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An Inequality for Monotonic Functions Generalizing Ostrowski and Related Results

S. S. DRAGOMIR AND A. SOFO
School of Computer Science and Mathematics
Victoria University
P.O. Box 14428
Melbourne City MC
VIC, Australia, 8001
<sever><sofo>@csm.vu.edu.au

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Abstract—In this paper, we establish a generalisation of the Ostrowski inequality for monotonic functions that also includes various recent results and apply it for quadrature formulae in numerical integration.

Keywords—Ostrowski's inequality, Monotonic mappings, Quadrature rules.

1. INTRODUCTION

In [1], Dragomir pointed out the following inequality for mappings of bounded variation generalising an Ostrowski type inequality first established in [2].

THEOREM 1. *Let $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ be a division of the interval $[a, b]$ and α_i $(i = 0, \dots, k+1)$ be "$k+2$" points such that $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1, \dots, k)$ and $\alpha_{k+1} = b$. If $f : [a, b] \to \mathbb{R}$ is of bounded variation, then we have the inequality,*

$$\begin{aligned}
&\left| \int_a^b f(x)\,dx - \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) \right| \\
&\quad \le \left[\frac{1}{2}\nu(h) + \max\left\{ \left| \alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right|,\ i = 0, \dots, k-1 \right\} \right] \bigvee_a^b (f) \qquad (1.1) \\
&\quad \le \nu(h) \bigvee_a^b (f),
\end{aligned}$$

where $\nu(h) := \max\{h_i \mid i = 0, \dots, k-1\}$, $h_i := x_{i+1} - x_i$ $(i = 0, \dots, k-1)$, and $\bigvee_a^b(f)$ is the total variation of f on the interval $[a, b]$.

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It is obvious that if we consider that $f : [a,b] \to \mathbb{R}$ is a monotonic mapping, then $\bigvee_a^b(f) = |f(b) - f(a)|$ and (1.1) becomes

$$\begin{aligned} &\left| \int_a^b f(x)\,dx - \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) \right| \\ &\leq \left[\frac{1}{2}\nu(h) + \max\left\{ \left| \alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right|,\ i = 0, \dots, k-1 \right\} \right] |f(b) - f(a)| \\ &\leq \nu(h) |f(b) - f(a)|. \end{aligned} \tag{1.2}$$

In the same paper [1], the author observed that the best inequality that could be obtained from (1.1) is that one for which $\alpha_{i+1} = (x_i + x_{i+1})/2$, $i = 0, \dots, k-1$, i.e., the following.

COROLLARY 1. *Let f and I_k be as in Theorem 1. Then, we have the inequality,*

$$\begin{aligned} \left| \int_a^b f(x)\,dx - \frac{1}{2}\left[(x_1 - a) f(a) + \sum_{i=1}^{k} (x_{i+1} - x_{i-1}) f(x_i) + (b - x_{n-1}) f(b) \right] \right| \\ \leq \frac{1}{2}\nu(h) \bigvee_a^b (f). \end{aligned} \tag{1.3}$$

In this case, if f is monotonic, obviously we can state that

$$\begin{aligned} \left| \int_a^b f(x)\,dx - \frac{1}{2}\left[(x_1 - a) f(a) + \sum_{i=1}^{k-1} (x_i - x_{i-1}) f(x_i) + (b - x_{n-1}) f(b) \right] \right| \\ \leq \frac{1}{2}\nu(h) |f(b) - f(a)|. \end{aligned} \tag{1.4}$$

If we consider the practical case where I_k is equidistant, i.e., let

$$I_k : x_i = a + (b-a)\frac{i}{k} \qquad (i = 0, \dots, k),$$

then, with f is as in Theorem 1, we have the inequality

$$\begin{aligned} \left| \int_a^b f(x)\,dx - \left[\frac{1}{k} \cdot \frac{f(a) + f(b)}{2}(b-a) + \frac{b-a}{k} \sum_{i=1}^{k-1} f\left[\frac{(k-i)a + ib}{k} \right] \right] \right| \\ \leq \frac{1}{2k}(b-a) \bigvee_a^b (f). \end{aligned} \tag{1.5}$$

If in this inequality, we assume that f is monotonic, then we can state that

$$\begin{aligned} \left| \int_a^b f(x)\,dx - \left[\frac{1}{k} \cdot \frac{f(a) + f(b)}{2}(b-a) + \frac{b-a}{k} \sum_{i=1}^{k-1} f\left[\frac{(k-i)a + ib}{k} \right] \right] \right| \\ \leq \frac{1}{2k}(b-a) |f(b) - f(a)|. \end{aligned} \tag{1.6}$$

For a comprehensive list of results related to, or generalising the above, see [3,4] where further references are given.

The main aim of this paper is to point out an improvement of the inequality (1.2) for monotonic mappings and, subsequently, for the particular cases (1.4) and (1.6). Applications for quadrature formulae will be given as well.

2. SOME INTEGRAL INEQUALITIES

We start with the following result.

THEOREM 2. *Let $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ be a division of the interval $[a, b]$, α_i $(i = 0, \dots, k+1)$ be "$k+2$" points such that $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1, \dots, k)$, and $\alpha_{k+1} = b$. If $f : [a, b] \to \mathbb{R}$ is monotonic nondecreasing on $[a, b]$, then we have the inequality,*

$$\begin{aligned} &\left| \int_a^b f(x)\, dx - \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) \right| \\ &\quad \le \sum_{i=0}^{k-1} \left[(x_{i+1} - \alpha_{i+1}) f(x_{i+1}) - (\alpha_{i+1} - x_i) f(x_i) \right] \\ &\qquad + \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} \operatorname{sgn}(\alpha_{i+1} - t) f(t)\, dt \\ &\quad \le \sum_{i=0}^{k-1} (x_{i+1} - \alpha_{i+1}) \left[f(x_{i+1}) - f(\alpha_{i+1}) \right] \\ &\qquad + \sum_{i=0}^{k-1} (\alpha_{i+1} - x_i) \left[f(\alpha_{i+1}) - f(x_i) \right] \\ &\quad \le \max_{i=\overline{0,k-1}} \left[\frac{1}{2} h_i + \left| \alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right| \right] [f(b) - f(a)] \\ &\quad \le \left[\frac{1}{2} \nu(h) + \max_{i=\overline{0,k-1}} \left| \alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right| \right] [f(b) - f(a)] \\ &\quad \le \nu(h) [f(b) - f(a)], \end{aligned} \tag{2.1}$$

where $h_i := x_{i+1} - x_i$ $(i = 0, \dots, k-1)$ and $\nu(h) = \max_{i=\overline{0,k-1}} h_i$.

PROOF. Consider the mapping $K : [a, b] \to \mathbb{R}$ given by (see also [1])

$$K(t) := \begin{cases} t - \alpha_1, & t \in [a, x_1), \\ t - \alpha_2, & t \in [x_1, x_2), \\ \quad \vdots \\ t - \alpha_{k-1}, & t \in [x_{k-2}, x_{k-1}), \\ t - \alpha_k, & t \in [x_{k-1}, b]. \end{cases}$$

Integrating by parts in the Riemann-Stieltjes integral, we deduce [1]

$$\int_a^b f(t)\, dt = \sum_{i=0}^{k-1} (\alpha_{i+1} - \alpha_i) f(x_i) - \int_a^b K(t)\, df(t). \tag{2.2}$$

It is well known that if $p : [c, d] \to \mathbb{R}$, $m : [c, d] \to \mathbb{R}$ are such that m is monotonic decreasing and p is continuous on $[c, d]$, then p is Riemann-Stieltjes integrable with respect to m on $[c, d]$ and

$$\left| \int_c^d p(x)\, dm(x) \right| \le \int_c^d |p(x)|\, dm(x). \tag{2.3}$$

Therefore, applying this property on each subinterval $[x_i, x_{i+1}]$, we can state that

$$\begin{aligned}
\left|\int_a^b K(t)\, df(t)\right| &= \left|\sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}} K(t)\, df(t)\right| \le \sum_{i=0}^{k-1}\left|\int_{x_i}^{x_{i+1}} K(t)\, df(t)\right| \\
&\le \sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}} |K(t)|\, df(t) = \sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}} |t-\alpha_{i+1}|\, df(t) \\
&= \sum_{i=0}^{k-1}\left[\int_{x_i}^{\alpha_{i+1}} (\alpha_{i+1}-t)\, df(t) + \int_{\alpha_{i+1}}^{x_{i+1}} (t-\alpha_{i+1})\, df(t)\right] \\
&= \sum_{i=0}^{k-1}\left[(\alpha_{i+1}-t) f(t)\Big|_{x_i}^{\alpha_{i+1}} + \int_{x_i}^{\alpha_{i+1}} f(t)\, dt\right. \\
&\quad \left. + (t-\alpha_{i+1}) f(t)\Big|_{\alpha_{i+1}}^{x_{i+1}} - \int_{\alpha_{i+1}}^{x_{i+1}} f(t)\, dt\right] \\
&= \sum_{i=0}^{k-1}\left[-(\alpha_{i+1}-x_i) f(x_i) + (x_{i+1}-\alpha_{i+1}) f(x_{i+1})\right. \\
&\quad \left. + \int_{x_i}^{x_{i+1}} \operatorname{sgn}(\alpha_{i+1}-t) f(t)\, dt\right] \\
&= \sum_{i=0}^{k-1}\left[(x_{i+1}-\alpha_{i+1}) f(x_{i+1}) - (\alpha_{i+1}-x_i) f(x_i)\right] \\
&\quad + \sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}} \operatorname{sgn}(\alpha_{i+1}-t) f(t)\, dt
\end{aligned} \tag{2.4}$$

and the first inequality in (2.1) is proved.

To prove the second inequality in (2.1), we observe that since f is monotonic decreasing on $[a, b]$, hence, we can state that

$$\int_{x_i}^{\alpha_{i+1}} f(t)\, dt \le (\alpha_{i+1}-x_i) f(\alpha_{i+1}), \qquad i = 0, \ldots, n-1, \tag{2.5}$$

and

$$\int_{\alpha_{i+1}}^{x_{i+1}} f(t)\, dt \ge (x_{i+1}-\alpha_{i+1}) f(\alpha_{i+1}), \qquad i = 0, \ldots, n-1.$$

That is,

$$-\int_{\alpha_{i+1}}^{x_{i+1}} f(t)\, dt \le -(x_{i+1}-\alpha_{i+1}) f(\alpha_{i+1}) \tag{2.6}$$

and then, by (2.5) and (2.6), we have

$$\begin{aligned}
&\sum_{i=0}^{k-1}\left[-(\alpha_{i+1}-x_i) f(x_i) + (x_{i+1}-\alpha_{i+1}) f(x_{i+1}) + \int_{x_i}^{\alpha_{i+1}} f(t)\, dt - \int_{\alpha_{i+1}}^{x_{i+1}} f(t)\, dt\right] \\
&\quad \le \sum_{i=0}^{k-1}\left[-(\alpha_{i+1}-x_i) f(x_i) + (x_{i+1}-\alpha_{i+1}) f(x_{i+1})\right. \\
&\quad \left. + (\alpha_{i+1}-x_i) f(\alpha_{i+1}) - (x_{i+1}-\alpha_{i+1}) f(\alpha_{i+1})\right] \\
&\quad = \sum_{i=0}^{k-1}(x_{i+1}-\alpha_{i+1})\left[f(x_{i+1}) - f(\alpha_{i+1})\right] + \sum_{i=0}^{k-1}(\alpha_{i+1}-x_i)\left[f(\alpha_{i+1}) - f(x_i)\right]
\end{aligned}$$

and the second inequality in (2.1) is proved.

For the last inequality, we observe that

$$
\begin{aligned}
&(x_{i+1}-\alpha_{i+1})\left(f(x_{i+1})-f(\alpha_{i+1})\right)+(\alpha_{i+1}-x_i)\left(f(\alpha_{i+1})-f(x_i)\right)\\
&\quad\le \max\left\{x_{i+1}-\alpha_{i+1},\alpha_{i+1}-x_i\right\}\left[f(x_{i+1})-f(\alpha_{i+1})+f(\alpha_{i+1})-f(x_i)\right]\\
&\quad=\left[\frac{1}{2}h_i+\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|\right]\left(f(x_{i+1})-f(x_i)\right)
\end{aligned}
$$

and then

$$
\begin{aligned}
&\sum_{i=0}^{k-1}\left[(x_{i+1}-\alpha_{i+1})\left(f(x_{i+1})-f(\alpha_{i+1})\right)+(\alpha_{i+1}-x_i)\left(f(\alpha_{i+1})-f(x_i)\right)\right]\\
&\quad\le \max_{i=\overline{0,k-1}}\left[\frac{1}{2}h_i+\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|\right]\sum_{i=0}^{k-1}\left[f(x_{i+1})-f(x_i)\right]\\
&\quad= \max_{i=\overline{0,k-1}}\left[\frac{1}{2}h_i+\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|\right]\left[f(b)-f(a)\right]\\
&\quad\le \left[\frac{1}{2}\nu(h)+\max_{i=\overline{0,k-1}}\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|\right]\left[f(b)-f(a)\right]\\
&\quad\le \nu(h)\left[f(b)-f(a)\right].
\end{aligned}
$$

The theorem is completely proved.

Now, if we assume that the points of the division I_k are given, then the best inequality we can get from Theorem 2 is embodied in the following corollary.

COROLLARY 2. *Let f, I_k be as above. Then, we have the inequality,*

$$
\begin{aligned}
&\left|\int_a^b f(x)\,dx-\frac{1}{2}\left[(x_1-a)f(a)+\sum_{i=1}^{k-1}(x_{i+1}-x_{i-1})f(x_i)+(b-x_{n-1})f(b)\right]\right|\\
&\quad\le\frac{1}{2}\sum_{i=0}^{k-1}h_i\Delta f(x_i)+\sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}}\operatorname{sgn}\left(\frac{x_i+x_{i+1}}{2}-t\right)f(t)\,dt\\
&\quad\le\frac{1}{2}\sum_{i=0}^{k-1}h_i\Delta f(x_i)\le\frac{1}{2}\nu(h)\left[f(b)-f(a)\right],
\end{aligned}
\tag{2.7}
$$

where $h_i := x_{i+1}-x_i$ and $\Delta f(x_i)=f(x_{i+1})-f(x_i)$, $i=0,\dots,k-1$.

PROOF. We choose in Theorem 2, $\alpha_{i+1}=(x_i+x_{i+1})/2$, $i=0,\dots,k-1$, to obtain

$$
\begin{aligned}
\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i)&=\left(\frac{a+x_1}{2}-a\right)f(a)+\left(\frac{x_1+x_2}{2}-\frac{a+x_1}{2}\right)f(x_1)\\
&\quad+\cdots+\left(\frac{x_{k-1}+b}{2}-\frac{x_{k-2}+x_{k-1}}{2}\right)f(x_{k-1})+\left(b-\frac{x_{k-1}+b}{2}\right)f(b)\\
&=\frac{1}{2}\left[(x_1-a)f(a)+\sum_{i=1}^{k-1}(x_{i+1}-x_{i-1})f(x_i)+(b-x_{n-1})f(b)\right].
\end{aligned}
$$

Now, (2.7) follows immediately from (2.1) and we omit the details.

The case of equidistant partitioning is important in practice.

COROLLARY 3. *Let $I_k : x_i = a + i \cdot (b-a)/k$ $(i = 0, \dots, k)$ be an equidistant partitioning of $[a,b]$. If f is as above, then we have the inequality,*

$$\begin{aligned} &\left| \int_a^b f(x)\,dx - \left[\frac{1}{k} \cdot \frac{f(a)+f(b)}{2}(b-a) + \frac{b-a}{k}\sum_{i=1}^{k-1} f\left[\frac{(k-i)a+ib}{k}\right]\right]\right| \\ &\le \frac{1}{2}\cdot\frac{(b-a)}{k}[f(b)-f(a)] + \sum_{i=0}^{k-1}\int_{a+i\cdot(b-a/k)}^{a+(i+1)\cdot(b-a/k)} \operatorname{sgn}\left(a+\frac{2i+1}{2}\cdot\frac{b-a}{k}-t\right) f(t)\,dt \\ &\le \frac{1}{2k}(b-a)[f(b)-f(a)]. \end{aligned} \tag{2.8}$$

3. THE CONVERGENCE OF A GENERAL QUADRATURE FORMULA

Let $\Delta_n : a = x_0^{(n)} < x_1^{(n)} < \cdots < x_{n-1}^{(n)} < x_n^{(n)} = b$ be a sequence of divisions of $[a,b]$ and consider the sequence of numerical integration formulae

$$I_n\left(f, \Delta_n, \mathbf{w}_n\right) := \sum_{j=0}^{n} w_j^{(n)} f(x_j^{(n)}),$$

where $w_j^{(n)}$ $(j = 0, \dots, n)$ are the quadrature weights with the property that $\sum_{j=0}^{n} w_j^{(n)} = b - a$. The following theorem contains a sufficient condition for the weights $w_j^{(n)}$ such that $I_n(f, \Delta_n, \mathbf{w}_n)$ approximates the integral $\int_a^b f(x)\,dx$ with an error expressed in terms of the difference $f(b) - f(a)$ and the norm of the division Δ_n.

THEOREM 3. *Let $f : [a,b] \to \mathbb{R}$ be a mapping that is monotonic nondecreasing on $[a,b]$. If the quadrature weights $w_j^{(n)}$ $(j = 0, \dots, n)$ satisfy the condition,*

$$x_i^{(n)} - a \le \sum_{j=0}^{i} w_j^{(n)} \le x_{i+1}^{(n)} - a, \qquad \text{for all } i = 0, \dots, n-1; \tag{3.1}$$

then we have the estimate

$$\begin{aligned} &\left| I_n(f, \Delta_n, \mathbf{w}_n) - \int_a^b f(x)\,dx \right| \\ &\le \sum_{i=0}^{n-1}\left[\left(x_{i+1}^{(n)} - a - \sum_{j=0}^{i} w_j^{(n)}\right) f\left(x_{i+1}^{(n)}\right) \right. \\ &\quad \left. - \left(a + \sum_{j=0}^{i} w_j^{(n)} - x_i^{(n)}\right) f\left(x_i^{(n)}\right)\right] \\ &\quad + \sum_{i=0}^{n-1}\int_{x_i^{(n)}}^{x_{i+1}^{(n)}} \operatorname{sgn}\left(a + \sum_{j=0}^{i} w_j^{(n)} - t\right) f(t)\,dt \\ &\le \sum_{i=0}^{n-1}\left[\left(x_{i+1}^{(n)} - a - \sum_{j=0}^{i} w_j^{(n)}\right)\left(f\left(x_{i+1}^{(n)}\right) - f\left(a + \sum_{j=0}^{i} w_j^{(n)}\right)\right)\right] \\ &\quad + \sum_{i=0}^{n-1}\left(a + \sum_{j=0}^{i} w_j^{(n)} - x_i^{(n)}\right)\left[f\left(a + \sum_{j=0}^{i} w_j^{(n)}\right) - f\left(x_i^{(n)}\right)\right] \end{aligned} \tag{3.2}$$

$$\le \max_{i=\overline{0,n-1}} \left[\frac{1}{2} h_i^{(n)} + \left| a + \sum_{j=0}^{i} w_j^{(n)} - \frac{x_i^{(n)} + x_{i+1}^{(n)}}{2} \right| \right] [f(b) - f(a)]$$

$$\le \left[\frac{1}{2} \nu \left(h^{(n)} \right) + \max_{i=\overline{0,n-1}} \left| a + \sum_{j=0}^{i} w_j^{(n)} - \frac{x_i^{(n)} + x_{i+1}^{(n)}}{2} \right| \right] [f(b) - f(a)] \quad (3.2)(\text{cont.})$$

$$\le \nu \left(h^{(n)} \right) [f(b) - f(a)],$$

where $\nu(h^{(n)}) := \max \left\{ h_i^{(n)} : i = 0, \ldots, n-1 \right\}$ *and* $h_i^{(n)} := x_{i+1}^{(n)} - x_i^{(n)}$, $i = 0, \ldots, n-1$. *Moreover, we have*

$$\lim_{\nu\left(h^{(n)}\right) \to 0} I_n(f, \Delta_n, \mathbf{w}_n) = \int_a^b f(x)\, dx$$

uniformly by the influence of the weights $\mathbf{w}_n$.

PROOF. Define the sequence of real numbers,

$$\alpha_0^{(n)} = a,$$

$$\alpha_{i+1}^{(n)} := a + \sum_{j=0}^{i} w_j^{(n)}, \qquad i = 0, \ldots, n.$$

Note that

$$\alpha_{n+1}^{(n)} = a + \sum_{j=0}^{n} w_j^{(n)} = a + b - a = b.$$

By the assumption (3.1), we have $\alpha_{i+1}^{(n)} \in [x_i^{(n)}, x_{i+1}^{(n)}]$ for all $i = 0, \ldots, n-1$. Observe that

$$\alpha_1^{(n)} - \alpha_0^{(n)} = w_0^{(n)},$$

$$\alpha_{i+1}^{(n)} - \alpha_i^{(n)} = a + \sum_{j=0}^{i} w_j^{(n)} - a - \sum_{j=1}^{i-1} w_j^{(n)} = w_j^{(n)} \qquad (i = 1, \ldots, n-1)$$

and

$$\alpha_{n+1}^{(n)} - \alpha_n^{(n)} = a + \sum_{j=0}^{n} w_j^{(n)} - a - \sum_{j=0}^{n-1} w_j^{(n)} = w_n^{(n)}.$$

Consequently,

$$\sum_{i=0}^{n} \left(\alpha_{i+1}^{(n)} - \alpha_i^{(n)} \right) f\left(x_i^{(n)} \right) = \sum_{i=0}^{n} w_i^{(n)} f\left(x_i^{(n)} \right) = I_n(f, \Delta_n, \mathbf{w}_n).$$

Applying the inequality (2.1), we deduce the estimate (3.2). The limit follows by the last inequality in (3.2).

The case where the partitioning is equidistant is important in practice. Consider then, the partitioning,

$$E_n : x_i^{(n)} = a + i \cdot \frac{b-a}{n} \qquad (i = 0, \ldots, n),$$

and define the sequence of numerical quadrature formulae,

$$I_n(f, \mathbf{w}_n) := \sum_{i=0}^{n} w_i^{(n)} f\left(a + i \cdot \frac{b-a}{n} \right), \qquad \sum_{i=0}^{n} w_i^{(n)} = b - a.$$

The following result holds.

COROLLARY 4. *Let $f : [a,b] \to \mathbb{R}$ be monotonic nondecreasing on $[a,b]$. If the quadrature weights satisfy the estimate,*

$$\frac{i}{n} \le \frac{1}{b-a} \sum_{j=0}^{i} w_j^{(n)} \le \frac{i+1}{n} \qquad (i = 0, \dots, n-1),$$

then we have the inequality,

$$\begin{aligned}
&\left| I_n(f, \mathbf{w}_n) - \int_a^b f(x)\, dx \right| \\
&\le \sum_{i=0}^{n-1} \left[\left(\frac{i+1}{n} \cdot (b-a) - \sum_{j=0}^{i} w_j^{(n)} \right) f\left(a + \frac{i+1}{n} \cdot (b-a) \right) \right. \\
&\quad \left. - \left(\sum_{j=0}^{i} w_j^{(n)} - \frac{i}{n}(b-a) \right) f\left(a + \frac{i}{n}(b-a) \right) \right] \\
&\quad + \sum_{i=0}^{n-1} \int_{a+\frac{i}{n}(b-a)}^{a+\frac{i+1}{n}\cdot(b-a)} \operatorname{sgn}\left(a + \sum_{j=0}^{i} w_j^{(n)} - t \right) f(t)\, dt \\
&\le \sum_{i=0}^{n-1} \left[\frac{i+1}{n} \cdot (b-a) - \sum_{j=0}^{i} w_j^{(n)} \right] \left[f\left(a + \frac{i+1}{n}(b-a) \right) - f\left(a + \sum_{j=0}^{i} w_j^{(n)} \right) \right] \\
&\quad + \sum_{i=0}^{n-1} \left[\sum_{j=0}^{i} w_j^{(n)} - \frac{i}{n}(b-a) \right] \left[f\left(a + \sum_{j=0}^{i} w_j^{(n)} \right) - f\left(a + \frac{i}{n}(b-a) \right) \right] \\
&\le \left[\frac{1}{2} \cdot \frac{b-a}{n} + \max_{i=\overline{0,n-1}} \left| \sum_{j=0}^{i} w_j^{(n)} - \frac{2i+1}{2} \cdot \frac{b-a}{n} \right| \right] [f(b) - f(a)] \\
&\le \frac{b-a}{n} [f(b) - f(a)].
\end{aligned}$$

Moreover, we have the limit

$$\lim_{n\to\infty} I_n(f, \mathbf{w}_n) = \int_a^b f(x)\, dx$$

uniformly by the influence of $\mathbf{w}_n$.

4. SOME PARTICULAR INEQUALITIES

In the section we point out some particular inequalities which generate classical results such as: the rectangle inequality, trapezoid inequality, Ostrowski's inequality, midpoint inequality, Simpson's inequality, and others for monotonic nondecreasing mappings.

PROPOSITION 1. *Let $f : [a,b] \to \mathbb{R}$ be monotonic nondecreasing on $[a,b]$. Then, we have the inequality [5],*

$$\begin{aligned}
&\left| \int_a^b f(x)\, dx - [(\alpha - a) f(a) + (b - \alpha) f(b)] \right| \\
&\le (b-\alpha) f(b) - (\alpha - a) f(a) + \int_a^b \operatorname{sgn}(\alpha - t) f(t)\, dt \\
&\le (b-\alpha)[f(b) - f(\alpha)] + (\alpha - a)[f(\alpha) - f(a)] \\
&\le \left[\frac{1}{2}(b-a) + \left| \alpha - \frac{a+b}{2} \right| \right] [f(b) - f(a)].
\end{aligned} \tag{4.1}$$

for all $\alpha \in [a,b]$.

PROOF. Follows from Theorem 2 by choosing $x_0 = a$, $x_1 = b$, $\alpha_0 = a$, $\alpha_1 = \alpha \in [a,b]$ and $\alpha_2 = b$.

REMARK 1. If, in (4.1), we put $\alpha = (b+a)/2$, then we get the "trapezoid inequality" as noted in [5],

$$\begin{aligned} \left|\int_a^b f(x)\,dx - (b-a)\frac{f(a)+f(b)}{2}\right| &\le \frac{b-a}{2}[f(b)-f(a)] + \int_a^b \operatorname{sgn}\left(\frac{a+b}{2}-t\right) f(t)\,dt \\ &\le \frac{1}{2}(b-a)[f(b)-f(a)]. \end{aligned} \tag{4.2}$$

Another particular integral inequality with many applications is the following one.

PROPOSITION 2. *Let $f:[a,b]\to\mathbb{R}$ be as above. Then, we have the inequality established in [6].*

$$\begin{aligned} &\left|\int_a^b f(x)\,dx - [(\alpha_1-a)f(a)+(\alpha_2-\alpha_1)f(x_1)+(b-\alpha_2)f(b)]\right| \\ &\quad\le (b-\alpha_2)f(b) + 2\left(x_1-\frac{\alpha_1+\alpha_2}{2}\right)f(x_1) - (\alpha_1-a)f(a) \\ &\qquad + \int_a^{x_1}\operatorname{sgn}(\alpha_1-t)f(t)\,dt + \int_{x_1}^b \operatorname{sgn}(\alpha_2-t)f(t)\,dt \\ &\quad\le (x_1-\alpha_1)[f(x_1)-f(\alpha_1)] + (b-\alpha_2)[f(b)-f(\alpha_2)] \\ &\qquad + (\alpha_1-a)[f(\alpha_1)-f(a)] + (\alpha_2-x_1)[f(\alpha_2)-f(x_1)] \\ &\quad\le \max\left\{\frac{1}{2}(x_1-a)+\left|\alpha_1+\frac{a+x_1}{2}\right|, \frac{1}{2}(b-x_1)+\left|\alpha_2+\frac{x_1+b}{2}\right|\right\}\times[f(b)-f(a)] \\ &\quad\le \frac{1}{2}\left[\max\{x_1-a, b-x_1\} + \max\left\{\left|\alpha_1+\frac{a+x_1}{2}\right|, \left|\alpha_2+\frac{x_1+b}{2}\right|\right\}\right]\times[f(b)-f(a)], \end{aligned} \tag{4.3}$$

provided that $a \le \alpha_1 \le x_1 \le \alpha_2 \le b$.

PROOF. Follows by Theorem 2 on choosing the division $a = x_0 \le x_1 \le x_2 = b$ and the numbers $\alpha_0 = a$, $\alpha_1 \in [a,x_1]$, $\alpha_2 \in [x_1,b]$, $\alpha_3 = b$.

REMARK 2.

(a) If, in (4.3), we choose $\alpha_2 = b$, $\alpha_1 = a$, then we get the inequality obtained in [7]

$$\begin{aligned} &\left|\int_a^b f(x)\,dx - (b-a)f(x_1)\right| \\ &\quad\le 2\left(x_1-\frac{a+b}{2}\right) + \int_a^{x_1}\operatorname{sgn}(a-t)f(t)\,dt + \int_{x_1}^b \operatorname{sgn}(b-t)f(t)\,dt \\ &\quad\le (x_1-a)[f(x_1)-f(a)] + (b-x_1)[f(b)-f(x_1)] \\ &\quad\le \left[\frac{1}{2}(b-a)+\left|x_1-\frac{a+b}{2}\right|\right][f(b)-f(a)] \end{aligned} \tag{4.4}$$

for all $x_1 \in [a,b]$.

(b) If, in (4.3), we choose $\alpha_2 = \alpha_1 = \alpha\ (= x_1) \in [a,b]$, then we get (4.1).

(c) If we choose $x_1 = (a+b)/2$ in (4.3), then we obtain, for $a \le \alpha_1 \le (a+b)/2 \le \alpha_2 \le b$

$$\begin{aligned} &\left|\int_a^b f(x)\,dx - \left[(\alpha_1-a)f(a)+(\alpha_2-\alpha_1)f\left(\frac{a+b}{2}\right)+(b-\alpha_2)f(b)\right]\right| \\ &\quad\le (b-\alpha_2)f(b) + (a+b-\alpha_1-\alpha_2)f\left(\frac{a+b}{2}\right) - (\alpha_1-a)f(a) \end{aligned} \tag{4.5}$$

$$
\begin{aligned}
&+\int_a^{(a+b)/2} \operatorname{sgn}(\alpha_1 - t) f(t)\, dt + \int_{(a+b)/2}^b \operatorname{sgn}(\alpha_2 - t) f(t)\, dt \\
&\le \left(\frac{a+b}{2} - \alpha_1\right)\left[f\left(\frac{a+b}{2}\right) - f(\alpha_1)\right] + (b-\alpha_2)\left[f(b) - f(\alpha_2)\right] \\
&\quad + (\alpha_1 - a)\left[f(\alpha_1) - f(a)\right] + \left(\alpha_2 - \frac{a+b}{2}\right)\left[f(\alpha_2) - f\left(\frac{a+b}{2}\right)\right] \\
&\le \frac{1}{2}\left[(b-a) + \max\left\{\left|\alpha_1 - \frac{3a+b}{4}\right|, \left|\alpha_2 - \frac{a+3b}{4}\right|\right\}\right]\left[f(b) - f(a)\right].
\end{aligned} \tag{4.5)(cont.}
$$

It is obvious that the best inequality we can obtain from (4.5) is the one for which $\alpha_1 = (3a+b)/4$ and $\alpha_2 = (a+3b)/4$, getting

$$
\begin{aligned}
&\left|\int_a^b f(x)\, dx - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\right| \\
&\quad \le \frac{(b-a)\left(f(b) - f(a)\right)}{4} + \int_a^{(a+b)/2} \operatorname{sgn}\left(\frac{3a+b}{2} - t\right) f(t)\, dt \\
&\quad + \int_{(a+b)/2}^b \operatorname{sgn}\left(\frac{a+3b}{4} - t\right) f(t)\, dt \\
&\quad \le \frac{b-a}{4}\left[f(b) - f(a)\right].
\end{aligned} \tag{4.6}
$$

If in (4.5) we choose $\alpha_1 = (5a+b)/6$, $\alpha_2 = (a+5b)/6$, then we get Simpson's inequality (see also [6]),

$$
\begin{aligned}
&\left|\int_a^b f(x)\, dx - \frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \\
&\quad \le \frac{(b-a)\left(f(b) - f(a)\right)}{6} + \int_a^{(a+b)/2} \operatorname{sgn}\left(\frac{5a+b}{6} - t\right) f(t)\, dt \\
&\quad + \int_{(a+b)/2}^b \operatorname{sgn}\left(\frac{a+5b}{6} - t\right) f(t)\, dt \\
&\quad \le \frac{b-a}{6}\left[f(b) - f(a) + f\left(\frac{a+5b}{6}\right) - f\left(\frac{5a+b}{6}\right)\right] \\
&\quad \le \frac{11}{8}(b-a)\left[f(b) - f(a)\right].
\end{aligned} \tag{4.7}
$$

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