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A conjecture of Palamodov about the functors Ext^k in the category of locally convex spaces

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Abstract

In 1971 Palamodov proved that in the category of locally convex spaces the derived functors $\mathrm{Ext}^k(E,X)$ of $\mathrm{Hom}(E,\cdot)$ all vanish if E is a (DF)-space, X is a Fréchet space, and one of them is nuclear. He conjectured a "dual result", namely that $\mathrm{Ext}^k(E,X)=0$ for all $k\in\mathbb{N}$ if E is a metrizable locally convex space, X is a complete (DF)-space, and one of them is nuclear. Assuming the continuum hypothesis we give a complete answer to this conjecture: If X is an infinite-dimensional nuclear (DF)-space, then

(1) There is a normed space E such that $\mathrm{Ext}^1(E,X)\neq 0$.
(2) $\mathrm{Ext}^2(\mathbb{K}^{\mathbb{N}},X)\neq 0$ where $\mathbb{K}^{\mathbb{N}}$ is a countable product of lines.
(3) $\mathrm{Ext}^k(E,X)=0$ for all $k\geqslant 3$ and all locally convex spaces E.

1. Introduction

In the fundamental work [7], Palamodov used methods from homological algebra and in particular derived functors to study a variety of classical analytical problems. For instance, the derivatives of the projective limit functor are a powerful tool to unify and simplify methods which construct a global solution from local ones and the derivatives $\mathrm{Ext}^k(E,\cdot)$ of $\mathrm{Hom}(E,\cdot)$ can be used in many situations to find even solution operators.

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For a fixed locally convex space (l.c.s.) E the functor $L(E,\cdot)$ assigns to each l.c.s. X the vector space $L(E,X)$ of continuous linear maps from E to X and to a continuous linear map $T: X \to Y$ the "composition operator" $L(E,T): f \mapsto T \circ f$. The category of locally convex spaces has enough injective objects (supplied by the Hahn–Banach theorem) to construct the derived functors $\mathrm{Ext}^k(E,\cdot)$ using injective resolutions. The following proposition (essentially due to Palamodov [7, Proposition 9.1], see also [10, Proposition 5.1.3]) covers as well the typical situations for applications as the properties of $\mathrm{Ext}^k(E,\cdot)$ which are mainly used in the present article.

Theorem 1.1. (1) *For l.c.s. E and X the following are equivalent*:

(i) $\mathrm{Ext}^1(E,X) = 0$.

(ii) *Each exact sequence* $0 \to X \xrightarrow{i} Y \xrightarrow{q} E \to 0$ (*i.e. i is a topological embedding onto the kernel of q which is a quotient map*) *splits, i.e. q has a right inverse* $r: E \to Y$ *with* $q \circ r = id_E$.

(iii) *For each exact sequence* $0 \to X \to Y \xrightarrow{q} Z \to 0$ *and* $T \in L(E,Z)$ *there is a lifting* $S \in L(E,Y)$ *with* $q \circ S = T$.

(2) *For l.c.s. E and X and* $k \geqslant 2$ *we have* $\mathrm{Ext}^k(E,X) = 0$ *if and only if the class* $\{Y : \mathrm{Ext}^{k-1}(E,Y) = 0\}$ *is stable with respect to forming quotients by subspaces isomorphic to X*.

The action of $L(E,\cdot)$ on the subcategory of Fréchet spaces can be investigated using the derivatives of the projective limit functor. For a countable spectrum $\mathscr{X} = (X_n, \varrho^n_m)$ of vector spaces (i.e. $\varrho^n_m : X_m \to X_n$ are linear for $m \geqslant n$ with $\varrho^n_n = id_{X_n}$ and $\varrho^n_m \circ \varrho^m_k = \varrho^n_k$ for $n \leqslant m \leqslant k$) we have the canonical complex

$$0 \to \operatorname{Proj} \mathscr{X} \to \prod_{n\in\mathbb{N}} X_n \xrightarrow{d} \prod_{n\in\mathbb{N}} X_n \to 0, \qquad (\star)$$

where $d((x_n)_{n\in\mathbb{N}}) = (\varrho^n_{n+1} x_{n+1} - x_n)_{n\in\mathbb{N}}$ is the difference map. Using an abstract Mittag–Leffler procedure, Palamodov proved that this complex is exact (i.e. d is surjective) whenever X_n are all Fréchet spaces and ϱ^n_m are continuous with dense range. Since every Fréchet space can be represented as a projective limit $\operatorname{Proj} \mathscr{X}$ of Banach spaces the calculation of $\mathrm{Ext}^k(E,X)$ can be first localized (which requires to find $\mathrm{Ext}^k(E,X_n)$) and then one can again use the Proj-functor (or a Mittag–Leffler procedure) to find $\mathrm{Ext}^k(E,X)$. In this way Palamodov [7, Theorem 9.1] proved:

Theorem 1.2. *If E is a* (DF)-*space, X is a Fréchet space, and one of these spaces is nuclear then* $\mathrm{Ext}^k(E,X) = 0$ *for all* $k \geqslant 1$.

(This result can also be deduced from Grothendieck's work about tensor products (e.g. from [4, Chapter II, Section 3, Proposition 10.3]) since each continuous linear operator from a (DF)-space into an metrizable space is bounded, i.e. maps some 0-neighbourhood into a bounded set. In (the English version of) [7] this is erroneously translated with "continuous".)

Palamodov conjectured [7, p. 54] "it is natural to expect that the following proposition 'dual' to Theorem 1.2 is valid: $\mathrm{Ext}^i(E, X) = 0$, $i \geqslant 1$, if E is metric, X is a complete dual metric space, and one of them is a nuclear space". The aim of the present article is to settle this conjecture.

2. Positive results

Palamodov's method to prove Theorem 1.2 suggests to represent a complete (DF)-space X as a projective limit of Banach spaces (every complete l.c.s. has such a representation which is countable precisely for Fréchet spaces) and to construct a resolution in analogy to ($\bigstar$) to which one can apply the functor $L(E, \cdot)$.

Let us first consider projective spectra of linear spaces. For a directed set $(I, \leqslant)$ an I-spectrum $\mathscr{X} = (X_\alpha, \varrho_\beta^\alpha)$ consists of linear spaces X_α for $\alpha \in I$ and linear spectral maps $\varrho_\beta^\alpha : X_\beta \to X_\alpha$ for $\alpha \leqslant \beta$ such that $\varrho_\beta^\alpha \circ \varrho_\gamma^\beta = \varrho_\gamma^\alpha$ for $\alpha \leqslant \beta \leqslant \gamma$ and $\varrho_\alpha^\alpha = id_{X_\alpha}$. A morphism $f = (f_\alpha)_{\alpha \in I} : (X_\alpha, \varrho_\beta^\alpha) \to (Y_\alpha, \sigma_\beta^\alpha)$ is a family of linear maps $f_\alpha : X_\alpha \to Y_\alpha$ commuting with the spectral maps, i.e. $\sigma_\beta^\alpha \circ f_\beta = f_\alpha \circ \varrho_\beta^\alpha$ for $\alpha \leqslant \beta$. The assignment of the projective limit $\mathrm{Proj}(X_\alpha, \varrho_\beta^\alpha) = \{(x_\alpha)_\alpha \in \prod_{\alpha \in I} X_\alpha : \varrho_\beta^\alpha x_\beta = x_\alpha\}$ is then a functor on the category of I-spectra. The derived functors Proj^k can be constructed using injective resolutions (see [10, Chapter 4]) but we will take a more direct way here. For an I-spectrum $\mathscr{X} = (X_\alpha, \varrho_\beta^\alpha)$ we define differentials

$$d_k : \prod_{\alpha_0 \leqslant \cdots \leqslant \alpha_k} X_{\alpha_0} \to \prod_{\alpha_0 \leqslant \cdots \leqslant \alpha_{k+1}} X_{\alpha_0} \quad \text{by } (x_{\alpha_0, \dots, \alpha_k})_{\alpha_0 \leqslant \cdots \leqslant \alpha_k}$$
$$\mapsto \left(\varrho_{\alpha_1}^{\alpha_0}(x_{\alpha_1, \dots, \alpha_{k+1}}) + \sum_{j=1}^{k+1} (-1)^j x_{\alpha_0, \dots, \hat{\alpha}_j, \dots, \alpha_{k+1}} \right)_{\alpha_0 \leqslant \cdots \leqslant \alpha_{k+1}},$$

where the hat indicates that this index is omitted. Since $d_{k+1} \circ d_k = 0$ we obtain a complex of linear spaces

$$0 \to \mathrm{Proj}\, \mathscr{X} \to \prod_{\alpha \in I} X_\alpha \xrightarrow{d_0} \prod_{\alpha_0 \leqslant \alpha_1} X_{\alpha_0} \xrightarrow{d_1} \prod_{\alpha_0 \leqslant \alpha_1 \leqslant \alpha_2} X_{\alpha_0} \xrightarrow{d_2} \cdots \qquad (\bigstar\bigstar)$$

and define $\mathrm{Proj}^k\, \mathscr{X} = \ker d_k / \mathrm{im}\, d_{k-1}$ for $k \geqslant 1$.

In 1972, Mitchell [5] showed that if the cardinality of the index set is less or equal than the nth infinite cardinality $\aleph_n$ then $\mathrm{Proj}^k\, \mathscr{X} = 0$ for all $k \geqslant n+2$. We will now improve this results for spectra consisting of Fréchet spaces and continuous spectral maps. We call such a spectrum reduced if for all $\alpha \in I$ there is $\beta \geqslant \alpha$ such that $\varrho_\beta^\alpha X_\beta \subseteq \overline{\varrho^\alpha\, \mathrm{Proj}\, \mathscr{X}}$ where the bar denotes the closure in X_α and $\varrho^\alpha :\, \mathrm{Proj}\, \mathscr{X} \to X_\alpha$ is the canonical map $(x_\delta)_{\delta \in I} \mapsto x_\alpha$.

Theorem 2.1. *If $\mathscr{X}$ is a reduced I-spectrum of Fréchet spaces and continuous spectral maps then $|I| \leqslant \aleph_n$ implies $\operatorname{Proj}^k \mathscr{X} = 0$ for all $k \geqslant n+1$.*

Proof. We prove the theorem by induction on $n \in \mathbb{N}_0$. Let us first observe that $\operatorname{im} d_k$ is dense in $\ker d_{k+1}$ (where the spaces in (★★) are endowed with the product topology). To prove this one only needs to solve finitely many of the equations $y_{\alpha_0,\dots,\alpha_{k+1}} = \sum_{j=0}^{k+1} (-1)^j x_{\alpha_0,\dots,\hat{\alpha}_j,\dots,\alpha_{k+1}}$ (we omitted the spectral map for the 0th summand to simplify notation) for given $(y_{\alpha,\dots,\alpha_{k+1}})_{\alpha_0 \leqslant \cdots \leqslant \alpha_{k+1}} \in \ker d_{k+1}$ and this can be done explicitly. Moreover, an elementary calculation shows that the reducedness of $\mathscr{X}$ implies that d_0 is open onto its range (see [10, Theorem 4.3.1]) and the same holds for d_k and $k \geqslant 1$ even without the assumption that $\mathscr{X}$ is reduced.

If now $|I| \leqslant \aleph_0$, the spaces $F_k = \prod_{\alpha_0 \leqslant \cdots \leqslant \alpha_k} X_{\alpha_0}$ are countable products of Fréchet spaces hence itself Fréchet and since quotients of Fréchet spaces are complete we obtain that $\operatorname{im} d_{k-1}$ is a dense complete subspace of $\ker d_k$ which yields $\operatorname{im} d_{k-1} = \ker d_k$ and thus proves the case $n = 0$.

We now suppose that the theorem is true for J-spectra with $|J| \leqslant \aleph_{n-1}$. If $|I| = \aleph_n$ we choose a limit ordinal ω and $I_\nu \subset I_\mu \subseteq I =: I_\omega$ for $\nu \leqslant \mu < \omega$ such that $|I_\nu| \leqslant \aleph_{n-1}$ and $I_\mu = \bigcup_{\nu<\mu} I_\nu$ for limit ordinals $\mu \leqslant \omega$ (this can be done by choosing a bijective mapping $h : \omega \to I$ and setting $I_\nu = h(\nu)$).

Next, we will replace I_ν by directed subsets J_ν satisfying the same conditions. Since $\mathscr{X}$ is reduced there is a mapping $f : I \times I \to I$ with $f(\alpha,\beta) \geqslant \alpha$, $f(\alpha,\beta) \geqslant \beta$, and such that $\varrho^\alpha_\gamma X_\gamma \subseteq \overline{\varrho^\alpha \operatorname{Proj} \mathscr{X}}$ and $\varrho^\beta_\gamma X_\gamma \subseteq \overline{\varrho^\beta \operatorname{Proj} \mathscr{X}}$ for $\gamma \geqslant f(\alpha,\beta)$. If now $M \subseteq I$ is any subset we define $M^0 = M$, $M^j = M^{j-1} \cup \{f(\alpha,\beta) : \alpha,\beta \in M_{j-1}\}$, and $\tilde{M} = \bigcup_{j \in \mathbb{N}_0} M^j$. Then $\tilde{M}$ is directed with respect to the order of I and it is either countable or has the same cardinality as M.

We set $J_\nu = \tilde{I}_\nu$ and obtain reduced J_ν-spectra $\mathscr{X}^\nu = (X_\alpha, \varrho^\alpha_\beta)_{\alpha \in J_\nu}$. The corresponding differentials as in (★★) are denoted by d^ν_k.

Let now $k \geqslant n+1$ and $y = (y_{\alpha_0,\dots,\alpha_k})_{\alpha_0 \leqslant \cdots \leqslant \alpha_k} \in \ker d_k$ be given. We construct a solution $x = (x_{\alpha_0,\dots,\alpha_{k-1}})_{\alpha_0 \leqslant \cdots \leqslant \alpha_{k-1}}$ of $d_{k-1}(x) = y$ by transfinite induction. Let $\mu < \omega$ and assume that for all $\nu < \mu$ we have $x^\nu_{\alpha_0,\dots,\alpha_{k-1}} \in X_{\alpha_0}$ for $\alpha_0,\dots,\alpha_{k-1} \in J_\nu$ with $\alpha_0 \leqslant \cdots \leqslant \alpha_{k-1}$ such that

(1) $\sum_{j=0}^k (-1)^j x^\nu_{\alpha_0,\dots,\hat{\alpha}_j,\dots,\alpha_k} = y_{\alpha_0,\dots,\alpha_k}$ for $\alpha_0 \leqslant \cdots \leqslant \alpha_k \in J_\nu$ and
(2) $x^\nu_{\alpha_0,\dots,\alpha_{k-1}} = x^\lambda_{\alpha_0,\dots,\alpha_{k-1}}$ for $\lambda < \nu$ and $\alpha_0,\dots,\alpha_{k-1} \in J_\lambda$.

If μ is a limit ordinal we have $J_\mu = \bigcup_{\nu<\mu} J_\nu$ and can thus define $x^\mu_{\alpha_0,\dots,\alpha_{k-1}} = x^\nu_{\alpha_0,\dots,\alpha_{k-1}}$ if $\alpha_0,\dots,\alpha_{k-1} \in J_\nu$. Let now $\mu = \lambda + 1$ be a successor ordinal. Since $|J_\mu| \leqslant \aleph_{n-1}$ we have $\operatorname{Proj}^k \mathscr{X}^\mu = 0$ and can thus find $z = (z_{\alpha_0,\dots,\alpha_{k-1}})_{\alpha_0 \leqslant \cdots \leqslant \alpha_{k-1} \in J_\mu}$ with $\sum_{j=0}^k (-1)^j z_{\alpha_0,\dots,\hat{\alpha}_j,\dots,\alpha_k} = y_{\alpha_0,\dots,\alpha_k}$ for all $\alpha_0 \leqslant \cdots \leqslant \alpha_{k-1} \in J_\mu$. Now, $u_{\alpha_0,\dots,\alpha_{k-1}} = x^\lambda_{\alpha_0,\dots,\alpha_{k-1}} - z_{\alpha_0,\dots,\alpha_{k-1}}$ for $\alpha_0 \leqslant \cdots \leqslant \alpha_{k-1} \in J_\lambda$ defines an element $u \in \ker d^\lambda_{k-1}$ and since $|J_\lambda| \leqslant \aleph_{n-1}$ we can apply $\operatorname{Proj}^{k-1} \mathscr{X}^\lambda = 0$ to find $v = (v_{\alpha_0,\dots,\alpha_{k-2}})_{\alpha_0 \leqslant \cdots \leqslant \alpha_{k-2} \in J_\lambda}$ with

$u = d^{\lambda}_{k-2}(v)$. For $\alpha_0 \leqslant \cdots \leqslant \alpha_{k-2} \in J_\mu$ we set

$$\tilde{v}_{\alpha_0,\ldots,\alpha_{k-2}} = \begin{cases} v_{\alpha_0,\ldots,\alpha_{k-2}} & \text{if } \alpha_0, \ldots, \alpha_{k-2} \in J_\lambda, \\ 0 & \text{else} \end{cases}$$

and $x^\mu = z + d^{\mu}_{k-2}(\tilde{v})$. Then x^μ satisfies (1) and (2). Finally, $x_{\alpha_0,\ldots,\alpha_{k-1}} = x^\mu_{\alpha_0,\ldots,\alpha_{k-1}}$ if $\alpha_0, \ldots, \alpha_{k-1} \in J_\mu$ defines a solution of $d_{k-1}(x) = y$. $\square$

To apply this result to calculate $\mathrm{Ext}^k(E, X)$ for a complete l.c.s. X we represent $X = \mathrm{Proj}\,\mathscr{X}$, where $\mathscr{X} = (X_\alpha, \varrho^\alpha_\beta)$ is a reduced projective spectrum of Banach spaces. For instance, we can take a fundamental system $(||\cdot||_\alpha)_{\alpha \in I}$ of continuous seminorms and define Banach spaces X_α as the completion of $(X, ||\cdot||_\alpha)/\ker ||\cdot||_\alpha$.

Theorem 2.2. *Let E and X be locally convex spaces such that one of them is nuclear. If X has a 0-neighbourhood basis of cardinality less or equal than $\aleph_n$ then $\mathrm{Ext}^k(E, X) = 0$ for all $k \geqslant n + 2$.*

Proof. Let Z be any l.c.s., $\tilde{Z}$ its completion and $r \geqslant 2$. Since

$$0 \to Z \to \tilde{Z} \to Z/\tilde{Z} \to 0$$

is exact and $Z/\tilde{Z}$ is an injective object in the category of l.c.s. (it carries the coarsest topology since Z is dense in $\tilde{Z}$) we obtain $\mathrm{Ext}^r(E, Z) = \mathrm{Ext}^r(E, \tilde{Z})$.

Therefore, we may assume that X is complete and can represent $X = \mathrm{Proj}\,\mathscr{X}$ with a reduced I-spectrum $\mathscr{X} = (X_\alpha, \varrho^\alpha_\beta)$ consisting of Banach spaces and $|I| \leqslant \aleph_n$. If X is nuclear we can assume $X_\alpha \cong \ell_\infty$ and then have $\mathrm{Ext}^k(E, X_\alpha) = 0$ for all $k \in \mathbb{N}$ since ℓ_∞ is an injective object in the category of l.c.s. If E is nuclear $\mathrm{Ext}^k(E, X_\alpha) = 0$ holds too, e.g. by results of Grothendieck quoted above.

Let us consider complex ($\bigstar\bigstar$). As noted in the proof of Theorem 2.1 $\mathrm{im}\, d_{k-1}$ is a dense topological subspace of $\ker d_k$ for all $k \in \mathbb{N}$. Let now $k \geqslant n + 2$. To prove $\mathrm{Ext}^k(E, X) = 0$ we have to show $\mathrm{Ext}^{k-1}(E, \mathrm{im}\, d_0) = 0$.

As $\ker d_1$ is the completion of $\mathrm{im}\, d_0$ we have

$$\mathrm{Ext}^{k-1}(E, \mathrm{im}\, d_0) = \mathrm{Ext}^{k-1}(E, \ker d_1) \quad \text{if } k - 1 \geqslant 2.$$

Iterating this argument we see that we have to prove $\mathrm{Ext}^{k-n}(E, \ker d_n) = 0$. Theorem 2.1 implies that

$$0 \to \ker d_n \to \prod_{\alpha_0 \leqslant \cdots \leqslant \alpha_n} X_{\alpha_0} \xrightarrow{d_n} \prod_{\alpha_0 \leqslant \cdots \leqslant \alpha_{n+1}} X_{\alpha_0} \xrightarrow{d_{n+1}} \cdots$$

is an exact E-acyclic resolution of $\ker d_n$ (this means that all objects but $\ker d_n$ satisfy $\mathrm{Ext}^i(E,\cdot)=0$ for all $i\in\mathbb{N}$) which yields

$$\mathrm{Ext}^r(E,\ker d_n)=\ker L(E,d_{n+r})/\mathrm{im}\,L(E,d_{n+r-1})$$

$$=\mathrm{Proj}^{n+r}(L(E,X_\alpha),L(E,\varrho^\alpha_\beta))=0$$

for $r\geqslant 2$ by Mitchell's result mentioned before Theorem 2.1. Since $k-n\geqslant 2$ this gives the conclusion. □

Let now $X=\mathrm{ind}\,X_n$ be a (non-normable) (LB)-space. A basis of the 0-neighbourhood filter is given by

$$\left\{\Gamma\left(\bigcup_{n\in\mathbb{N}}\frac{1}{m_n}B_n\right):(m_n)_n\in\mathbb{N}^{\mathbb{N}}\right\},$$

where B_n is the unit ball of the Banach space X_n. An application of the preceding theorem thus requires knowledge about the cardinality $|\mathbb{N}^{\mathbb{N}}|=|\mathbb{R}|$.

Corollary 2.3. *Let E be a l.c.s. and X an* (LB)*-space such that one of them is nuclear. Then the continuum hypothesis implies* $\mathrm{Ext}^k(E,X)=0$ *for all* $k\geqslant 3$.

The next result was obtained jointly with Leonhard Frerick (unpublished). A proof (under weaker assumptions on X) is contained in [10, Proposition 5.8].

Theorem 2.4. *Let E be a topological subspace of $\mathbb{K}^{\mathbb{N}}$ containing the unit vectors and X a nuclear* (LB)*-space. Then* $\mathrm{Ext}^1(E,X)=0$.

3. Negative results

To obtain further results in the situation of Corollary 2.3 we need more information about complex ($\bigstar\bigstar$) if $\mathscr{X}$ is a spectrum representing a nuclear (LB)-space.

Proposition 3.1. *Let X be an infinite-dimensional nuclear* (LB)*-space with the resolution*

$$0\to X\to\prod_{\alpha\in I}X_{\alpha_0}\xrightarrow{d_0}\prod_{\alpha_0\leqslant\alpha_1}X_{\alpha_0}\to\cdots$$

as above. Assuming the continuum hypothesis there is a bounded sequence in $\mathrm{im}\,d_0$ *which is not contained in the image of a bounded subset of* $\prod_{\alpha\in I}X_\alpha$.

Proof. Let us first observe that the lifting property in the proposition does not depend on the particular resolution as long as all spaces X_α are quasinormable Fréchet spaces (a Fréchet space F is quasinormable if and only if for each exact sequence

$$0 \to F \to G \xrightarrow{q} G/F \to 0$$

q lifts bounded sets, see e.g. [6, Theorem 26.17] if G is a Fréchet space or [10, Theorem 7.5]). One can prove this e.g. by considering the functor ℓ_∞ assigning to a l.c.s. X the space of bounded sequences in X.

We will first construct a particular resolution of X. Let ω_1 be the first uncountable ordinal and $\mathscr{U} = \{U_\alpha : \alpha < \omega_1\}$ a basis of the 0-neighbourhood filter with cardinality $\aleph_1$ (since X is infinite dimensional $\mathscr{U}$ cannot be countable). We denote by X_U the local Banach space associated to U, i.e. the completion of X endowed with the Minkowski functional of U as a norm (as nuclear (LB)-spaces have continuous norms we may assume that these are indeed norms rather than merely seminorms). We take an arbitrary $U \in \mathscr{U}$ and set $X_0 = X_U$. If for some $\beta < \omega_1$ all X_α with $\alpha < \beta$ are constructed and $\beta = \gamma + 1$ is a successor ordinal we choose $V \in \mathscr{U}$ such that $V \subseteq U_\beta$ and the canonical map $X_V \to X_\alpha$ is compact for all $\alpha < \beta$ (which is possible since X has the countable neighbourhood condition and is a Schwartz space). We then set $X_\beta = X_V$. If β is a limit ordinal we set $X_\beta = \operatorname{Proj}_{\alpha<\beta} X_\alpha$. With the obvious spectral maps (the continuous extensions of the identical maps) we obtain an ω_1-spectrum $\mathscr{X} = (X_\alpha, \varrho^\alpha_\beta)$ consisting of either Banach or else Fréchet–Schwartz spaces with $\operatorname{Proj}\mathscr{X} = X$, $\varrho^\alpha X$ dense in X_α, $X_\beta = \operatorname{Proj}(X_\alpha, \varrho^\alpha_\gamma)_{\alpha<\beta}$ for limit ordinals β, and $\varrho^\beta_{\beta+1}$ compact for all $\beta < \omega_1$.

By transfinite induction we will now construct a sequence $x^n = (x^n_\alpha)_{\alpha<\omega_1}$ in $\prod_{\alpha<\omega_1} X_\alpha$ such that $d_0(x^n)$ is bounded in $\operatorname{im}(d_0)$ (which is a topological subspace of $\prod_{\alpha_0 \leqslant \alpha_1} X_{\alpha_0}$ as noted in the proof of Theorem 2.1) but cannot be written as $d_0(x^n) = d_0(y^n)$ for a bounded sequence $(y^n)_{n\in\mathbb{N}}$ in $\prod_{\alpha<\omega_1} X_\alpha$.

Since X can be embedded into a separable Banach space the cardinality of $X^{\mathbb{N}}$ is $|\mathbb{R}^{\mathbb{N}}| = \aleph_1$, hence we can choose a surjective map $f : \omega_1 \to X^{\mathbb{N}}$.

Assume that for $\gamma < \omega_1$ we have already $x^n_\alpha \in X_\alpha$ for $\alpha < \gamma$ and $n \in \mathbb{N}$ such that

(1) for all $\alpha < \beta < \gamma$ the sequence $(\varrho^\alpha_\beta x^n_\beta - x^n_\alpha)_{n\in\mathbb{N}}$ is bounded in X_α and
(2) if $\beta + 1 < \gamma$ there is $\alpha < \gamma$ such that $(x^n_\alpha)_{n\in\mathbb{N}} - f(\beta)$ is unbounded in X_α (more precisely, this means that $(x^n_\alpha - \varrho^\alpha(y^n))_{n\in\mathbb{N}}$ is unbounded where $(y^n)_{n\in\mathbb{N}} = f(\beta) \in X^{\mathbb{N}}$).

If γ is a limit ordinal the sequence $((\varrho^\alpha_\beta\ x^n_\beta - x^n_\alpha)_{\alpha\leqslant\beta<\gamma})_{n\in\mathbb{N}}$ is bounded in $\operatorname{im} d^\gamma_0$ (where d^γ_0 again denotes the differential according to the spectrum $\mathscr{X}^\gamma = (X_\alpha, \varrho^\alpha_\beta)_{\alpha<\gamma}$) because of (1) and the kernel of d^γ_0 is the quasinormable Fréchet space $X_\gamma = \operatorname{Proj}\mathscr{X}^\gamma$. Hence there is a bounded sequence $(y^n)_{n\in\mathbb{N}} = ((y^n_\alpha)_{\alpha<\gamma})_{n\in\mathbb{N}}$ in $\prod_{\alpha<\gamma} X_\alpha$ with

$d_0^\gamma(y^n) = d_0^\gamma((x_\alpha^n)_{\alpha<\gamma})$. We set

$$x_\gamma^n = y^n - (x_\alpha^n)_{\alpha<\gamma} \in \ker d_0^\gamma = \operatorname{Proj} \mathscr{X}^\gamma = X_\gamma.$$

Then (1) holds for $\alpha<\beta<\gamma+1$ since $(y^n)_{n\in\mathbb{N}}$ is bounded and condition (2) is trivially satisfied since $\beta+1<\gamma+1$ implies $\beta+1<\gamma$ as γ is a limit ordinal.

If $\gamma=\beta+1$ is a successor ordinal we distinguish two cases. If there is $\alpha<\gamma$ such that $(x_\alpha^n)_{n\in\mathbb{N}} - f(\beta)$ is unbounded we choose a sequence $(x_\gamma^n)_{n\in\mathbb{N}}$ in X_γ such that $(\varrho_\gamma^\beta x_\gamma^n - x_\beta^n)_{n\in\mathbb{N}}$ is bounded in X_β (this can easily be done using the metrizability of X_β and the density of $\operatorname{im}\varrho^\beta \subseteq \operatorname{im}\varrho_\gamma^\beta$). Then (1) and (2) hold for $\gamma+1$.

If on the other hand $(x_\alpha^n)_{n\in\mathbb{N}} - f(\beta)$ is bounded in X_α for all $\alpha<\gamma$ we choose an unbounded sequence $(r^n)_{n\in\mathbb{N}}$ in X_γ such that $(\varrho_\gamma^\beta r^n)_{n\in\mathbb{N}}$ is bounded in X_β. This is possible since otherwise ϱ_γ^β would be an isomorphism onto its range. We set $(x_\gamma^n)_{n\in\mathbb{N}} = (r^n)_{n\in\mathbb{N}} + f(\beta)$. For $\alpha<\gamma$

$$(\varrho_\gamma^\alpha x_\gamma^n - x_\alpha^n)_{n\in\mathbb{N}} = (\varrho_\beta^\alpha \circ \varrho_\gamma^\beta(r^n))_{n\in\mathbb{N}} - ((x_\alpha^n)_{n\in\mathbb{N}} - f(\beta))$$

is then bounded in X_α, hence (1) holds for $\gamma+1$ and (2) holds since $(r^n)_{n\in\mathbb{N}}$ is unbounded in X_γ.

This completes the induction. Now, (1) implies that $(d_0(x^n))_{n\in\mathbb{N}}$ with $x^n = (x_\alpha^n)_{\alpha<\omega_1}$ is a bounded sequence in $\operatorname{im} d_\gamma$. If $y^n = (y_\alpha^n)_{\alpha<\omega_1}$ is any sequence in $\prod_{\alpha<\omega_1} X_\alpha$ with $d_0(x^n) = d_0(y^n)$ then there is $\beta<\omega_1$ with $(x^n - y^n)_{n\in\mathbb{N}} = f(\beta)$. Choosing $\alpha<\omega_1$ according to (2) we conclude that $(y_\alpha^n)_{n\in\mathbb{N}} = (x_\alpha^n)_{n\in\mathbb{N}} - f(\beta)$ is unbounded in X_α and thus, $(y^n)_{n\in\mathbb{N}}$ is unbounded. $\square$

Theorem 3.2. *There is a normed space E such that under the assumption of the continuum hypothesis* $\operatorname{Ext}^1(E,X)\neq 0$ *for each infinite-dimensional nuclear* (LB)-*space X.*

Proof. Let E be the subspace of ℓ_1 spanned by the unit vectors e_n. If $(y^n)_{n\in\mathbb{N}}$ is a bounded sequence in $\operatorname{im} d_0$ which cannot be lifted we define $T: E \to \operatorname{im} d_0$ by $T(e_n) = y^n$ and linear extension. Then T is continuous and

$$0 \to X \to \prod_{\alpha\in I} X_\alpha \xrightarrow{d_0} \operatorname{im} d_0 \to 0$$

is an exact sequence such that $T: E \to \operatorname{im} d_0$ does not factorize as a continuous linear map over $\prod_{\alpha\in F} X_\alpha$. Theorem 1.1 gives $\operatorname{Ext}^1(E,X)\neq 0$. $\square$

Remark 3.3. In the situation of Proposition 3.1, we can assume that all spaces X_α are reflexive. If $(y^n)_{n\in\mathbb{N}}$ is bounded in $\operatorname{im} d_0$ without lifting then there is not even a bounded set B in $\prod_{\alpha\in I} X_\alpha$ with $\{y^n : n\in\mathbb{N}\} \subseteq \overline{d_0(B)}$ (since $\prod_{\alpha\in I} X_\alpha$ is reflexive we can assume that B is weakly compact and then $d_0(B)$ is weakly compact hence closed in $\operatorname{im} d_0$).

As a consequence, the transposed map $d_0^t : (\operatorname{im} d_0)' \to (\prod_{\alpha\in I} X_\alpha)'$ is not open onto its range if the dual spaces are both endowed with their strong topologies. If D^1 denotes the derivative of the duality functor (see [7, Section 8] or [10, Chapter 7]) this means that under the continuum hypothesis $\mathrm{D}^1(X) \neq 0$ holds for each infinite-dimensional nuclear (LB)-space X. In particular, this answers Palamodov's unsolved problem [7, Section 12.5].

A similar proof as for Proposition 3.1 (see [10, Theorem 4.3.9]) yields

Theorem 3.4. *If X is an infinite-dimensional nuclear* (LB)*-space and $X = \operatorname{Proj} \mathscr{X}$ where $\mathscr{X}$ is a reduced projective spectrum of Fréchet spaces then the continuum hypothesis implies* $\operatorname{Proj}^1 \mathscr{X} \neq 0$.

This result has a consequence for the completeness of quotients. The special case $X = \varphi$, the space of finite sequences endowed with the strongest locally convex topology, is due to Schmerbeck [8].

Corollary 3.5. *Let X be an infinite-dimensional nuclear* (LB)*-space and X_α Fréchet spaces such that X is a topological subspace of $\prod_{\alpha\in I} X_\alpha$. Assuming the continuum hypothesis, the quotient space $\prod_{\alpha\in I} X_\alpha / X$ is incomplete.*

Proof. The completeness of $\prod_{\alpha\in I} X_\alpha / X$ again does not depend on the particular choice of X_α, it is thus enough to show that $\operatorname{im} d_0$ is incomplete with d_0 as above. But $\operatorname{im} d_0$ is a dense topological subspace of $\ker d_1$ and $\operatorname{Proj}^1 \mathscr{X} \neq 0$ implies $\operatorname{im} d_0 \neq \ker d_1$. Hence $\operatorname{im} d_0$ is not complete. □

The combination of this negative result with the positive one from Theorem 2.4 enables us to prove:

Theorem 3.6. *Let X be an infinite-dimensional nuclear* (LB)*-space. Assuming the continuum hypothesis we have* $\operatorname{Ext}^2(\mathbb{K}^{\mathbb{N}}, X) \neq 0$.

Proof. By Corollary 3.5 there are Banach spaces X_α and an exact sequence

$$0 \to X \to \prod_{\alpha\in I} X_\alpha \xrightarrow{q} Z \to 0$$

such that Z is not complete. We do not know whether Z is quasi- or locally complete. However, Theorem 2.4 yields that Z indeed satisfies a very weak completeness condition. If $\mathscr{T}$ denotes the group topology having $\{\ker p : p$ continuous seminorm on $Z\}$ as a basis of the 0-neighbourhood filter then $(Z, \mathscr{T})$ is sequentially complete (this "sequential seminorm kernel completeness" is shared by all complete locally convex spaces as well as, trivially, by all l.c.s. having a continuous norm). Indeed, if $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(Z,\mathscr{T})$ and $\Phi = [\{e_n : n\in\mathbb{N}\}]\subseteq\mathbb{K}^{\mathbb{N}}$ denotes the space of finite sequences endowed with the topology of pointwise convergence we define $T:\Phi\to Z$ by $T(e_n) = z_n - z_{n-1}$ (where $z_0 = 0$) and linear extension. Then T is continuous (if Z is endowed with its original topology) and because of $\mathrm{Ext}^1(\Phi,Z) = 0$ there is a lifting $R\in L(\Phi,\prod_{\alpha\in I} X_\alpha)$ with $q\circ R = T$. Since $\prod_{\alpha\in I} X_\alpha$ is complete and Φ is dense in $\mathbb{K}^{\mathbb{N}}$ there is an extension $\tilde{R}\in L(\mathbb{K}^{\mathbb{N}},\prod_{\alpha\in I} X_\alpha)$, i.e. $\tilde{R}|_\Phi = R$. It is then easy to check that $(z_n)_{n\in\mathbb{N}}$ converges to $z = \tilde{R}((1,1,1,\ldots))$.

Let us now assume that $\mathrm{Ext}^2(\mathbb{K}^{\mathbb{N}},X) = 0$ holds which, by Theorem 1.1, implies $\mathrm{Ext}^1(\mathbb{K}^{\mathbb{N}},Z) = 0$. We choose $z\in\tilde{Z}\backslash Z$ and obtain an exact sequence

$$0\to Z\xrightarrow{I} Z+[z]\xrightarrow{Q}[z]\to 0,$$

where $Z+[z]$ is endowed with the relative topology of $\tilde{Z}$, I is the inclusion and the one-dimensional space $[z]$ carries the trivial topology $\{[z],\emptyset\}$ since Z is dense in $Z+[z]$. Using e.g. a Hamel basis of $\mathbb{K}^{\mathbb{N}}$ we find a non-zero linear map $T:\mathbb{K}^{\mathbb{N}}\to[z]$ with $T|_\Phi = 0$. $\mathrm{Ext}^1(\mathbb{K}^{\mathbb{N}},Z) = 0$ yields a continuous linear lifting $\tilde{T}:\mathbb{K}^{\mathbb{N}}\to Z+[z]$. We have $Q\circ\tilde{T}|_\Phi = T|_\Phi = 0$, hence there is $S:\Phi\to Z$ with $I\circ S = \tilde{T}$. Since Z is sequentially seminorm kernel complete there is an extension $\tilde{S}:\mathbb{K}^{\mathbb{N}}\to Z$ of S. Now $I\circ\tilde{S}$ and $\tilde{T}$ coincide on the dense subspace Φ of $\mathbb{K}^{\mathbb{N}}$ and continuity implies $\tilde{T} = I\circ\tilde{S}$, hence the contradiction $T = Q\circ\tilde{T} = Q\circ I\circ\tilde{S} = 0$. □

4. Concluding remarks

We do not know whether our solution to Palamodov's conjecture logically depends on the continuum hypothesis—at least our proofs used it at two essential steps (a spectrum indexed by $\mathbb{N}^{\mathbb{N}}$ consists of $\aleph_1$ spaces and separable Banach spaces have $\aleph_1$ elements).

In [8] Schmerbeck proved that there are Banach spaces X_α and a closed subspace $X\subseteq\prod_{\alpha\in I} X_\alpha$ such that the quotient $\prod_{\alpha\in I} X_\alpha/X$ fails to be complete under the continuum hypothesis but it is complete assuming Martin's axiom and the negation of the continuum hypothesis. Using similar techniques it is possible to construct locally convex spaces E and X such that the behaviour of $\mathrm{Ext}^k(E,X)$ depends on set theoretic axioms. However, it seems that also the decision whether e.g. X is an (LB)-space would require those axioms.

Anyway, our results strongly suggest that there in not much hope for a splitting theory using the functors $\mathrm{Ext}^k(E,X)$ in the category of locally convex spaces beyond the case where X is a Fréchet space (for this case there are powerful results of Vogt [9] and e.g. in [3,10]). It is therefore reasonable to restrict oneselves to smaller categories. The appropriate category for distribution theory is that of PLS-spaces which had been worked out by Domański and Vogt [1]. Strong splitting results for this category were obtained in [1,2,11].

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