Omega + \Omega U)\boldsymbol{x}^{(k)} + (I - \Omega L)^{-1}\boldsymbol{b}, \tag{1.4}$$

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and also gave a convergence condition for this method. However, the author was only able to show by numerical computation that optimal parameters ω_{ii} exist, and did not describe a means of estimating these optimal parameters. In practice, the problem of estimating the optimal ω_{ii} is difficult. In 1994, Davey and Rosindale [2] proposed a straightforward means of estimating the optimum Ω using an upper triangular form of matrix Ω. The elements of $\Omega = (\omega_{ij})$ are constructed by $\omega_{ii} = 1$, $\omega_{ii+1} > 0$ $(i = 1, \dots, n-1)$ and $\omega_{ij} = 0$ otherwise. The optimum parameters ω_{ii+1} are given by

$$\omega_{ii+1} = \frac{-(a_{ii+1} + \sum_{j=i+2}^{n} a_{i+1j}a_{ij})}{1 + \sum_{j=i+2}^{n} a_{i+1j}^2}, \quad i = 1, 2, \dots, n-1. \tag{1.5}$$

(Further details may be found in [2]). We consider Davey et al.'s estimate of the optimal ω_{ii+1} to be reasonable, and call this scheme the extended GSOR (ESOR) method, because Ω is an upper triangular matrix. In order to analyze other schemes in the next section, it is convenient to express the preconditioner corresponding to Eq. (1.5) as $P_{S'} = I + S'$, where

$$S' = (s'_{ij}) = \begin{cases} \omega_{ii+1}, & i = 1, 2, \dots, n-2,\ j = i+1, \\ -a_{ii+1}, & i = n-1,\ j = n, \\ 0, & \text{otherwise.} \end{cases}$$

In an alternative approach, in order to improve the rate of convergence of the Gauss–Seidel method, Gunawardena et al. [5] proposed the algorithm given by applying the Gauss–Seidel method to the preconditioned linear system $P_S A\boldsymbol{x} = P_S\boldsymbol{b}$, where $P_S = I + S$, and S is defined by

$$S = (s_{ij}) = \begin{cases} -a_{ii+1}, & i = 1, 2, \dots, n-1,\ j = i+1, \\ 0, & \text{otherwise.} \end{cases}$$

The preconditioned matrix $A_S = (I + S)A = M_S - N_S$ can then be written as

$$\begin{aligned} A_S &= (a^S_{ij}) = I - L - SL - U + S - SU \\ &= (I - D) - (L + E) - (U - S + SU), \end{aligned} \tag{1.6}$$

where D and E are the diagonal and strictly lower triangular parts of SL, respectively. Whenever $a_{ii+1}a_{i+1i} \neq 1\,(1 \le i \le n-1)$, $M_S^{-1} = (I - D) - (L + E)^{-1}$ exists. The preconditioned Gauss–Seidel iterative matrix for A_S therefore becomes

$$T_S = \{(I - D) - (L + E)\}^{-1}(U - S + SU),$$

which is referred to as the modified Gauss–Seidel iterative matrix. From a matrix analysis perspective, they proved the following inequality [1]:

$$\rho(T_S) \le \rho(T) < 1,$$

where $\rho(T)$ denotes the spectral radius of the Gauss–Seidel iterative matrix T for A.

We use the notation $A \ge B$ for two real matrices $A = (a_{ij}),\ B = (b_{ij})$ of equal order if each entry of the difference $A - B$ is nonnegative, that is, $a_{ij} \ge b_{ij}$ for all i, j. We write $A > B$ if each entry of the difference is positive. A matrix $A \ge O$ $(A > O)$ is called nonnegative (positive). We define $|A| = (|a_{ij}|)$. Since $S' \le S$, it easily follows that $\rho(T_{S'}) \ge \rho(T_S)$, where $T_{S'}$ is Gauss–Seidel matrix for $(I + S')A$. By using matrix analysis, a comparison theorem for the spectral radii of the iterative schemes is easily obtained. Note that P_S has the same form as Ω_B^2 [5]. A suitable choice of the preconditioner is therefore important, because it has a strong influence on the convergence behavior of the iterative method. As a consequence, many alternative preconditioners have been proposed [5,7,8,10,11,15,18]. In the present study we will consider the properties of the principal examples of these preconditioners. The remainder of the present paper is organized as follows: Section 2 summarizes existing preconditioners. Section 3 discusses a reordering scheme, before defining some splittings associated with the iterative schemes. Finally, we present and prove convergence and comparison theorems. In Section 4, we report several numerical examples that demonstrate the validity of our analysis. The last section is devoted to some concluding remarks.

2. Existing preconditioners

In 1996, Kotakemori et al. [11] proposed $P_m = I + S_m$, where S_m is defined by

$$S_m = (s^{(m)}_{ij}) = \begin{cases} -a_{ik_i}, & 1 \le i < n,\ i+1 \le j \le n, \\ 0, & \text{otherwise,} \end{cases}$$

and where $k_i = \min I_i$, $I_i = \{j\ :\ |a_{ij}| \text{ is maximal for } i+1 \le j \le n\}$, for $1 \le i < n$. Then $A_m = (I + S_m)A$ can be expressed in the form

$$A_m = I - L - U + S_m - S_mL - S_mU = M_m - N_m, \tag{2.7}$$

where $M_m = (I - D_m) - (L + E_m)$, $N_m = U - S_m + F_m + S_mU$ and D_m, E_m and F_m are the diagonal, strictly lower and strictly upper triangular parts of S_mL, respectively. If M_m is nonsingular, then the Gauss–Seidel iterative matrix is defined by

$$T_m = M_m^{-1}N_m = \{(I - D_m) - (L + E_m)\}^{-1}(U - S_m + F_m + S_mU).$$

Under the following condition

$$a_{ii+1}a_{i+1j} \le a_{ik_i}a_{k_ij}, \quad 1 \le i \le n-2,\ j \le i, \tag{2.8}$$

$M_m^{-1} \ge M_S^{-1}$ holds. Moreover, assume that $(A_m - A_S)\boldsymbol{x} \ge 0$, $A_S\boldsymbol{x} \ge 0$, where $\boldsymbol{x}$ is the eigenvector of T_S. Then, from Theorem 3.7 ([13], Theorem 3.15) it follows that $\rho(T_m) \le \rho(T_S)$. Note that the proof in [8] is insufficient since it has not supposed that $(A_m - A_S)\boldsymbol{x} \ge 0$.

Note 2.1. *$P_m = I + S_m$ has the same form as $\Omega_L^{(2)}$ proposed by Davey and Rosindale [2].*

To remove the need for these hypotheses, Morimoto et al. [15] has proposed the following preconditioner,

$$P_{s+m} = (a_{ij}^{(s+m)}) = I + S + S_m.$$

In this preconditioner, S_m is defined by

$$S_m = (s_{ij}^{(m)}) = \begin{cases} -a_{il_i}, & 1 \le i < n-1,\ i+1 < j \le n, \\ 0, & \text{otherwise}, \end{cases}$$

where $l_i = \min I_i$, $I_i = \{j : |a_{ij}| \text{ is maximal for } i+1 < j \le n\}$, for $1 \le i < n-1$. The preconditioned matrix $A_{s+m} = (I + S + S_m)A$ can then be written as

$$\begin{aligned} A_{s+m} &= I - L - U + S - SL - SU + S_m - S_mL - S_mU \\ &= \{(I - D_{s+m}) - (L + E_{s+m})\} - (U - S - S_m + SU + S_mU + F_{s+m}), \end{aligned}$$

where D_{s+m}, E_{s+m} and F_{s+m} are the diagonal, strictly lower and strictly upper triangular parts of $(S + S_m)L$, respectively. Assume that the following inequalities (A) are satisfied:

$$\text{(A)} \quad \begin{cases} 0 < -a_{ii+1}a_{i+1i} + a_{il_i}a_{l_ii} < 1, & 1 \le i < n-1, \\ 0 < a_{ii+1}a_{i+1i} < 1, & i = n-1. \end{cases}$$

Then $\{(I - D_{s+m}) - (L + E_{s+m})\}$ is nonsingular. The preconditioned Gauss–Seidel iterative matrix T_{s+m} for A_{s+m} is then defined by

$$T_{s+m} = \{(I - D_{s+m}) - (L + E_{s+m})\}^{-1}(U - S - S_m + SU + S_mU + F_{s+m}).$$

By applying Theorem 3.7 ([13], Theorem 3.15), Morimoto et al. [15] proved that $\rho(T_{s+m}) \le \rho(T_S)$ and $\rho(T_{s+m}) \le \rho(T_m)$.

In order to solve the linear systems arising from applications of the boundary element method, Kotakemori et al. [12] proposed two-step preconditioning matrices $I + \alpha S$ and $I + BU$, where α is a positive number and B is a nonnegative diagonal matrix.

Since these preconditioners are constructed from part of the upper triangular part of A, the preconditioning does not act on the last row of matrix A. We shall call these preconditioners 'upper' preconditioners.

To extend the preconditioning effect to the last row, Morimoto et al. [14] proposed the preconditioner

$$P_R = I + R,$$

where R is defined by

$$R = (r_{nj}) = \begin{cases} -a_{nj}, & 1 \le j \le n-1, \\ 0, & \text{otherwise}. \end{cases}$$

The elements a_{nj}^R of A_R are given by

$$A_R = (I + R)A = (a_{ij}^R), \quad a_{ij}^R = \begin{cases} a_{ij}, & 1 \le i < n, 1 \le j \le n, \\ a_{nj} - \displaystyle\sum_{k=1}^{n-1} a_{nk}a_{kj}, & 1 \le j \le n. \end{cases} \tag{2.9}$$

And they proved that $\rho(T_R) \le \rho(T)$ holds, where T_R is the iterative matrix for A_R. Morimoto et al. [14] also presented combined preconditioners, which are given by combinations of R with any upper preconditioner, and they showed that the convergence rate of the combined methods are better than those of the Gauss–Seidel method applied with other upper preconditioners. In [16], Niki et al. considered the preconditioner $P_{SR} = (I + S + R)$. Denote $A_{SR} = M_{SR} - N_{SR}$. In [17], Niki et al. proved that if the following inequality is satisfied,

$$a_{nj} \le a_{nj}^R = a_{nj} - \sum_{k=1,k=\ne j}^{n-1} a_{nk}a_{kj}, \quad j = 1, 2, \ldots, n-1, \tag{2.10}$$

$\rho(T_{SR}) \leq \rho(T_S)$ holds, where T_{SR} is the iterative matrix for A_{SR}. For matrices that do not satisfy Eq. (2.10), by putting $R = (r_{nj}) = -\sum_{k=1,\neq j}^{n-1} a_{nk}a_{kj} - a_{nj}$, $1 \leq j < n$, Eq. (2.10) is satisfied. Therefore, Niki et al. [17] proposed a new preconditioner $P_G = I + \gamma G (\gamma \geq 1)$ where

$$G = (g_{nj}) = \begin{cases} -\displaystyle\sum_{k=1,k\neq j}^{n-1} a_{nk}a_{kj} - a_{nj}, & 1 \leq j \leq n-1, \\ 0, & \text{otherwise.} \end{cases}$$

Put $A_G = P_G A = (a_{ij}^G) = M_G - N_G$, and $T_G = M_G^{-1} N_G$.

Upon resetting P_G to $P_G = I + S + S_m + \gamma G$, and setting $\gamma = 1$, the Gauss–Seidel splitting of A_G can be written as

$$A_G = \{(I - D_{s+m}) - (L + E_{s+m}) - (G(L+U) - G)\} - (U - S - S_m + SU + S_m U + F_{s+m}), \tag{2.11}$$

where $G(L+U) - G$ is constructed by the elements $a_{nj}^G = g_{nj}a_{jn}$. Thus if the preconditioner P_G is used, then all of the rows of A are subject to preconditioning. Niki et al. [17] proved that under the condition $\gamma_u > \gamma$, $\rho(T_R) \geq \rho(T_G)$, where γ_u is the upper bound of those values of γ for which $\rho(T_G) < 1$. By setting $a_{nj}^G = 0$, we have

$$\gamma = a_{nj} \Big/ \sum_{k=1,k\neq j}^{n-1} g_{nk}a_{kj}. \tag{2.12}$$

Niki et al. [17] proved that the preconditioner P_G satisfies Eq. (2.10) unconditionally. Moreover, they reported that the convergence rate of the Gauss–Seidel method using preconditioner P_G is better than that of the SOR method using the optimum ω found by numerical computation, and reported that there is an optimum $\gamma(\gamma_{\text{opt}})$ in the range $\gamma_u > \gamma_{\text{opt}} > \gamma_m$, that produced an extremely small $\rho(T_{\gamma_{\text{opt}}})$, where γ_m is the upper bound of the values of γ for which $a_{nj}^G \geq 0$, for all j. If there are non-diagonally dominant columns, it may be the case that diagonal elements satisfying $a_{nn}^G < 0$ may appear before γ_m is obtained. In this case, γ_{opt} exists in the range $\gamma_m > \gamma_{\text{opt}} > \gamma = 1$. In the next section, we clarify that the structure of the Gauss–Seidel method with the existing preconditioner described above corresponds to that of the accelerated Gauss–Seidel (AGS) method. It will also be shown that the AGS method can obtain a better rate of convergence than that of the GSOR scheme using the optimum Ω proposed by Davey and Rosindale [2] theoretically.

3. The accelerated GS method

3.1. The reordering

If Ω is restricted to an upper triangular matrix, then the ESOR method is expressed as $(I + (\Omega A)_{sl})\boldsymbol{x}^{(k+1)} = (I - (\Omega A)_u)\boldsymbol{x}^{(k)} + \Omega \boldsymbol{b}$, where $(\Omega A)_u$ and $(\Omega A)_{sl}$ are matrices consisting of upper triangular and strictly lower triangular parts of ΩA, respectively. Davey et al. proposed the use of a reordering scheme before performing the iterative method [2,3]. We summarize the reordering scheme in this section. For the sake of brevity, in the explanation of scheme, we shall assume that A is an $\boldsymbol{M}$-matrix, namely, that the maximum term is located on the diagonal. First, the vector 1-norm is calculated for each row, and these values are compared. The row with the minimum vector 1-norm is re-assigned to be the first row. The procedure continues by calculating a new value of the norm for each row, excluding the first row and column. For the reduced matrix with $n-1$ rows and columns, the vector 1-norm values are again compared, with the row with the minimum of these values assigned to be the second row. This procedure is continued in a similar manner for the remaining rows. It should be recognized that the procedure may not give rise to smallest value of $\|U\|_\infty$ that is achievable. By using matrix notation, this scheme can be written as

$$Q_k \ldots Q_1 \Omega A Q_1 \ldots Q_k,$$

where Q_l $(1 \leq l \leq k)$ is a permutation matrix. The basic idea of this scheme is to use row and column ordering to place the smaller terms of A in U so as to minimize either the Frobenius norm $\|U\|_F$ or the ∞ norm $\|U\|_\infty$. Hence performing this scheme can improve the rate of convergence. By applying P_S to the following example A, we have:

$$A = \begin{pmatrix} 1 & -0.5 & -0.2 & -0.1 \\ -0.3 & 1 & -0.2 & -0.3 \\ -0.2 & -0.2 & 1 & -0.1 \\ -0.4 & -0.3 & -0.3 & 1 \end{pmatrix}, \quad P_S A = \begin{pmatrix} 0.85 & 0 & -0.3 & -0.25 \\ -0.34 & 0.96 & 0 & -0.32 \\ -0.24 & -0.23 & 0.97 & 0 \\ -0.4 & -0.3 & -0.3 & 1 \end{pmatrix}.$$

Then $\rho(T_S) = 0.4245$, and $\rho(T) = 0.6125$. By performing the reordering scheme of [3], the following matrix is produced:

$$A^r = \begin{pmatrix} 1 & -0.2 & -0.2 & -0.1 \\ -0.2 & 1 & -0.3 & -0.3 \\ -0.2 & -0.5 & 1 & -0.1 \\ -0.3 & -0.3 & -0.4 & 1 \end{pmatrix},$$

and we have $\rho(T_S^r) = 0.4284$. On the other hand, $\rho(T^r) = 0.5827$. Next, we try to perform the reordering so that $|a_{ii+1}| = \max_{j>i} |a_{ij}|$. Then the following matrix is produced:

$$A^{r_m} = \begin{pmatrix} 1 & -0.5 & -0.1 & -0.2 \\ -0.3 & 1 & -0.3 & -0.2 \\ -0.4 & -0.3 & 1 & -0.3 \\ -0.2 & -0.2 & -0.1 & 1 \end{pmatrix},$$

and $\rho(T_S^{r_m}) = 0.386$ and $\rho(T^{r_m}) = 0.6210$. As can be seen in the above results, $\rho(T_S^{r_m}) < \rho(T_S) < \rho(T_S^r)$. If possible, the matrix should be reordered so that $|a_{ii+1}| = \max |a_{ij}|, j > i$. Accordingly, there exists an appropriate reordering corresponding to the preconditioner Ω. Moreover, we know that when the Gauss–Seidel method is directly applied to a matrix obtained by the reordering scheme [3], this reordering scheme is effective. Note that for the Jacobi iterative method, the reordering effect is not obtained.

3.2. The comparison theorems

We now review some known results used in this section.

Definition 3.1. The real $n \times n$ matrix, called the comparison matrix, defined by

$$\langle A \rangle = (m_{ij}) = \begin{cases} |a_{ii}|, & i = j, \\ -|a_{ij}|, & i \neq j. \end{cases}$$

Definition 3.2 (*Axelsson [1], Definition 6.3*). A square matrix A is said to be generalized diagonally dominant if

$$|a_{ii}|x_i \geq \sum_{j \neq i}^{n} |a_{ij}|x_j, \quad i = 1, 2, \ldots, n, \tag{3.13}$$

for some positive vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ and generalized strictly diagonally dominant if (3.13) is valid with strict inequality.

Definition 3.3 (*Frommer and Szyld [4], Definition 3.3*). Let A be a real matrix. The representation $A = M - N$ is called

(1) regular if $M^{-1} \geq O$ and $N \geq O$,
(2) weak regular if $M^{-1} \geq O$ and $M^{-1}N \geq O$,
(3) an M-splitting if $M$is an $\boldsymbol{M}$-matrix and $N \geq O$,
(4) an H- splitting if $\langle M \rangle - |N|$ is an $\boldsymbol{M}$-matrix,
(5) an H-compatible splitting if $\langle A \rangle = \langle M \rangle - |N|$.

Lemma 3.4. *Let A be an $\boldsymbol{M}$-matrix with unit diagonal. Let $A = M - N$ be the Gauss–Seidel convergent regular splitting with $T = M^{-1}N$. Then there exists a nonzero eigenvector $\boldsymbol{x} > 0$ such that $T\boldsymbol{x} = \rho(T)\boldsymbol{x}$. We then have $A\boldsymbol{x} \geq 0$.*

Proof. Since A is an $\boldsymbol{M}$-matrix and a nonnegative matrix T (denoted $T \geq O$), there exists a Perron vector $\boldsymbol{x}$ such that $T\boldsymbol{x} = \rho(T)\boldsymbol{x}$, and so we have

$$M\boldsymbol{x} = \frac{1}{\rho(T)} N\boldsymbol{x} \geq 0.$$

We also therefore have the following inequality,

$$A\boldsymbol{x} = (M - N)\boldsymbol{x} = \frac{1 - \rho(T)}{\rho(T)} N\boldsymbol{x} \geq 0. \quad \square$$

Theorem 3.5 (*Frommer and Szyld [4], Theorem 3.4*).

(1) *If the splitting is regular or weak regular, then $\rho(M^{-1}N) < 1$ if and only if $A^{-1} \geq O$.*
(2) *If the splitting is an M-splitting, then $\rho(M^{-1}N) < 1$ if and only if A is an $\boldsymbol{M}$-matrix.*
(3) *If the splitting is an H-splitting, then A and M are $\boldsymbol{H}$-matrices and $\rho(M^{-1}N) \leq \rho(\langle M \rangle^{-1}|N|) < 1$.*
(4) *If the splitting is an M-splitting, then it is a regular splitting.*
(5) *If the splitting is an M-splitting and A is an $\boldsymbol{M}$-matrix, then it is an H-splitting and also an H-compatible splitting.*
(6) *If the splitting is an H-compatible splitting and A is an $\boldsymbol{H}$-matrix, then it is an H-splitting and thus convergent.*

Definition 3.6 (*Niki et al. [16], Definition 1.7*). $A = M - N$ is called the Gauss–Seidel splitting of A if $M = D - E$ and $N = F$, where D is the diagonal parts and $-E$ and $-F$ are strictly lower and upper triangular parts of A, respectively. In addition, the splitting is called

(1) Gauss–Seidel convergent if $\rho(M^{-1}N) < 1$,
(2) Gauss–Seidel regular if $M^{-1} \geq O$ and $N \geq O$,
(3) Gauss–Seidel weak regular if $M^{-1} \geq O$ and $M^{-1}N \geq O$.

Let B be a real Banach space, B' its dual and $L(B)$ the space of all bounded linear operators mapping B into itself. We assume that B is generated by a normal cone K [13]. As is defined in [13], the operator $A \in L(B)$ has the property "d" if its dual, A', possesses a Frobenius eigenvector in the dual cone K' which is defined by

$$K' = \{\boldsymbol{x}' \in B' : \langle \boldsymbol{x}, \boldsymbol{x}' \rangle = \boldsymbol{x}' \geq 0 \text{ for all } \boldsymbol{x} \in K\}.$$

As is remarked in [13,15], when $B = \mathbb{R}^n$ and a generating cone $K = \mathbb{R}^n_+$ (the set of nonnegative vectors), all $n \times n$ matrices have the property "d". Therefore, the case that we discuss fulfills the property "d". For the space of all $n \times n$ matrices, the theorem of Marek and Szyld can be stated as follows:

Theorem 3.7 (*Marek and Szyld [13], Theorem 3.15*). *Let $A_1 = M_1 - N_1$ and $A_2 = M_2 - N_2$ be weak regular splittings with $T_1 = M_1^{-1}N_1$, $T_2 = M_2^{-1}N_2$ having the property "d". Let $\boldsymbol{x} \geq 0$, $\boldsymbol{y} \geq 0$ be such that $T_1\boldsymbol{x} = \rho(T_1)\boldsymbol{x}$ and $T_2\boldsymbol{y} = \rho(T_2)\boldsymbol{y}$. If*

$$M_1^{-1} \geq M_2^{-1},$$

and either $(A_1 - A_2)\boldsymbol{x} \geq 0,\ A_1\boldsymbol{x} \geq 0,$ or $(A_1 - A_2)\boldsymbol{y} \geq 0,\ A_1\boldsymbol{y} \geq 0$ with $\boldsymbol{y} > 0$, then

$$\rho(T_1) \leq \rho(T_2).$$

Moreover, if $M_1^{-1} > M_2^{-1}$ and $N_1 \neq N_2$, then

$$\rho(T_1) < \rho(T_2).$$

Theorem 3.8 (*Varga [21], Theorem 3.36*). *Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of A, where $A^{-1} \geq O$. If $M_2^{-1} \geq M_1^{-1} \geq O$, then*

$$1 > \rho(M_2^{-1}N_2) \geq \rho(M_1^{-1}N_1) \geq 0.$$

If, moreover, $A^{-1} > O$ and if $M_2^{-1} > M_1^{-1} \geq O$, equality excluded, then

$$1 > \rho(M_2^{-1}N_2) > \rho(M_1^{-1}N_1) > 0.$$

Saad [19] shows that an over-relaxation can be based on the splitting:

$$\omega A = \omega I - \omega L - \omega U + (I - I) = (I - \omega L) - ((1 - \omega)I + \omega U),$$

and the corresponding SOR method is given in Eq. (1.1).

We now define some splittings based on this splitting scheme of Saad [19].

Definition 3.9. Let $\omega A = \omega A + (I - I)$. Then $\omega A = (I + (\omega A)_{sl}) - (I - (\omega A)_u)$ is called the SOR splitting, where $(\omega A)_{sl}$ and $(\omega A)_u$ are the strictly lower and upper parts of ωA, respectively. Let $\Omega A = \Omega A + (I - I)$, where $\Omega = (\omega_{ij})$ is the diagonal matrix constructed by $\omega_{ii} > 0$. Then we shall call $\Omega A = (I + (\Omega A)_{sl}) - (I - (\Omega A)_u)$ the GSOR splitting. If Ω is an upper triangular matrix, we shall call ΩA the ESOR splitting.

From the above definition, if $\Omega = I + S$, then the ESOR splitting A_{ESOR} has the form

$$\begin{aligned} A_{ESOR} &= \Omega A = (I + (\Omega A)_{sl}) - (I - (\Omega A)_u) \\ &= (I - (E + L)) - (D + U - S + SU), \end{aligned} \tag{3.14}$$

where D and E are the diagonal and strictly lower parts of SL, respectively. By contrast, Davey and Bounds [3] is not using Eq. (1.3), and from the ESOR splitting the following scheme is derived:

$$\boldsymbol{x}^{(k+1)} = (I - (E' + L))^{-1}(D' + U - S' + S'U)\boldsymbol{x}^{(k)} + (I - (E' + L))^{-1}(I + S')\boldsymbol{b}, \tag{3.15}$$

where D' and E' are the diagonal and strictly lower parts of $S'L$, respectively. We shall call this scheme the extended GSOR (ESOR) method, because Ω is an upper triangular matrix. By substituting $\Omega = I + S$ to ω in Eq. (1.3), we have

$$\boldsymbol{x}^{(k+1)} = (I - D - (E + L))^{-1}(U - S + SU)\boldsymbol{x}^{(k)} + (I - D - (E + L))^{-1}(I + S)\boldsymbol{b}. \tag{3.16}$$

Eq. (3.16) is simply the modified Gauss–Seidel iterative scheme, which is applied to the Gauss–Seidel method by the splitting Eq. (1.6). Eq. (3.16) is of course derived from the Gauss–Seidel splitting not the ESOR splitting. Hereafter, we call this splitting the accelerated Gauss–Seidel (AGS) splitting, and we shall call the Gauss–Seidel method produced from the AGS splitting the AGS method with preconditioner, because this scheme is derived from the accelerated formula Eq. (1.3). Applying $\Omega = I + S'$ to Eq. (1.3), we have

$$\boldsymbol{x}^{(k+1)} = (I - D' - (E' + L))^{-1}(U - S' + S'U)\boldsymbol{x}^{(k)} + (I - D' - (E' + L))^{-1}(I + S')\boldsymbol{b}. \tag{3.17}$$

This scheme is the AGS method with the preconditioner $(I + S')$. From the results above, we can now define the splitting corresponding to the Gauss–Seidel iterative method.

Definition 3.10. Let Ω be an upper triangular matrix, for example, P_S, P_m, P_{s+m}. We shall call the Gauss–Seidel splitting of ΩA the AGS splitting. Let Ω be a nonsingular matrix, for example P_{smr} or P_G. As shown by Eq. (2.11), since all the rows of matrix A are subject to a preconditioning effect, the Gauss–Seidel splitting of ΩA is called the EGS splitting.

Next, let us discuss the comparison theorems for the schemes obtained above.

Theorem 3.11. *Let A be an $\boldsymbol{M}$-matrix. For $\Omega' = I+S'$, put $A'_{ESOR} = M'_{ESOR} - N'_{ESOR}$ and $T'_{ESOR} = (I-(E'+L))^{-1}(D'+U-S'+S'U)$. Let $T\boldsymbol{x} = \rho(T)\boldsymbol{x}$, $\boldsymbol{x} \geq 0$. Then the following inequality holds:*

$$\rho(T) \geq \rho(T'_{ESOR}).$$

Proof. From the assumption, the following inequality holds,

$$M^{-1} = (I-L)^{-1} = I + L + L^2 + \cdots + L^{n-1} \geq O.$$

Since $E' + L \geq O$, we easily obtain

$$(M')^{-1}_{ESOR} = I + (E'+L) + (E'+L)^2 + \cdots + (E'+L)^{n-1} \geq O. \tag{3.18}$$

Clearly $(M')^{-1}_{ESOR} \geq M^{-1} \geq O$. Since $N'_{ESOR} \geq O$, $A'_{ESOR} = M'_{ESOR} - N'_{ESOR}$ is a regular splitting and, from Theorem 3.5, a convergent splitting. Since the inequality $A'_{ESOR}\boldsymbol{x} - A\boldsymbol{x} = (I+S')A\boldsymbol{x} - A\boldsymbol{x} = S'A\boldsymbol{x} \geq 0$ holds, then from Lemma 3.4, $\boldsymbol{x}$ is a Perron vector of T. Finally, from Theorem 3.7 $\rho(T) \geq \rho(T'_{ESOR})$ holds. □

Theorem 3.12. *Let A be an $\boldsymbol{M}$-matrix. For $\Omega = I + S$, put $A_{ESOR} = M_{ESOR} - N_{ESOR}$. Then the following inequality holds:*

$$\rho(T'_{ESOR}) \geq \rho(T_{ESOR}).$$

Proof. Since $E + L \geq O$, we easily obtain

$$(M)^{-1}_{ESOR} = I + (E+L) + (E+L)^2 + \cdots + (E+L)^{n-1} \geq O. \tag{3.19}$$

Since $S \geq S'$, $E \geq E'$ and $M^{-1}_{ESOR} \geq (M'_{ESOR})^{-1} \geq O$ hold. Furthermore, we have $A_{ESOR}\boldsymbol{x} - A'_{ESOR}\boldsymbol{x} = (S - S')A\boldsymbol{x} \geq 0$. From Lemma 3.4, $\boldsymbol{x}$ is an eigenvector of T, and $\boldsymbol{x}$ is also a Perron vector of T. Therefore, from Theorem 3.7 $\rho(T'_{ESOR}) \geq \rho(T_{ESOR})$ holds. □

Theorem 3.13. *Let A be an $\boldsymbol{M}$-matrix. Put $T_S = (I - D - (E+L))^{-1}(U - S + SU)$. Then the following inequality holds:*

$$\rho(T_{ESOR}) \geq \rho(T_S). \tag{3.20}$$

Proof. Since $(I-D)^{-1} \geq I$ and $E + L \geq O$, then

$$M_S^{-1} = [I + (I-D)^{-1}(E+L) + \{(I-D)^{-1}(E+L)\}^2 + \cdots + \{(I-D)^{-1}(E+L)\}^{n-1}](I-D)^{-1} \geq O. \tag{3.21}$$

Clearly $M_S^{-1} \geq M^{-1}_{ESOR} \geq O$. In this case, $A_S\boldsymbol{x} = (I+S)A\boldsymbol{x} = A_{ESOR}\boldsymbol{x}$ holds. From Theorem 3.8 it follows that $\rho(T_{ESOR}) \geq \rho(T_S)$. □

Denote $A_{smr} = (I + S + S_m + R)A$ and let T_{smr} be the iterative matrix associated with A_{smr}. Then it easily follows that $\rho(T_{s+m}) \geq \rho(T_{smr})$. In summary, we have proven the following inequalities:

$$\rho(T) \geq \rho(T'_{ESOR}) \geq \rho(T_{ESOR}) \geq \rho(T_S) \geq \rho(T_{s+m}) \geq \rho(T_{smr}) \geq \rho(T_G).$$

4. Numerical results

In this section, we test simple examples to compare and contrast the characteristics of the different preconditioners. Consider first the matrix

$$A = \begin{pmatrix} 1 & -0.2 & -0.3 & -0.2 \\ -0.2 & 1 & -0.3 & -0.1 \\ -0.1 & -0.2 & 1 & -0.3 \\ -0.2 & -0.3 & -0.2 & 1 \end{pmatrix},$$

which satisfies Eq. (2.10). Applying the Gauss–Seidel method, we have $\rho(T) = 0.4431$. By using preconditioner $P_S = I + S$, we find that A_S and T_S have the following forms:

$$A_S = \begin{pmatrix} 0.96 & 0 & -0.36 & -0.22 \\ -0.23 & 0.94 & 0 & -0.19 \\ -0.16 & -0.29 & 0.94 & 0 \\ -0.2 & -0.3 & -0.2 & 1 \end{pmatrix}, \qquad T_S = \begin{pmatrix} 0 & 0 & 0.3750 & 0.2292 \\ 0 & 0 & 0.0918 & 0.2582 \\ 0 & 0 & 0.0921 & 0.1187 \\ 0 & 0 & 0.1210 & 0.1470 \end{pmatrix},$$

and $\rho(T_S) = 0.2425$. By using the ESOR method with $\Omega = I + S$, the iterative matrix T_{ESOR} is found to be

$$T_{ESOR} = \begin{pmatrix} 0.0400 & 0 & 0.3600 & 0.2200 \\ 0.0092 & 0.0600 & 0.0828 & 0.2406 \\ 0.0091 & 0.0174 & 0.1416 & 0.1050 \\ 0.0126 & 0.0215 & 0.1252 & 0.1372 \end{pmatrix},$$

and $\rho(T_{ESOR}) = 0.3051$. Next, for the AGS method with $P_{S'}$, we have

$$A_{S'} = \begin{pmatrix} 0.9836 & -0.1182 & -0.3245 & -0.2082 \\ -0.2248 & 0.9505 & -0.0523 & -0.1743 \\ -0.16 & -0.29 & 0.94 & 0 \\ -0.2 & -0.3 & -0.2 & 1 \end{pmatrix}, \qquad T_{S'} = \begin{pmatrix} 0 & 0.1202 & 0.3299 & 0.2116 \\ 0 & 0.0284 & 0.1330 & 0.2334 \\ 0 & 0.0292 & 0.0972 & 0.1080 \\ 0 & 0.0384 & 0.1253 & 0.1340 \end{pmatrix},$$

and $\rho(T_{S'}) = 0.2836$. Using the preconditioner P_{s+m}, we obtain

$$A_{s+m} = \begin{pmatrix} 0.93 & -0.06 & -0.06 & -0.31 \\ -0.25 & 0.91 & -0.02 & -0.09 \\ -0.16 & -0.29 & 0.94 & 0 \\ -0.2 & -0.3 & -0.2 & 1 \end{pmatrix}, \qquad T_{s+m} = \begin{pmatrix} 0 & 0.0645 & 0.0645 & 0.3333 \\ 0 & 0.0177 & 0.0397 & 0.1905 \\ 0 & 0.0164 & 0.0232 & 0.1155 \\ 0 & 0.0215 & 0.0295 & 0.1470 \end{pmatrix},$$

and $\rho(T_{s+m}) = 0.1966$. For $P_{smr} = I + S + S_m + R$, we have

$$A_{smr} = \begin{pmatrix} 0.93 & -0.06 & -0.06 & -0.31 \\ -0.25 & 0.91 & -0.02 & -0.09 \\ -0.16 & -0.29 & 0.94 & 0 \\ -0.08 & -0.08 & -0.15 & 0.87 \end{pmatrix}, \qquad T_{smr} = \begin{pmatrix} 0 & 0.0645 & 0.0645 & 0.3333 \\ 0 & 0.0177 & 0.3970 & 0.1905 \\ 0 & 0.0165 & 0.0232 & 0.1155 \\ 0 & 0.0104 & 0.0136 & 0.0681 \end{pmatrix},$$

and $\rho(T_{smr}) = 0.1176$. From the results above, we have $(G)_{nj} = (0.28, 0.38, 0.35, 0)$. Then $A_G(\gamma = 1)$ and T_G have the forms:

$$A_G = \begin{pmatrix} 0.93 & 0.06 & -0.06 & -0.31 \\ -0.25 & 0.91 & -002 & -0.09 \\ -0.16 & -0.29 & 0.94 & 0 \\ -0.031 & -0.046 & -0.048 & 0.801 \end{pmatrix}, \qquad T_G = \begin{pmatrix} 0 & 0.0645 & 0.0645 & 0.3333 \\ 0 & 0.0177 & 0.0397 & 0.1905 \\ 0 & 0.0164 & 0.0232 & 0.1155 \\ 0 & 0.0045 & 0.0062 & 0.0308 \end{pmatrix},$$

and $\rho(T_G) = 0.0787$. Since the preconditioned matrices differ only in the values of their last rows, the related matrices also differ only in these values, as shown in the results above. Therefore, we hereafter show only the last row.

By putting $\gamma_1 = 1.1811$, the value of a_{42} is zero. Then $(A_{G_{\gamma_1}})_{nj}$ and $(T_{G_{\gamma_1}})_{nj}$ have the following forms:

$$(A_{G_{\gamma_1}})_{nj} = (-0.00039, 0, -0.02047, 0.76500), \qquad (T_{G_{\gamma_1}})_{nj} = (0, 0.00047, 0.00065, 0.00326),$$

and $\rho(T_{G_{\gamma_1}}) = 0.0497$. For $\gamma_2 = 1.18343$, we have:

$$(A_{G_{\gamma_2}})_{nj} = (0, 0.0006, -0.0201, 0.7645), \qquad (T_{G_{\gamma_2}})_{nj} = (0, 0.00042, 0.00058, 0.00289),$$

and $\rho(T_{G_{\gamma_2}}) = 0.0493$.

Remark 4.1. Clearly $(A_{G_{\gamma_1}})$ is not a $\boldsymbol{Z}$-matrix. But since the value of a^G_{42} is very small, $M^{-1}_{G_{\gamma_2}}$ is nonnegative. Thus $T_{G_{\gamma_2}} \geq O$.

For $\gamma_3 = 1.31579$, we have the following results:

$$(A_{G_{\gamma_3}})_{nj} = (0.02237, 0.03421, 0, 0.73816), \qquad (T_{G_{\gamma_3}})_{nj} = (0, -0.00278, -0.00380, -0.01893),$$

and $\rho(T_{G_{\gamma_3}}) = 0.0241$. In this case, since $M^{-1}_{G_{\gamma_3}}$ is not an $\boldsymbol{M}$-matrix, $T_{G_{\gamma_3}}$ is not nonnegative. When the value of γ increases further, there exists an optimum γ corresponding to the minimum $\rho(T_G)$. For example, for $\gamma = 1.36807$ we have:

$$(A_{G_{\gamma_{\text{opt}}}})_{nj} = (0.03120, 0.04749, 0.00795, 0.72775),$$
$$(T_{G_{\gamma_{\text{opt}}}})_{nj} = (0, -0.00410, -0.00561, -0.02798),$$

and $\rho(T_{G_{\gamma_{\text{opt}}}}) = 0.00649$. For the SOR method we obtain $\rho(T_{\text{opt}\omega=1.1488}) = 0.24660$ by numerical computation. By performing the reordering [3] to example A, the following matrix is derived:

$$A^r = \begin{pmatrix} 1 & -0.3 & -0.1 & -0.2 \\ -0.2 & 1 & -0.3 & -0.1 \\ -0.3 & -0.2 & 1 & -0.2 \\ -0.2 & -0.3 & -0.2 & 1 \end{pmatrix}.$$

We cannot obtain better results than these for example matrix A. For example

$$A^r_{smr} = \begin{pmatrix} 0.93 & -0.06 & -0.23 & -0.03 \\ -0.31 & 0.91 & -0.02 & -0.06 \\ -0.34 & -0.269 & 0.964 & 0 \\ -0.12 & -0.10 & -0.11 & 0.89 \end{pmatrix}, \qquad T^r_{smr} = \begin{pmatrix} 0 & 0.06667 & 0.25556 & 0.33333 \\ 0 & 0.01227 & 0.10904 & 0.07729 \\ 0 & 0.01298 & 0.12004 & 0.03274 \\ 0 & 0.01522 & 0.06154 & 0.01722 \end{pmatrix},$$

and $\rho(T^r_{smr}) = 0.16745$. Moreover, we have $\rho(T^r_G)_{\gamma=1} = 0.1572$, while for $\gamma_3 = 1.26582$, we get $\rho(T^r_{G_{\gamma 3}}) = 0.091152$. For $\gamma_{\text{opt}} = 3.4109$, $\rho(T^r_{G_{\gamma opt}}) = 0.0116$.

Table 1
Spectral radii for each method for test matrices B and C.

Method		B		C
	$\gamma/\rho(.)$	*non-re.*	*re.*	*non-re.*
ESOR	$\rho(.)$	0.5321	0.5318	0.2468
AGS, P_S	$\rho(.)$	0.4888	0.4900	0.18090
P_{s+m}	$\rho(.)$	0.4028	0.41800	0.10205
EGS, P_{smr}	$\rho(.)$	0.3706	0.3915	0.1056
$P_G, \gamma = 1$	$\rho(.)$	0.3362	0.3616	0.0634
$\gamma = \gamma_m$	γ_m	1.19848	[a]	0.52575
	$\rho(.)$	0.17888	[a]	0.0842
$\gamma = \gamma_{\text{opt.}}$	γ_{opt}	2.2770	2.349	1.111
	$\rho(.)$	0.1093	0.0976	0.0461

[a] Under the condition $\gamma_u > \gamma,\ \gamma_m$ does not exist. Non-reordering is denoted by 'non-re'.

The following test matrix is a randomly generated $\boldsymbol{M}$-matrix,

$$B = \begin{pmatrix} 1 & -0.1897 & -0.1179 & -0.3462 & -0.1256 \\ -0.2283 & 1 & -0.1811 & -0.0787 & -0.2803 \\ -0.0755 & -0.2736 & 1 & -0.1038 & -0.1623 \\ -0.1918 & -0.1633 & -0.3306 & 1 & -0.1778 \\ -0.1562 & -0.1742 & -0.2865 & -0.3362 & 1 \end{pmatrix}.$$

We performed the reordering scheme based on [3], and display the results in Table 1. The next test matrix is a linear system of equations arising from the boundary element method for the Laplace problem (For further details, see [20]),

$$C = \begin{pmatrix} 1.193 & 0.369 & 0.111 & -0.030 & -0.058 & -0.005 & 0.124 & 0.514 \\ 0.369 & 1.193 & 0.514 & 0.124 & -0.005 & -0.058 & -0.030 & 0.111 \\ 0.124 & 0.514 & 1.193 & 0.369 & 0.111 & -0.030 & -0.058 & -0.005 \\ -0.030 & 0.111 & 0.369 & 1.193 & 0.514 & 0.124 & -0.005 & -0.058 \\ -0.058 & -0.005 & 0.124 & 0.514 & 1.193 & 0.369 & 0.111 & -0.030 \\ -0.005 & -0.058 & -0.030 & 0.111 & 0.369 & 1.193 & 0.514 & 0.124 \\ 0.111 & -0.030 & -0.058 & -0.005 & 0.124 & 0.514 & 1.193 & 0.369 \\ 0.514 & 0.124 & -0.005 & -0.058 & -0.03 & 0.111 & 0.369 & 1.193 \end{pmatrix}.$$

This matrix is clearly not an $\boldsymbol{H}$-matrix. Therefore, it is impossible to apply the Gauss- Seidel iterative method. To obtain a diagonally dominant matrix, Sakakihara et al. proposed the application of two preconditioners [20], for which the resultant matrix has a spectral radius of 0.078. On the other hand, the preconditioned matrix P_SA can produce an $\boldsymbol{H}$-matrix (a detailed explanation of this procedure is described in the next section). For this example, the reordering scheme is not used, because all the rows have the same vector 1-norm values. Numerical results are shown in Table 1.

Finally, we test the matrix A having the special form

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -a & 0 & 0 & 1 \end{pmatrix},$$

where $0 < |a| < 1$. It is well known that A^{T} is a p- *cyclic* matrix [21]. Then A_S and T_S have the following forms:

$$A_S = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -a & 0 & 1 & 0 \\ -a & 0 & 0 & 1 \end{pmatrix}, \qquad T_S = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & a & 0 \\ 0 & 0 & a & 0 \end{pmatrix}.$$

Clearly, $\rho(T_S) = a$. In this case, the spectral radius of T_S corresponds to the value of a_{41}. On the other hand, A_R and T_R have the following forms:

$$A_R = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -a & 0 & 1 & 0 \\ 0 & -a & 0 & 1 \end{pmatrix}, \qquad T_R = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}.$$

And $\rho(T_R) = a$. Set

$$P_G = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ a & a & a & 1 \end{pmatrix}.$$

Then we have

$$A_G = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1-a \end{pmatrix}, \qquad T_R = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Clearly $\rho(T_G) = 0$.

5. Concluding remarks

(1) Matrix B:

The diagonally dominant ratio is defined by $t_i = \frac{\sum_{j=1,j\neq i}^{n} |a_{ij}|}{|a_{ii}|}$, for all i. Moreover, the average diagonally dominant ratio is defined by $t_{av} = \frac{\sum_{i=1}^{n} t_i}{n}$. For matrix A, $t_{av} = 0.275$, while for B, $t_{av} = 0.84158$. Thus the preconditioning effect on A is better than that on B. Moreover, since the matrix B does not satisfy Eq. (2.8), we have $\rho(T_S) = 0.4888 < \rho(T_m) = 0.5032$. Hence, in contrast to the case for matrix A, we cannot obtain an extremely small value of $\rho(T_{s+m})$.

Matrix C:

Put $\boldsymbol{y} = (t_1, \ldots, t_n)$. As shown below, the preconditioned matrix $P_S C$ is not diagonally dominant only in the last row: $\boldsymbol{y} = (0.2597, 0.4168, 0.2597, 0.4168, 0.2597, 0.4168, 0.2597, -.0151)$. Put $\boldsymbol{y} = (1, 1, 1, 1, 1, 1, 1, 1.1)$. Then from the inequality of Definition 3.2 we have $t_n = 0.849$. Thus $P_S C$ is an $\boldsymbol{H}$-matrix. Readers interested in the criterion of $\boldsymbol{H}$-matrices may refer to [9] for further details. The use of other upper preconditioners can also produce an $\boldsymbol{H}$-matrix. From 6 in Theorem 3.5, C is an $\boldsymbol{H}$-compatible splitting.

(2) As shown in Table 1, for example with reference to the results for C, $\rho(T_{smr}) \leq \rho(T_{sm})$ does not hold. We now investigate the reasons for these results. Since C is not an $\boldsymbol{H}$-matrix, $\sum_{k=1,k\neq j}^{n-1} |a_{nk}a_{kj}|$ is of large value. Therefore, since the value of a_{nn}^{smr} is small, the relation $\rho(T_{smr}) > \rho(T_{sm})$ can be satisfied. For this matrix we have the numerical results, $a_{nn}^{sm} = 1$, $a_{nn}^{smr} = 0.693$ and $a_{nn}^{G} = 0.7702$. To confirm the validity of our assertion, we put $P_{smr'} = (I+S+S_m+0.9R)$. Then we have $a_{nn}^{smr'} = 0.86470$ and $\rho(T_{smr'}) = 0.0922$. Since matrix C is not a $\boldsymbol{Z}$-matrix, the description of γ_m is meaningless, but we have included these values in Table 1 for reference.

(3) From the numerical results, we have seen that the EGS method with the preconditioner $P_G(\gamma_{\text{opt}})$ produces an extremely small spectral radius. A development of a simple estimation scheme for γ_{opt} is a subject for future study.

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