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A remark on Liouville's formula on small time scales ☆

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Abstract

We present a new proof of the Liouville formula for a d-dimensional linear dynamic system $x^{\Delta} = A(t)x$ on a time scale $\mathbb{T}$, where $\mathbb{T}$ is in a sense small. Our proof demonstrates that Liouville's formula on small time scales is a direct consequence of its well-known counterpart for ordinary differential equations.

Keywords: Time scales; Liouville's formula; Unification of difference and differential calculus

1. Introduction

The development of the theory of time scales was initiated by Hilger in his PhD thesis in 1988 as a theory capable to contain both difference and differential calculus in a consistent way. Since then we have been witnesses of great efforts in the field of time scales, especially in unifying the theory of differential equations and the theory of difference equations (see monographs [3,4]), where many results important in the theory of ordinary differential equations have been already proved in the time scale setting. One of these results is the

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Liouville formula concerning the time development of the phase space under the flow of a linear differential system

$$\dot{x} = A(t)x, \qquad {}^{\cdot} := \frac{d}{dt}.$$

Both monographs [3,4] contain only proofs for $d = 2$, the general Liouville's formula for a (generalized) time scale was proved by Cormani [5].

In this article we give another proof of the general Liouville's formula for linear dynamic system on a time scales which are in a sense small. Therefore the result itself is hardly surprising. More interesting is the idea of the proof—we use the idea of embedding or more explicitly the idea that solutions of dynamic systems on small time scales are nothing but restrictions of solutions of suitable ordinary differential equations.

2. Hypotheses and auxiliary results

Throughout this paper we shall use the standard notation widely used in the theory of ordinary differential equations (e.g., [6]). A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. The most prominent examples are $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, but of course time scales can be much more complicated objects, e.g., the well-known middle third Cantor set is a time scale.

For any time scale $\mathbb{T}$ we define the *graininess* function $\mu : \mathbb{T} \to [0, \infty)$ by

$$\mu(t) := \inf\{s \in \mathbb{T}\colon\ s > t\} - t,$$

so if $\mathbb{T} = \mathbb{R}$, then $\mu(t) \equiv 0$ and if $\mathbb{T} = \mathbb{Z}$, then $\mu(t) \equiv 1$, a time scale could have nonconstant graininess.

We say that a function $f : \mathbb{T} \to \mathbb{R}$ is *delta differentiable* at $t \in \mathbb{T}$, if there exists a real number, denoted as $f^{\Delta}(t)$, called the *delta derivative* of f at t, such that for all $\varepsilon > 0$ in a neighbourhood $\Omega(t)$ of t,

$$\bigl| f\bigl(t + \mu(t)\bigr) - f(s) - f^{\Delta}(t)\bigl(t + \mu(t) - s\bigr) \bigr| \leqslant \varepsilon \bigl| t + \mu(t) - s \bigr|,$$

for every $s \in \Omega(t)$. Let f be differentiable at t;

(1) if $\mu(t) = 0$, then $f^{\Delta}(t) = \lim_{s \to t, s \in \mathbb{T}} [f(t) - f(s)]/(t - s)$,
(2) if $\mu(t) \neq 0$, then $f^{\Delta}(t) = [f(t + \mu(t)) - f(t)]/\mu(t)$,
(3) $f(t + \mu(t)) = f(t) + \mu(t) f^{\Delta}(t)$.

Any interval in $\mathbb{T}$ will be denoted by the subscript $\mathbb{T}$, e.g., $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T}\colon\ a \leqslant t \leqslant b\}$. If $\mathcal{S}$ is the system of all left closed and right open intervals $[a, b)_{\mathbb{T}}$ (we put $[a, a)_{\mathbb{T}} = \emptyset$), then the set function $\xi : \mathcal{S} \to [0, \infty]$, $\xi([a, b)_{\mathbb{T}}) := b - a$ is a countably additive measure on $\mathcal{S}$. The standard Carathéodory extension of ξ yields the *Lebesgue delta measure* on $\mathbb{T}$, so it is possible to work with the *Lebesgue delta integral* of f on a set $A \subseteq \mathbb{T}$ which is usually denoted as $\int_A f(t)\,\Delta t$. Many theorems of the abstract Lebesgue integration theory holds [4].

For a time scale $\|\mathbb{T}\|$, we define its *norm* as $\|\mathbb{T}\| := \sup\{\mu(t)\colon\ t \in \mathbb{T}\}$. Clearly $\|\mathbb{T}\| \in [0, +\infty]$, e.g., $\|\mathbb{R}\| = 0$, $\|\mathbb{Z}\| = 1$, if $a, b > 0$, then $\| \bigcup_{k=1}^{\infty} [k(a + b), k(a + b) + a] \| = b$, and $\|\{n^2\colon\ n \in \mathbb{N}\}\| = +\infty$.

Moreover, we shall use $\mathrm{Mat}(m \times n)$ for the set of all (real) $m \times n$ matrices, I_n for the n-dimensional identity matrix, and $\|A\|$ for the spectral norm of matrix A. That is $\|A\|$ is the square root of the largest eigenvalue of A^*A, where A^* is the conjugate transpose of A. To emphasize that $A(t)$ is a matrix function depending on t, we shall write $A(\cdot)$ in expressions like $e_{A(\cdot)}(t,s)$. Eigenvalues of A will be denoted as $\lambda(A)$, respectively $\lambda_i(A)$, so $\|A\| = \sqrt{\max_{i=1,\dots,n}\{\lambda_i(A^*A)\}}$.

Throughout this paper we shall work with analytic functions H and $\hat{H}$ (see the proof of our main Theorem 3.1) which are derived from the following function $h : \mathbb{R} \times \mathrm{Mat}(d \times d) \to \mathrm{Mat}(d \times d)$,

$$h(t,A) := A - \frac{t}{2}A^2 + \frac{t^2}{3}A^3 + \cdots + (-1)^n \frac{t^n}{n+1}A^{n+1} + \cdots. \tag{1}$$

This means, we need to ensure that the infinite series (1) is convergent. Since we shall consider effectively only the case when $0 \leqslant t < \|\mathbb{T}\|$, the easiest way to achieve the convergence of (1) is via the following hypothesis:

(H) *The time scale* $\mathbb{T}$ *is sufficiently small in the sense that* $\|\mathbb{T}\| < \|A\|^{-1}$.

Clearly if $A = 0$, then the hypothesis (H) holds trivially. Because the standard existence and uniqueness theorem for the initial value problem

$$x^{\Delta} = A(t)x, \qquad x(t_0) = x_0 \in \mathbb{R}^d, \quad t, t_0 \in \mathbb{T},$$

on $\mathbb{T}$ [3, Theorem 5.8] requires *regressivity* of the matrix $A(t)$, that is the invertibility of

$$I_d + \mu(t)A(t),$$

for all $t \in \mathbb{T}$, the hypothesis (H) is very natural in the setting of dynamic equations on time scales as we can see from the following lemma.

Lemma 2.1. *Let* $\mathbb{T}$ *be a time scale, and* $A(t)$ *be a* $d \times d$ *matrix function on* $\mathbb{T}$ *such that* (H) *holds. Then* $A(t)$ *is regressive on* $\mathbb{T}$.

Proof. Since $\|\mathbb{T}\| < \|A(t)\|^{-1}$ on $\mathbb{T}$, the spectral radius ϱ of $\mu(t)A(t)$ satisfies

$$\varrho\big(\mu(t)A(t)\big) \leqslant \big\|\mu(t)A(t)\big\| = \mu(t)\big\|A(t)\big\| \leqslant \|\mathbb{T}\| \cdot \|A\| < 1,$$

and any eigenvalue λ of $I_d + \mu(t)A(t)$ satisfies

$$\Re\lambda\big(I_d + \mu(t)A(t)\big) = 1 + \Re\lambda\big(\mu(t)A(t)\big) \geqslant 1 - \varrho\big(\mu(t)A(t)\big) > 0.$$

Therefore $A(t)$ is regressive on the time scale $\mathbb{T}$. $\square$

Moreover, for such small time scales

$$h(t,A) = \begin{cases} \log(I_d + tA)/t & \text{for } 0 < t < \|\mathbb{T}\|, \\ A & \text{for } t = 0, \end{cases}$$

so, the other way round, we may suppose that the function $t \mapsto \log(I_d + tA)/t$ is well-defined for $t \geqslant 0$ (in the sense of continuous extension) and write directly $h(t,A) = \log(I_d + tA)/t$ for $t \geqslant 0$ (so $h(0,A) = A$).

In our proof of Liouville's formula we shall use the concept of a normal partition of a time scale interval. Let $\mathbb{T}$ be a time scale. By a partition of an interval $[a,b]_{\mathbb{T}}$ we understand any finite ordered set $P := \{t_0, t_1, \ldots, t_n\} \subseteq [a,b]_{\mathbb{T}}$, where $a = t_0 < t_1 < \cdots < t_n = b$. It is possible to prove ([1] or [4]) the following lemma.

Lemma 2.2. *Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$ and $a \leqslant b$. Then for every $n \in \mathbb{N}$ there exists a partition $P_n := \{t_0, t_1, \ldots, t_{\omega(n)}\}$, $a = t_0 < t_1 < \cdots < t_{\omega(n)} = b$ such that*

$$\text{for each } i \in \{0, \ldots, \omega(n)-1\}$$
$$\text{either} \quad t_{i+1} - t_i \leqslant \frac{1}{n} \quad \text{or} \quad t_{i+1} - t_i > \frac{1}{n} \quad \text{and} \quad \sigma(t_i) = t_{i+1}. \tag{2}$$

We will call any partitions satisfying (2) *normal partitions* P_n. It is clear that if P_n is a normal partition of $[a,b]_{\mathbb{T}}$, then for any $t \in [a,b]_{\mathbb{T}}$ the set $P_n \cup \{t\}$ is again a normal partition of $[a,b]_{\mathbb{T}}$.

Because of this if $(P_n)_{n \geqslant 1}$ is a sequence of normal partitions of $[t_0, T]_{\mathbb{T}}$, then we may suppose that this sequence is *nondecreasing* in the sense that $P_n \subseteq P_{n+1}$. Of course, because $\mathbb{T} \cap [a,b]_{\mathbb{T}}$ could be a finite set, we cannot exclude the case when for some $n_0 \in \mathbb{N}$ this sequence is stationary, that is $P_{n_0+i} = P_{n_0}$ for $i = 0, 1, \ldots$.

3. Main result

Theorem 3.1 (Liouville's formula). *Let $\mathbb{T}$ be a time scale, $t_0, t \in \mathbb{T}$, $t_0 \leqslant t \leqslant T < \infty$, and $A(t)$ be a $d \times d$ matrix function continuous on $[t_0, T]_{\mathbb{T}}$ such that* (H) *holds. Then the solution $e_{A(\cdot)}(t, t_0)$ of an initial matrix time scale problem*

$$X^{\Delta} = A(t)X, \qquad X(t_0) = I_d, \quad t, t_0 \in \mathbb{T},$$

satisfies

$$\det\bigl(e_{A(\cdot)}(t, t_0)\bigr) = \prod_{i=1}^{d} e_{\lambda_i(A(s))}(T, t_0) = e_{(\lambda_1 \oplus \cdots \oplus \lambda_d)(A(\cdot))}(t, t_0),$$

where $\lambda_i(A(t))$, $i = 1, \ldots, d$, are the eigenvalues of the matrix $A(t)$.

Proof. It follows from Lemma 2.1, that $A(t)$ is regressive on $\mathbb{T}$, hence $e_{A(\cdot)}(t, t_0)$ is well-defined on $\mathbb{T}$. Clearly we may suppose that $t = T$.

Let $(P_n)_{n \geqslant 1}$ be a nondecreasing sequence of normal partitions $P_n := \{t_i^n\}_{i=0}^{\omega(n)}$ of $[t_0, T]_{\mathbb{T}}$. Then, according to [1, Theorem 3.1],

$$e_{A(\cdot)}(T, t_0) = \lim_{n\to\infty} e_{A(t^n_{\omega(n)-1})}\bigl(T, t^n_{\omega(n)-1}\bigr) \ldots e_{A(t_i^n)}\bigl(t^n_{i+1}, t^n_i\bigr) \ldots e_{A(t_0)}\bigl(t^n_1, t_0\bigr).$$

Hence

$$\begin{aligned}
\det\bigl(e_{A(\cdot)}(T, t_0)\bigr) &= \det\Bigl(\lim_{n\to\infty} \ldots e_{A(t_i^n)}\bigl(t^n_{i+1}, t^n_i\bigr) \ldots\Bigr) \\
&= \lim_{n\to\infty} \det\bigl(\ldots e_{A(t_i^n)}\bigl(t^n_{i+1}, t^n_i\bigr) \ldots\bigr) \\
&= \lim_{n\to\infty} \ldots \det\bigl(e_{A(t_i^n)}\bigl(t^n_{i+1}, t^n_i\bigr)\bigr) \ldots .
\end{aligned}$$

On any interval $[t_i^n, t_{i+1}^n]_{\mathbb{T}}$ we are working with a time independent linear system with a matrix $A_i^n := A(t_i^n)$, hence we can use the explicit representation of its solution developed in [1]; we have

$$\det\bigl(e_{A(\cdot)}(T, t_0)\bigr) = \lim_{n\to\infty} \ldots \det\left(\exp\left\{\int_{t_i^n}^{t_{i+1}^n} \hat{H}\bigl(s, A_i^n\bigr)\,\Delta s\right\}\right)\ldots,$$

where the function $\hat{H} : \mathbb{T} \times \mathrm{Mat}(d \times d) \to \mathrm{Mat}(d \times d)$ is defined by

$$\hat{H}(t, A) := \frac{\log(I_d + \mu(t)A)}{\mu(t)},$$

in the sense of continuous extension, so if for some $t_0 \in \mathbb{T}$ the graininess $\mu(t_0) = 0$, we set $\hat{H}(t_0, A) = A$.

If we replace the graininess function $\mu(t)$ by a function $m : \mathbb{R} \to \mathbb{R}$,

$$m(t) := \mu\bigl(\sup\{s \in \mathbb{T}\colon\ s \leqslant t\}\bigr),$$

we obtain the function $H : \mathbb{R} \times \mathrm{Mat}(d \times d) \to \mathrm{Mat}(d \times d)$, defined by

$$H(t, A) := \frac{\log(I_d + m(t)A)}{m(t)},$$

again in the sense of continuous extension, so if for some $t_0 \in \mathbb{T}$, $m(t_0) = 0$, we set $H(t_0, A) = A$.

It is proved in [1] that for a constant matrix $A \in \mathrm{Mat}(d \times d)$ the hypothesis (H) implies that the equality

$$\int_a^b \hat{H}(s, A)\,\Delta s = \int_a^b H(s, A)\,ds$$

holds for any $a, b \in \mathbb{T}$, $a \leqslant b$ and that $Y(t) := \exp\{\int_{t_0}^t H(s, A)\,ds\}$ is the principal fundamental matrix of the matrix linear ordinary differential equation

$$\dot{Y} = H(t, A)Y, \quad t \in \mathbb{R},$$

at t_0. Therefore

$$\begin{aligned}
\det\bigl(e_{A(\cdot)}(T, t_0)\bigr) &= \lim_{n\to\infty} \ldots \det\left(\exp\left\{\int_{t_i^n}^{t_{i+1}^n} H\bigl(s, A_i^n\bigr)\,ds\right\}\right)\ldots \\
&= \lim_{n\to\infty} \ldots \exp\left\{\int_{t_i^n}^{t_{i+1}^n} \mathrm{tr}\bigl(H\bigl(s, A_i^n\bigr)\bigr)\,ds\right\}\ldots \\
&= \lim_{n\to\infty} \ldots \exp\left\{\int_{t_i^n}^{t_{i+1}^n} \mathrm{tr}\bigl(\hat{H}\bigl(s, A_i^n\bigr)\bigr)\,\Delta s\right\}\ldots
\end{aligned}$$

$$= \lim_{n\to\infty} \ldots \exp\left\{ \int_{t_i^n}^{t_{i+1}^n} \frac{\log\det(I_d + \mu(s)A_i^n)}{\mu(s)}\,\Delta s \right\} \ldots$$

$$= \lim_{n\to\infty} \ldots \exp\left\{ \int_{t_i^n}^{t_{i+1}^n} \frac{\log\prod_{j=1}^d (1 + \mu(s)\lambda_j(A_i^n))}{\mu(s)}\,\Delta s \right\} \ldots$$

$$= \prod_{j=1}^d \lim_{n\to\infty} \ldots \exp\left\{ \int_{t_i^n}^{t_{i+1}^n} \frac{\log(1 + \mu(s)\lambda_j(A_i^n))}{\mu(s)}\,\Delta s \right\} \ldots$$

$$= \prod_{j=1}^d e_{\lambda_j(A(\cdot))}(T, t_0)$$

$$= e_{(\lambda_1\oplus\cdots\oplus\lambda_d)(A(\cdot))}(T, t_0). \qquad \square$$

Example 3.1. The case $\mathbb{T} = \mathbb{R}$. Straightforward calculation gives

$$\det\bigl(e_{A(\cdot)}(t, t_0)\bigr) = \prod_{i=1}^d \exp\left\{ \int_{t_0}^t \lambda_i\bigl(A(s)\bigr)\,ds \right\} = \exp\left\{ \int_{t_0}^t \sum_{i=1}^d \lambda_i\bigl(A(s)\bigr)\,ds \right\}$$

$$= \exp\left\{ \int_{t_0}^t \operatorname{tr}\bigl(A(s)\bigr)\,ds \right\},$$

which is the well-known Liouville's formula.

Example 3.2. The case $\mathbb{T} = h\mathbb{Z}$, where $h > 0$ is so small, that the hypothesis (H) holds. Because for any $i = 1, \ldots, n$ the solution of the initial time scale problem

$$x^\Delta = \lambda_i\bigl(A(t)\bigr)x, \qquad x(t_0) = 1, \quad t_0, t \in \mathbb{T}$$

is

$$e_{\lambda_i(A(\cdot))}(t, t_0) = \prod_{n=1}^{(t-t_0)/h} \Bigl(1 + h\lambda_i\Bigl(A\bigl(t_0 + (n-1)h\bigr)\Bigr)\Bigr),$$

we get the known result

$$\det\bigl(e_{A(\cdot)}(t, t_0)\bigr) = \prod_{n=1}^{(t-t_0)/h} \det\Bigl(I_d + h\Bigl(A\bigl(t_0 + (n-1)h\bigr)\Bigr)\Bigr)$$

$$= \det \prod_{n=1}^{(t-t_0)/h} \Bigl(I_d + h\Bigl(A\bigl(t_0 + (n-1)h\bigr)\Bigr)\Bigr).$$

Example 3.3. Let $\mathbb{T}$ be the middle third Cantor set, $t_0 = 0$, $t = 1$, and $A \in \mathrm{Mat}(d \times d)$ be a constant matrix such that $\|A\| < 1/3$. In this case computations similar to those in Example 3.2 (we use here the representation of the solution of $x^{\Delta}(t) = ax(t),\ x(0) = 1$, $a \in \mathbb{R},\ t \in \mathbb{T}$ from [2]) give

$$\det\bigl(e_A(1,0)\bigr) = \prod_{i=1}^{d}\prod_{n=1}^{\infty}\left(1 + \frac{\lambda_i(A)}{3}\left(\frac{2}{3}\right)^n\right) = \det\prod_{n=1}^{\infty}\left(I_d + \frac{2^n}{3^{n+1}}A\right).$$

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