Attractors for the Generalized Benjamin–Bona–Mahony Equation

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We consider the periodic initial-boundary value problem for a multidimensional generalized Benjamin–Bona–Mahony equation. We show the existence of the global attractor with a finite fractal dimension and the existence of the exponential attractor for the corresponding semigroup.

Key Words: attractor; exponential attractor; fractal dimension.

1. INTRODUCTION

We consider the equation

$$u_t - a A u_t - b A u + \nabla \cdot F(u) = h(x), \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}^+$$

(1)

with the initial condition

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^n$$

(2)

and the periodic boundary condition

$$u(x + L_i e_i, t) = u(x, t), \quad x \in \mathbb{R}^n, \quad t > 0, \quad L_i > 0, \quad i = 1, 2, \ldots, n$$

(3)
where $a$ and $b$ are positive constants; $e_1, \ldots, e_n$ is the canonical basis of $\mathbb{R}^n$, $u_0(x)$ and $b(x)$ are given functions, $\nabla \cdot F = \sum_{i=1}^n (\partial_i \otimes x_i) F_i$, and $F(s) = (F_1(s), F_2(s), \ldots, F_n(s))$ is a given vector field satisfying the following properties:

(i) $F_k(0) = 0$, $k = 1, 2, \ldots, n$;

(ii) the functions $F_k$, $k = 1, 2, \ldots, n$ are twice continuously differentiable in $\mathbb{R}^1$;

(iii) the functions $f_k(s) = (d/ds) F_k(s)$, $k = 1, 2, \ldots, n$, satisfy the growth conditions

$$|f_k(s)| \leq C(1 + |s|^m), \quad k = 1, 2, \ldots, n,$$

where $0 \leq m < \infty$ if $n = 2$, $0 \leq m < 2$ if $n = 3$ and $m = 0$ if $n \geq 4$. No growth condition is required if $n = 1$.

Using the standard Faedo–Galerkin method, it is not difficult to prove that if $h \in L^2(\Omega)$ and $u_0 \in H^1_{per}(\Omega)$, then the problem (1)-(3) has a unique solution $u \in C(\mathbb{R}^+; H^1_{per}(\Omega))$.

The Cauchy problem for the Benjamin–Bona–Mahony equation

$$u_t - uu_x + u_{xxx} + uu_x = 0 \quad (4)$$

and some of its generalizations has been investigated by several authors, such as Amick et al. [2], Bona and Dougalis [6], and Karch [11]. In these articles the problem of global unique solvability and long time behaviour of solutions are studied.

Kalantarov [10] has proved the existence of a global attractor for the semigroup generated by the initial-boundary value problem for the Kelvin–Voigt equations

$$\nu_t - \nu - \Delta \nu_x - v \Delta v + \text{grad} p + v \nu_x = h(x),$$

$$\text{div} \nu = 0. \quad (5)$$
On the other hand Wang [16–18] using the technique of Ghidaglia [8] has proved the existence of a global attractor for the semigroup generated by (1)–(3) in one dimensional case, that is, the periodic initial-boundary value problem for the equation

$$u_t - u_{xxxt} + f(u) u_x = g(x), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}^+ \quad (6)$$

In our studies, we have used the ideas of Hale [9] and Ladyzhenskaya [13] on attractors for asymptotically compact semigroups. In the sequel we will use the following theorems.

**Theorem 1** [9, 13]. If a semigroup \( V_t, t \in \mathbb{R}^+ \) acts on a Banach space \( X \), and \( V_t = W_t + Z_t \) in which \( W_t, t \in \mathbb{R}^+ \), is a family of operators, such that

$$\| W_t(B) \|_X \leq m_1(t) m_2(\| B \|_X), \quad (7)$$

where \( m_1(\cdot) \) and \( m_2(\cdot) \) are continuous functions on \( \mathbb{R}^+ \) and \( m_1(t) \to 0 \), as \( t \to \infty \), \( \| B \|_X = \sup_{v \in B} \| v \|_X \). While \( Z_t, t \in \mathbb{R}^+ \) maps bounded sets into precompact sets, then \( V_t, t \in \mathbb{R}^+ \) is asymptotically compact semigroup.

**Theorem 2** [9, 13]. Let \( V_t : X \to X \), \( t \in \mathbb{R}^+ \), be a continuous bounded point-dissipative asymptotically compact semigroup. Then for this semigroup there exists a non-empty global attractor \( \mathcal{A} \). It is compact, invariant, and connected.

**Theorem 3** [12]. Let \( B \) be a bounded set in a Hilbert space \( X \), and let there be defined a map \( V : B \to X \) such that \( B \subseteq V(B) \) and for all \( v, \tilde{v} \in B \)

$$\| V(v) - V(\tilde{v}) \|_X \leq \ell \| v - \tilde{v} \|_X, \quad (8)$$

and

$$\| Q_N V(v) - Q_N V(\tilde{v}) \|_X \leq \delta \| v - \tilde{v} \|_X, \quad \delta < 1, \quad (9)$$

where \( Q_N \) is the orthogonal projection of \( X \) onto the subspace \( X_N^\perp \) of codimension \( N \). Then for the fractal dimension of \( B \) the inequality

$$d_F(B) \leq N \log \left( \frac{8\kappa^2\ell^2}{1-\delta^2} \right) \left/ \log \frac{2}{1-\delta^2} \right. \quad (10)$$

is true, where \( \kappa \) is the Gauss constant.
2. EXISTENCE OF THE GLOBAL ATTRACTOR

First let us show that the semigroup $V_t$ is bounded dissipative in a phase space $X^1$; that is, it has an absorbing ball in $X^1$. Multiplying Eq. (1) by $u$ in $L_2(\Omega)$ we get

$$\frac{1}{2} \frac{d}{dt} \left[ \|u(\cdot, t)\|^2 + a \|\nabla u(\cdot, t)\|^2 \right] + b \|\nabla u(\cdot, t)\|^2 = (h, u). \quad (11)$$

We will use the notations $\|\cdot\|$, $(\cdot, \cdot)$ for the norm and inner product in $L_2(\Omega)$, respectively. Using the Poincaré–Friedrichs inequality

$$\|u\| \leq \lambda_1^{-1/2} \|\nabla u\|, \quad (12)$$

which is valid for each $x \in X^1$, we can easily get

$$|(h, u)| \leq \frac{b}{2} \|\nabla u\|^2 + \frac{\lambda_1^{-1}}{2b} \|h\|^2, \quad (13)$$

where $\lambda_1$ is the lowest eigenvalue of the periodic boundary value problem

$$-\Delta \psi(x) = \lambda \psi(x),$$

$$\psi(x + L_i e_i) = \psi(x), \quad i = 1, \ldots, n, \quad (E)$$

$$\int_{\Omega} \psi(x) \, dx = 0.$$

Due to (12) we have

$$\frac{b}{2} \|\nabla u(\cdot, t)\|^2 + \frac{b \lambda_1}{2} \|u(\cdot, t)\|^2 \leq b \|\nabla u(\cdot, t)\|^2. \quad (14)$$

By using (13), (14) we get from (11)

$$\frac{d}{dt} \left[ \|u(\cdot, t)\|^2 + a \|\nabla u(\cdot, t)\|^2 \right] + \frac{b}{2} \|\nabla u(\cdot, t)\|^2 + \frac{b \lambda_1}{2} \|u(\cdot, t)\|^2 \leq \frac{1}{b \lambda_1} \|h\|^2$$

or

$$\frac{d}{dt} \left[ \|u(\cdot, t)\|^2 + a \|\nabla u(\cdot, t)\|^2 \right] + K_0 \left[ \|u(\cdot, t)\|^2 + a \|\nabla u(\cdot, t)\|^2 \right] \leq \frac{1}{b \lambda_1} \|h\|^2, \quad (15)$$
where \( K_0 = \min\left\{ b\lambda_1/2, b/2a \right\} \). Integrating (15) we find
\[
\|\mathbf{u}(\cdot, t)\|^2 \leq \frac{1}{a} \left[ \|\mathbf{u}_0\|^2 + a \|\mathbf{Vu}_0\|^2 \right] e^{-K_0 t} + \frac{1}{bK_0\lambda_1} \|h\|^2.
\]
From this inequality it follows that
\[
B_0 := \left\{ u \in X^1 : \|u(\cdot, t)\|_{X^1} \leq \left( \frac{2}{\lambda_1 b K_0} \right)^{1/2} \|h\| \right\}
\]
is an absorbing ball for the semigroup \( V_t \) in \( X^1 \).

Now, we will prove that the semigroup \( V_t \) is asymptotically compact, that is, for each sequence \( \{t_k\} \rightarrow \infty \) and each bounded sequence \( \{v_k\} \subset X^1 \), the set \( \{ V_{t_k}(v_k) \} \) is precompact. To do this we will use Theorem 1. It is clear that the solution \( \mathbf{u}(x, t) \) of the problem (1)–(3) can be represented in the form
\[
\mathbf{u}(x, t) = \mathbf{w}(x, t) + \mathbf{z}(x, t),
\]
where \( \mathbf{w}(x, t) \) is a solution of the problem
\[
\begin{align*}
\mathbf{w}_t - a \mathbf{w} - b \mathbf{w} = 0, & \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}^+, \quad (16) \\
\mathbf{w}(x, 0) = \mathbf{u}_0(x), & \quad x \in \mathbb{R}^n, \quad (17) \\
\mathbf{w}(x, t) = \mathbf{w}(x + \mathbf{L}_i e_i, t), & \quad i = 1, \ldots, n, \quad t \in \mathbb{R}^+. \quad (18)
\end{align*}
\]
while \( \mathbf{z}(x, t) \) is a solution of the problem
\[
\begin{align*}
\mathbf{z}_t - a \mathbf{z} - b \mathbf{z} + \mathbf{V} \cdot \mathbf{F}(\mathbf{w} + \mathbf{z}) = h(x), & \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}^+ \quad (19) \\
\mathbf{z}(x, 0) = 0, & \quad x \in \Omega \quad (20) \\
\mathbf{z}(x, t) = \mathbf{z}(x + \mathbf{L}_i e_i, t), & \quad x \in \mathbb{R}^n, i = 1, \ldots, n, \quad t \in \mathbb{R}^+. \quad (21)
\end{align*}
\]
Thus, the semigroup \( V_t \) has the representation
\[
V_t = W_t + Z_t, \quad (22)
\]
where \( W_t \) is the semigroup generated by (16)–(18) and \( Z_t \) is a solution operator of the problem (19)–(21). Multiplying Eq. (16) by \( w \) in \( L_2(\Omega) \), after some elementary operations we can easily get
\[
\frac{d}{dt} \left[ \|\mathbf{w}(\cdot, t)\|^2 + a \|\nabla \mathbf{w}(\cdot, t)\|^2 \right] + k_1 \left[ \|\mathbf{w}(\cdot, t)\|^2 + a \|\nabla \mathbf{w}(\cdot, t)\|^2 \right] \leq 0. \quad (23)
\]
Integrating (23) and then using Poincaré–Friedrichs inequality we obtain
\[
\|\nabla u(\cdot, t)\|^2 \leq e^{-k_1 t} \left( \frac{1}{\lambda_1 a} + 1 \right) \|\nabla u(\cdot, 0)\|^2.
\]
That is, the semigroup \( W_t : X^1 \rightarrow X^1 \) satisfies the condition (7) of Theorem 1 with \( m_1(t) = e^{-k_1 t} (d/(\lambda_1 a) + 1) \) and \( m_2(t) = t \).

It remains now to show that \( Z_t : X^1 \rightarrow X^1 \) is precompact for each \( t > 0 \), when \( n = 3 \); the cases \( n = 1, 2 \) and \( n > 3 \) can be dealt with in a similar way.

In order to see this property, let us rewrite Eq. (19) in the form
\[
z_t - a \Delta z_t - b \Delta z = h(x) - \sum_{i=1}^n f_i(u) u_{x_i}
\]
\[= g(x, t). \quad (24)
\]
Let \( p = 6/(m + 3) \); using the Hölder’s inequality and the condition (iii) we can easily get the estimate
\[
\int_{\Omega} |f_i(u) u_{x_i}|^p \, dx \leq \left( C_1 + C_2 |u|^{mp} |u_{x_i}|^p \right) \int_{\Omega} |u_{x_i}|^p \, dx
\]
\[\leq C_3 \left( 1 + \int_{\Omega} |u_{x_i}|^2 \, dx \right) + C_2 \left( \int_{\Omega} |u_{x_i}|^2 \, dx \right)^{p/2} \left( \int_{\Omega} |u|^{mp(2-p)} \, dx \right)^{(2-p)/2}.
\]
Since \( mp(2/(2-p)) = 6 \), by using the well-known inequality [14, p. 45]
\[
\|u(\cdot, t)\|_{L^p(\Omega)} \leq c \|\nabla u(\cdot, t)\|_{L^2(\Omega)},
\]
which is valid for each \( u \in H^1_{per}(\Omega), \Omega \subset \mathbb{R}^3 \) we obtain
\[
\int_{\Omega} |f_i(u) u_{x_i}|^p \, dx \leq C_3 \left( 1 + \int_{\Omega} |u|^2 \, dx \right) + C_4 \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{3(2-p)/2}.
\]
Since \( V_t : X^1 \rightarrow X^1 \) is bounded dissipative
\[
\max_{t \in \mathbb{R}^+} \|\nabla u(\cdot, t)\|_{L^2(\Omega)} \leq C_5
\]
and \( h \in L_2(\Omega) \), we get \( g \in C(\mathbb{R}^+; L_2(\Omega)) \). By the embedding theorem (see Triebel [15, p. 327]) \( L_2(\Omega) \subset H_{\text{per}}^{1+\sigma}(\Omega), \sigma = 1 - (m/2) \), we have

\[
g = h + \sum_{i=1}^N f_i(u_i) u_{\xi_i} \in L_2(0, T; \dot{H}_{\text{per}}^{1+\sigma}(\Omega)), \quad \forall T > 0
\]

and the precompactness of the operator \( W_t : X^1 \to X^1 \) follows from

**Proposition 4.** If \( g \in L_2(0, T; \dot{H}_{\text{per}}^s(\Omega)) \) and \( v_0 \in \dot{H}_{\text{per}}^{s+2}(\Omega) \), then the initial value problem

\[
v_t - a \Delta v_t - b \Delta v = g(x, t), \quad x \in \mathbb{R}^n, \quad t \in (0, T)
v(x, 0) = v_0(x), \quad x \in \mathbb{R}^n,
v(x, t) = v(x + L_ie_i, t), \quad i = 1, \ldots, n, \quad t \in (0, T),
\]

has a solution \( v(x, t) \) in \( C(0, T; \dot{H}_{\text{per}}^{s+2}(\Omega)) \) for \( s \in \mathbb{R} \).

This proposition can be proved by using the standard Fourier method. Following the technique used in Babin and Vishik [4, Theorem 6.2] it can be proved that \( \mathcal{M} \) is bounded in \( X^2 = \dot{H}_{\text{per}}^2(\Omega) \cap \dot{H}_{\text{per}}^4(\Omega) \). So we have obtained

**Theorem 5.** Suppose that the vector field \( F \) satisfies the conditions (i)–(iii) and \( h \in L_2(\Omega) \). Then the semigroup \( V_t : X^1 \to X^1 \) has a global attractor \( \mathcal{M} \) which is compact, invariant and connected in \( X^1 \). \( \mathcal{M} \) is included and bounded in \( X^2 \).

### 3. Estimate of the Fractal Dimensions of the Attractor

Now we are going to show that for some \( t_1 > 0 \), the operator \( V = V_{t_1} \) satisfies the conditions of Theorem 3, from which we get the estimate of the dimension of the global attractor. Let \( u \) and \( v \) be two solutions of the problem (1)–(3) with \( u(x, 0) = u_0(x) \) and \( v(x, 0) = v_0(x) \) in \( \mathcal{M} \). Then from the Theorem 5, it follows that \( u(\cdot, t), v(\cdot, t) \in \mathcal{M}, \forall t \in \mathbb{R}^+ \). Let us define \( w = u - v; \) then \( w \) will satisfy the equation

\[
w_t - a \Delta w_t - b \Delta w + \nabla \cdot (F(u) - F(v)) = 0. \tag{26}
\]

Taking the inner product with \( w(x, t) \) in \( L_2(\Omega) \), we obtain

\[
\frac{1}{2} \frac{d}{dt} \left[ \| w(\cdot, t) \|^2 \right] + a \| \nabla w(\cdot, t) \|^2 + b \| \nabla w(\cdot, t) \|^2 + (\nabla \cdot (F(u) - F(v)), w) = 0.
\]
Now let us consider the last term,
\[
|\nabla \cdot (F(u) - F(v), w)| = \left| \sum_{i=1}^{n} (F_i(u) - F_i(v), \nabla w) \right|
\]
\[
= \left| \sum_{i=1}^{n} \left( \int_{0}^{1} \frac{d}{d\theta} F_i(\theta u + (1 - \theta)v) \, d\theta, \nabla w \right) \right|
\]
\[
\leq \sum_{i=1}^{n} \int_{0}^{1} \left| F_i(\theta u + (1 - \theta)v) \right| \, |w| \, |\nabla w| \, dx.
\]
Since
\[
|f_i(\theta u + (1 - \theta)v)| \leq C_6(1 + |u|^m + |v|^m), \quad i = 1, 2, ..., n,
\]
using the Hölder’s inequality and (25) we get
\[
|\nabla \cdot (F(u) - F(v), w)| \leq C_6 \sum_{i=1}^{n} \int_{0}^{1} (1 + |u|^m + |v|^m) \, |w| \, |\nabla w| \, dx \leq C_7 \|w\| \|\nabla w\|
\]
and utilizing Young’s inequality
\[
|\nabla \cdot (F(u) - F(v), w)| \leq C_7 \left[ \|w\|^2 + \frac{1}{4a} \|w\|^2 \right] \leq \mu [\|w\|^2 + a \|\nabla w\|^2],
\]
where \( \mu = C_7 \max\{1, 1/4a\} \). So we obtain
\[
\frac{d}{dt} \left[ \|w(\cdot, t)\|^2 + a \|\nabla w(\cdot, t)\|^2 \right] \leq \mu [\|w(\cdot, t)\|^2 + a \|\nabla w(\cdot, t)\|^2].
\]
Thus
\[
\|w(\cdot, t)\|^2 + a \|\nabla w(\cdot, t)\|^2 \leq \left[ \|w(\cdot, 0)\|^2 + a \|\nabla w(\cdot, 0)\|^2 \right] e^{\mu t}
\]
and
\[
\|\nabla w(\cdot, t)\| \leq (a + \lambda_1^{-1})^{1/2} \|\nabla w(\cdot, 0)\| \, e^{\mu t/2}.
\]
Now, let \( P_N \) denote the orthogonal projection to the subspace \( X_N^1 \) of \( X^1 \) spanned by the first \( N \) basis elements of \( X^1 \), that is, the first \( N \)
eigenfunctions of the problem \((E)\). Multiplying Eq. \((26)\) in \(L^2(\Omega)\) by \(Q_N w := (I - P_N) w\), we obtain
\[
(w_t (\cdot, t), Q_N w (\cdot, t)) - a (Dw_t (\cdot, t), Q_N w (\cdot, t)) + b \|\nabla Q_N w (\cdot, t)\|^2 = (\nabla \cdot F(u) - \nabla \cdot F(v), Q_N w)
\]
\[
= \left( \sum_{i=1}^n f_i(u) u_x - f_i(v) v_x, Q_N w \right)
\]
\[
= \left( \sum_{i=1}^n \left[ f_i(u) w_x + (f_i(u) - f_i(v)) v_x \right], Q_N w \right)
\]
\[
= \left( \sum_{i=1}^n f_i(u) w_x, Q_N w \right) + \left( \sum_{i=1}^n \int_0^1 f_i'(\theta u + (1 - \theta) v) \, d\theta \, w_x, Q_N w \right). \tag{28}
\]
Since the attractor \(\mathcal{M}\) is bounded in \(H^2(\Omega)\) we have
\[
\max_{x \in \Omega} |u|, \quad \max_{x \in \Omega} |v|, \quad |u|_{H^1(\Omega)}, \quad |v|_{H^1(\Omega)} \leq M_0. \tag{29}
\]
Using the condition (iii), the Hölder inequality (29), (25) we can estimate the right hand side of (28) as
\[
\left| \left( \sum_{i=1}^n f_i(u) w_x, Q_N w \right) + \left( \sum_{i=1}^n \int_0^1 f_i'(\theta u + (1 - \theta) v) \, d\theta \, w_x, Q_N w \right) \right|
\]
\[
\leq C_8 \int \|\nabla w(x, t)\| \|Q_N w(x, t)\| \, dx
\]
\[
+ C_9 \int \|w(x, t)\| \|\nabla w(x, t)\| \|Q_N w(x, t)\| \, dx
\]
\[
\leq C_8 \|\nabla w (\cdot, t)\| \|Q_N w (\cdot, t)\|
\]
\[
+ C_9 \left( \int \|w(x, t)\|^6 \, dx \right)^{1/6} \left( \int \|\nabla w(x, t)\|^3 \, dx \right)^{1/3} \left( \int \|Q_N w(x, t)\|^2 \, dx \right)^{1/2}
\]
\[
\leq C_8 \|\nabla w (\cdot, t)\| \|Q_N w (\cdot, t)\|
\]
\[
+ C_{10} \|\nabla w (\cdot, t)\| \|v (\cdot, t)\|_{H^1(\Omega)} \|Q_N w (\cdot, t)\|
\]
\[
\leq C_{11} \|\nabla w (\cdot, t)\| \|Q_N w (\cdot, t)\|.
\]
So (28) implies
\[ \frac{1}{2} \frac{d}{dt} \left[ \|Q_Nw(\cdot, t)\|^2 + a \|\nabla Q_Nw(\cdot, t)\|^2 \right] + b \|\nabla Q_Nw(\cdot, t)\|^2 \]
\[ \leq C_{11} \|\nabla w(\cdot, t)\| \|Q_Nw(\cdot, t)\|. \] (30)

By using the inequality
\[ \|Q_N\psi\| \leq \lambda_{N+1}^{-1/2} \|\nabla Q_N\psi\|, \quad \forall \psi \in (X_N')^4, \]
where \( \lambda_N \) is the \( N \)th eigenvalue of the problem \( (E) \), we can rewrite (30) as
\[ \frac{d}{dt} \left[ \|Q_Nw(\cdot, t)\|^2 + a \|\nabla Q_Nw(\cdot, t)\|^2 \right] \\
+ b \|\nabla Q_Nw(\cdot, t)\|^2 + \lambda_1 b \|Q_Nw(\cdot, t)\|^2 \\
\leq 2C_{11} \|\nabla w(\cdot, t)\| \|\nabla Q_Nw(\cdot, t)\| \lambda_{N+1}^{-1/2} \\
\leq C_{11} \lambda_{N+1}^{-1/2} \|\nabla w(\cdot, t)\|^2 + C_{11} \lambda_{N+1}^{-1/2} \|\nabla Q_Nw(\cdot, t)\|^2 \] (31)
or
\[ \frac{d}{dt} \left[ \|Q_Nw(\cdot, t)\|^2 + a \|\nabla Q_Nw(\cdot, t)\|^2 \right] \\
+ b \lambda_1 \|Q_Nw(\cdot, t)\|^2 \leq C_{11} \lambda_{N+1}^{-1/2} \|\nabla w(\cdot, t)\|^2. \]

Let us choose \( N \) large enough, so that \( b - C_{11} \lambda_{N+1}^{-1/2} > 0 \) and set
\[ \mu_1 = \min \left( \frac{b - C_{11} \lambda_{N+1}^{-1/2}}{a}, \lambda_1 b \right). \]

From the last inequality we get
\[ \frac{d}{dt} \left[ \|Q_Nw(\cdot, t)\|^2 + a \|\nabla Q_Nw(\cdot, t)\|^2 \right] \\
+ \mu_1 \left[ \|Q_Nw(\cdot, t)\|^2 + a \|\nabla Q_Nw(\cdot, t)\|^2 \right] \\
\leq C_{11} \lambda_{N+1}^{-1/2} \|\nabla w(\cdot, t)\|^2 \\
\leq (a + \lambda_1^{-1}) C_{11} \lambda_{N+1}^{-1/2} \|\nabla w(\cdot, 0)\|^2 \cdot e^{\mu t} \]
by use of (27). Integrating this inequality, and after some elementary operations we obtain
\[ \|Q_Nw(\cdot, t)\|^2 \leq a^{-1} (a + \lambda_1^{-1}) \left[ C_{11} \lambda_{N+1}^{-1/2} (\mu + \mu_1)^{-1} e^{\mu t} + e^{-\mu_1 t} \right] \|\nabla w(\cdot, 0)\|^2. \]
Now we can choose $N$ and $t_0 > 0$ so that
\[ a^{-1} (a + \lambda_1^{-1}) \left[ C_{11} \bar{\mu}_{N+1} (\mu + \mu_1)^{-1} e^{a t_0} + e^{-\mu t_0} \right] \leq \delta < 1. \]
Hence the conditions of the Theorem 3 are satisfied with $V = V_{t_0}$ and we obtain the estimate
\[ d_p (\mathcal{U}) \leq N \frac{\log(8 \kappa^2 / (1 - \delta^2))}{\log(2 / (1 + \delta^2))} \]
for the fractal dimension of the global attractor.
So we have established the following theorem:

**Theorem 6.** Let all conditions of the Theorem 5 be satisfied. Then the attractor of the semigroup $V_t : X^1 \to X^1$ has a finite fractal dimension

4. A REMARK ON THE EXISTENCE OF THE EXPONENTIAL ATTRACTOR

Consider now the one-dimensional version of the problem (1)–(3),
\begin{align*}
u_t - a \nu_{xx} - b \nu_{xx} + f(u) \nu_x &= h(x), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}^+, \quad (32) \\
u(x, 0) &= u_0(x), \quad x \in \mathbb{R}, \quad (33) \\
u(x, t) &= u(x + L, t), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}^+. \quad (34)
\end{align*}
It follows from the Theorem 5, that the problem (32)–(34) has an absorbing ball $B_0 \subset X^1$ and a global attractor $\mathcal{A}$, which is compact.

Now, assume that $u_0, v_0$ are arbitrary two elements of $B_0$, then for $w(\cdot, t) = V_t (u_0) - V_t (v_0) = u(\cdot, t) - v(\cdot, t)$ the inequality (27) is valid:
\[ \|w_x(\cdot, t)\|_{\mathcal{L}^2} \leq C_1 (a + \lambda_1^{-1})^{1/2} ||\nabla w(\cdot, 0)|| e^{at}/2. \quad (35) \]
It follows from (32) that $w$ satisfies the equation
\[ w_t - a w_{xx} - b w_{xx} + \int_0^1 f'(\theta u + (1 - \theta) v) \ d\theta \cdot w_x + f(v) \ w_x = 0. \quad (36) \]
Let us multiply (36) by $Q_N w$ in $L_2(0, L)$,
\begin{align*}
\frac{1}{2} & \frac{d}{dt} \|Q_N w\|^2 + a \frac{d}{dt} \|Q_N w_x\|^2 + b \|Q_N w_x\|^2 \\
+ & \int_0^1 f'(\theta u + (1 - \theta) v) \ d\theta \cdot w_x Q_N w \ dx + \int_0^L f(v) \ w_x Q_N w \ dx = 0. \quad (37)
\end{align*}
Due to the Sobolev inequality
\[ \max_{x \in [0, L]} |z(x)| \leq d_0 \|z\|, \quad \forall z \in H^1_{per}(0, L) \]
we get from the relation (37)
\[
\frac{1}{2} \frac{d}{dt} \left[ \|Q_N w(\cdot, t)\|^2 + a \|Q_N w_x(\cdot, t)\|^2 \right] + b \|Q_N w(\cdot, t)\|^2 \\
\leq C_{12} \max_{x \in [0, L]} |w(x, t)| |Q_N w| + C_{13} \|w_x\| \|Q_N w\| \\
\leq C_{14} \|w_x(\cdot, t)\| \|Q_N w(\cdot, t)\| \\
\leq \frac{1}{2} C_{14} \lambda N_{N+1}^{1/2} \|w_x(\cdot, t)\|^2 + \frac{1}{2} C_{14} \lambda N_{N+1}^{1/2} \|Q_N w(\cdot, t)\|^2.
\]
So we have got the inequality similar to (31). Therefore the following inequality holds:
\[
\|Q_N w(\cdot, t)\| \leq a^{-1}(a + \lambda)^{-1}[C_{14} \lambda N_{N+1}^{1/2} (\mu + \mu_t)^{-1} e^{\delta t} + e^{-\mu t}] \|Q_N w(\cdot, 0)\|^2.
\]
It follows from the last estimate that the semigroup \( V_t : X^1 \rightarrow X^1, t \in \mathbb{R}^+ \) satisfies the discrete squeezing property (see [7]), that is, there exists \( N_0 \) and \( t_1 \) such that the operator \( T := V_{t_1} \) satisfies the conditions

\[
\|T x - T y\|_{X^1} \leq \ell_0 \|x - y\|_{X^1}, \quad \forall x, y \in B_0
\]
and for some \( \delta \in (0, 1/\sqrt{2}) \)

\[
\|(I - P_{N_0})(T x - T y)\|_{X^1} \leq \delta \|x - y\|_{X^1}, \quad \forall x, y \in B_0.
\]
Therefore the semigroup \( V_t : X^1 \rightarrow X^1, t \in \mathbb{R}^+ \) has an exponential attractor \( \mathcal{M}_e \) (see [3, 7]), that is a compact set \( \mathcal{M}_e \) such that

(i) \( \mathcal{M} \subseteq \mathcal{M}_e \subseteq B_0 \),

(ii) \( V_t \mathcal{M} \subseteq \mathcal{M}_e \),

(iii) \( \mathcal{M}_e \) has finite fractal dimension,

(iv) there exist \( C_1 \) and \( C_2 \), which does not depend on \( x \) such that \( \forall x \in B \) and each \( t > 0 \)

\[
\text{dist}(V_t x, \mathcal{M}_e) \leq C_1 \exp \{-C_2 t\}.
\]
REFERENCES


