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Exact Solution to a Nonlinear Klein–Gordon Equation

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The nonlinear Klein–Gordon equation $\partial^\mu\partial_\mu\Phi + M^2\Phi + \lambda_1\Phi^{1-m} + \lambda_2\Phi^{1-2m} = 0$ has the exact formal solution $\Phi = [u^{2m} - \lambda_1 u^m/(m-2)M^2 + \lambda_1^2/(m-2)^2M^4 - \lambda_2/4(m-1)M^2]^{1/m}u^{-1}$, $m \neq 0, 1, 2$, where u and u^{-1} are solutions of the linear Klein–Gordon equation. This equation is a simple generalization of the ordinary second order differential equation satisfied by the homogeneous function $y = [au^m + b(uv)^{m/2} + cv^m]^{\kappa/m}$, where u and v are linearly independent solutions of $y'' + r(x)\,y' + q(x)\,y = 0$.

Recently it was shown in [1] that the homogeneous function

$$y = [au^m + b(uv)^{m/2} + cv^m]^{\kappa/m}, \qquad \kappa l = 1, \tag{1}$$

satisfies the nonlinear differential equation

$$y'' + r(x)\,y' + \kappa q(x)\,y = (1-l)\,y'^2y^{-1} + \kappa Q^* w^2 y^{1-2ml}, \tag{2}$$

Where primes denote differentiation with respect to x and

$$Q^* = (b/4)[(m-2)(au^m + cv^m)(uv)^{-m/2} - b + 4b^{-1}(m-1)\,ac](uv)^{m-2}. \tag{3}$$

The functions u and v satisfy the linear equation

$$y'' + r(x)\,y' + q(x)\,y = 0, \tag{4}$$

and $w \neq 0$ is the Wronskian of u and v. The exponents are real and nonzero, while the constants a, b, c are arbitrary.

The purpose of this note is to observe that (2) can be put in the form

$$\begin{aligned} y'' + r(x)\,y' + \kappa q(x)\,y = (1-l)\,y'^2y^{-1} &+ \kappa(b/4)(m-2)(uv)^{\frac{1}{2}m-2}\,w^2y^{1-ml} \\ &+ \kappa(m-1)(ac - b^2/4)(uv)^{m-2}\,w^2y^{1-2ml}. \end{aligned} \tag{5}$$

Moreover, we remark that (5) is physically important, especially in view of the generalization to partial differential equations.

We have obtained in [2] equations of the form

$$\sum_{i,j=1}^{n} f_{ij}(x) \frac{\partial^2 \Phi}{\partial x_i \, \partial x_j} + \sum_{i=1}^{n} g_i(x) \frac{\partial \Phi}{\partial x_j} + \kappa h(x)\Phi$$

$$= (1 - l)\, \Phi^{-1} \sum_{1,j=1}^{n} f_{ij}(x) \frac{\partial \Phi}{\partial x_j} \frac{\partial \Phi}{\partial x_j} + \kappa Q(u, v)\, W^2 \Phi^{1-2ml}, \tag{6}$$

where

$$W^2 = \sum_{i,j=1}^{n} f_{ij}(x) \times \left[u^2 \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} - uv \left(\frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \right) + v^2 \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right]. \tag{7}$$

The functions $u(x)$ and $v(x)$ are now solutions of the linear equation

$$\sum_{i,j=1}^{n} f_{ij}(x) \frac{\partial^2 \Phi}{\partial x_i \, \partial x_j} + \sum_{i=1}^{n} g_i(x) \frac{\partial \Phi}{\partial x_i} + h(x)\Phi = 0, \tag{8}$$

and $x = (x_1, ..., x_n)$ now denotes a set of independent variables. Thus by choosing $l = 1$, $v = u^{-1}$, and $W^2 = -4M^2$, we can extend (5) immediately to the nonlinear equation

$$\partial^\mu \partial_\mu \Phi + M^2 \Phi + \lambda_1 \Phi^{1-m} + \lambda_2 \Phi^{1-2m} = 0, \tag{9}$$

which has application to nonlinear quantum field theories [3]. The solution

$$\Phi = [u^{2m} - \lambda_1 u^m/(m-2)\, M^2 + \lambda_1^2/(m-2)^2 M^4 - \lambda_2/4(m-1)\, M^2]^{1/m} u^{-1} \tag{10}$$

is in terms of solutions of the Klein–Gordon equation

$$\partial^\mu \partial_\mu \Phi + M^2 \Phi = 0. \tag{11}$$

In (10) we have

$$a = 1, \qquad b = -\lambda_1/(m-2)\, M^2, \qquad c = b^2 - \lambda_2/4(m-1)\, M^2. \tag{12}$$

We showed in [2] that $W^2 = -4h(x)$ when the condition $v = u^{-1}$ is imposed on a solution of (8). With $v = u^{-1}$, a necessary and sufficient condition for W^2 to be constant is that $h(x) = M^2$, a constant.

Operator–valued solutions similar to (10) have been applied in [4, 5].

References

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