A Note on the Number of Solutions of the Generalized Ramanujan–Nagell Equation $D_1 x^2 + D_2 = 4p^n$

Maohua Le*

Department of Mathematics, Zhanjiang Teachers College, P.O. Box 524048, Zhanjiang, Guangdong, People’s Republic of China

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Let $D_1, D_2$ be positive integers with $2 | D_1 D_2$ and $\gcd(D_1, D_2) = 1$. Let $p$ be a prime with $p | D_1 D_2$. In this note we prove that the generalized Ramanujan–Nagell equation $D_1 x^2 + D_2 = 4p^n$ has at most two positive integer solutions $(x, n)$ except $(D_1, D_2, p) = (1, 7, 2), (3, 5, 2), (1, 11, 3), \text{ and } (1, 19, 5)$.

1. INTRODUCTION

Let $\mathbb{Z}, \mathbb{N}, \mathbb{Q}$ be the sets of integers, positive integers, and rational numbers, respectively. Let $D_1, D_2$ be positive integers with $2 | D_1 D_2$ and $\gcd(D_1, D_2) = 1$. Let $p$ be a prime with $p | D_1 D_2$. For a fixed pair $(D_1, D_2, p)$, let $N(D_1, D_2, p)$ denote the number of solutions $(x, n)$ of the generalized Ramanujan–Nagell equation

$$D_1 x^2 + D_2 = 4p^n,$$

$x, n \in \mathbb{N}$. (1)

There are many works concerned with $N(D_1, D_2, p)$, including the following:

1. (Nagell [7]). $N(1, 7, 2) = 5$.
2. (Beukers [2]). If $D_2 = 23$ or $D_2 = 2^r - 1$ for some $r \in \mathbb{N}$ with $r > 3$, then $N(1, D_2, 2) = 2$, otherwise $N(1, D_2, 2) \leq 1$ except $N(1, 7, 2) = 5$.
3. (Skinner [8]). If $p > 2$ and $D_2 = 4p^r - 1$ for some $r \in \mathbb{N}$, then $N(1, D_2, p) = 2$ except $N(1, 11, 3) = N(1, 19, 5) = 3$.
4. (Le [3]). If $p > 2$ and $D_2 \neq 4p^r - 1$, then $N(1, D_2, p) \leq 2$.
5. (Le [4]). If $D_1 > 1$, then $N(D_1, D_2, 2) \leq 2$ except $N(3, 5, 2) = 3$.

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6. (Bender and Herzberg [1]). If $D_1 > 1$, $p > 2$ and $(D_1, D_2) \neq (3, 4p' - 3)$ for some $r \in \mathbb{N}$, then $N(D_1, D_2, p) \leq 2$.

In this note we give out the upper bound for $N(D_1, D_2, p)$ for the remaining cases as follows:

**Theorem.** If $p > 2$ and $(D_1, D_3) = (3, 4p' - 3)$ for some $r \in \mathbb{N}$, then $N(D_1, D_2, p) \leq 2$.

On combining this with the above results, we obtain the following result immediately.

**Corollary.** For any pair $(D_1, D_2, p)$, we have $N(D_1, D_2, p) \leq 2$ except $N(1, 7, 2) = 5$ and $N(3, 5, 2) = N(1, 11, 3) = N(1, 19, 5) = 3$.

## 2. PRELIMINARIES

**Lemma 1** [6, Formula 3.76]. For any positive integer $t$ and any complex numbers $\alpha$ and $\beta$, we have

$$x'^2 + \beta^2 = \sum_{i=0}^{\lfloor t/2 \rfloor} \binom{t}{i} (\alpha + \beta)^{t-2i} (\alpha \beta)^i,$$

where

$$\binom{t}{i} = \frac{(t-i-1)! t}{(t-2i)! i!}, \quad i = 0, \ldots, \lfloor t/2 \rfloor$$

are positive integers.

**Lemma 2** [5, Proof of Theorem]. Let $\varepsilon = (a \sqrt{D_1} + \sqrt{-D_2})/2$ and $\varepsilon = (a \sqrt{D_1} - \sqrt{-D_2})/2$, where $a$ is a positive integer with $2 \nmid \{ a \}$. If $|\varepsilon - \varepsilon'| \leq |\varepsilon - \varepsilon|$ for some $t \in \mathbb{N}$, then we have $t < 8.5 \times 10^6$.

**Lemma 3** [1, Theorems 2 and 3]. For $D_1 > 1$, $D_2 > 1$, and $p > 2$, if the equation

$$D_1 X^2 + D_2 Y^2 = 4p'z, \quad X, Y, Z \in \mathbb{Z}, \quad \gcd(X, Y) = 1, \quad Z > 0,$$

has solutions $(X, Y, Z)$, then it has a unique solution $(X_1, Y_1, Z_1)$ such that $X_1 > 0, Y_1 > 0, \text{ and } Z_1 \leq Z$, where $Z$ through all solutions $(X, Y, Z)$ of (2).
(X₁, Y₁, Z₁) is called the least solution of (2). Then, every solution (X, Y, Z) of (2) can be expressed as

\[ Z = Z₁ t, \quad \frac{X \sqrt{D₁} + Y \sqrt{-D₂}}{2} = λ₁ \left( \frac{X \sqrt{D₁} + λ₂ Y₁ \sqrt{-D₂}}{2} \right)^t, \quad t \in \mathbb{N}, \quad \gcd(6, t) = 1, \]

\[ λ₁, λ₂ \in \{-1, 1\}. \]

**Lemma 4.** For D₁ > 1, D₂ > 1, and p > 2, if N(D₁, D₂, p) > 2, then (1) has two solutions (x₁, n₁) and (x₂, n₂) such that

\[ n₁ = Z₁ t₁, \quad n₂ = Z₁ t₂, \quad 3 < t₁ < t₂, \]

where (X₁, Y₁, Z₁) is the least solution of (2), t₁ and t₂ are odd primes satisfying

\[ \frac{π}{2} \mathrm{Arc} \sin(D₂/p^n)^{1/2} < t₂ < 8.5 \times 10^6, \]

where \( \mathrm{Arc} \sin z \) is the principal value of the anti-sine function \( \sin z \).

**Proof.** Let (x, n) be a solution of (1). Then (x, 1, n) is a solution of (2). Let (X₁, Y₁, Z₁) be the least solution of (2). By Lemma 3, we have

\[ n = Z₁ t, \quad t \in \mathbb{N}, \quad \gcd(6, t) = 1, \]

\[ \frac{x \sqrt{D₁} + \sqrt{-D₂}}{2} = λ₁ \left( \frac{X₁ \sqrt{D₁} + λ₂ Y₁ \sqrt{-D₂}}{2} \right)^t, \quad λ₁, λ₂ \in \{-1, 1\}. \]

(6)

Let \( x = λ₁(X₁ \sqrt{D₁} + λ₂ Y₁ \sqrt{-D₂})/2 \) and \( \beta = λ₁(X₁ \sqrt{D₁} - λ₂ Y₁ \sqrt{-D₂})/2 \). Since \( x - β = λ₁ λ₂ Y₁ \sqrt{-D₂} \) and \( xβ = p^n \), by Lemma 1, we get from (6) that

\[ 1 = \frac{λ₀ - λ₁}{\sqrt{-D₂}} = λ₁ λ₂ Y₁ \frac{λ₀ + ( - β)'}{λ₀ + ( - β)} \]

\[ = λ₁ λ₂ Y₁ \sum_{i = 0}^{(i - 1)/2} \left[ \frac{1}{i} ( - D₂ Y₁^{(i - 1)/2 - i} p^{n'i}). \right. \]

Since \( Y₁ \in \mathbb{N} \), this implies that (1) has solutions (x, n) if and only if \( Y₁ = 1 \). Clearly, if \( Y₁ = 1 \), then (1) has a solution (x, n) = (X₁, Z₁). Further, let

\[ ε = \frac{X₁ \sqrt{D₁} + \sqrt{-D₂}}{2}, \quad \bar{ε} = \frac{X₁ \sqrt{D₁} - \sqrt{-D₂}}{2}. \]

(7)
If (1) has a solution \((x, n)\) with \((x, n) \neq (X_1, Z_1)\), then we have
\[
n = Z_1 t, \quad t \in \mathbb{N}, \quad t > 1, \quad \gcd(6, t) = 1, \quad (8)
\]
\[
\frac{e' - \bar{e}^t}{e - \bar{e}} = \sum_{i=0}^{(t-1)/2} \left[ \begin{array}{c} t \\ i \end{array} \right] \left( -D_2 \right)^{(t-1)/2 - i} p^{i t} = \pm 1. \quad (9)
\]
Therefore, by (8), if \(N(D_1, D_2, p) > 2\), then (1) has two solutions \((x_1, n_1)\) and \((x_2, n_2)\) satisfying (3). Moreover, we may assume that (1) has no solution \((x, n)\) satisfying
\[
Z_1 < n < n_1 \quad \text{or} \quad n_1 < n < n_2. \quad (10)
\]
For any \(t \in \mathbb{N}\) with \(\gcd(6, t) = 1\), let \(Y_t = [(e' - \bar{e}^t)/(e - \bar{e})]\). Then, we see from (9) that (1) has a solution \((x, n)\) with (8) if and only if \(Y_t = 1\). Hence, by (3), we get
\[
Y_{n_1} = Y_{n_2} = 1, \quad (11)
\]
and by (10),
\[
Y_t \neq 1, \quad 1 < t < t_1 \quad \text{or} \quad t_1 < t < t_2. \quad (12)
\]
If \(t_1\) is not an odd prime, then \(t_1 = k_1 k_2\), where \(k_1, k_2 \in \mathbb{N}\) satisfy \(k_1 > 1, k_2 > 1\) and \(\gcd(6, k_1) = \gcd(6, k_2) = 1\). By Lemma 1, we get from (11) that
\[
1 = Y_{t_1} = \left[ \frac{e^{k_1} - \bar{e}^{k_1}}{e - \bar{e}} \left( \frac{(k_1 - 1) k_2 + (k_1 k_2 - 1) k_2}{k_1 - 1} \right) \right] = Y_{k_1} \sum_{j=0}^{(t_1 - 1)/2} \left[ \begin{array}{c} k_2 \\ j \end{array} \right] (-D_2 Y_{t_1}^2)^{(k_2 - 1)/2 - j} p^{j k_1}.
\]
This implies that \(Y_{k_1} = Y_{t_1}/Y_{t_1} = 1\), which contradicts with (12). Thus \(t_1\) must be an odd prime with \(t_1 > 3\), since \(\gcd(6, t_1) = 1\). By the same argument, if \(t_2\) is not an odd prime, then from (11) and (12) we get \(t_2 = t_1^2\). In this case, by Lemma 1, we have
\[
\frac{e^{t_1} - \bar{e}^{t_1}}{e - \bar{e}} = \sum_{i=0}^{(t_1-1)/2} \left[ \begin{array}{c} t_1 \\ i \end{array} \right] \left( -D_2 \right)^{(t_1-1)/2 - i} p^{i t_1} = \pm 1 \quad (13)
\]
and
\[
\frac{e^{t_1} - \bar{e}^{t_1}}{e^{t_1} - \bar{e}^{t_1}} = \sum_{i=0}^{(t_1-1)/2} \left[ \begin{array}{c} t_1 \\ i \end{array} \right] \left( -D_2 \right)^{(t_1-1)/2 - i} p^{i t_1} = \pm 1. \quad (14)
\]
From (13) and (14), there exists a suitable $\delta \in \{-1, 1\}$ such that

\[
(( -D_2)^{(t_1-1)/2} - \delta) + \left( \frac{t_1}{i} \right) ( -D_2)^{(t_1-3)/2} p^{u_1} + \left( \frac{t_1}{2} \right) ( -D_2)^{(t_1-5)/2} p^{2u_1}
\equiv 0 \pmod{p^{3u_1}}
\]  
(15)

and

\[
( -D_2)^{(t_1-1)/2} - \delta \equiv 0 \pmod{p^{u_1}},
\]
(16)
respectively. Since $t_1 > 3$, by (16), $( -D_2)^{(t_1-1)/2} - \delta \equiv 0 \pmod{p^{u_1}}$, and by (15),

\[
t_1 \left( D_2 - \left( \frac{t_1-3}{2} \right) p^{u_1} \right) \equiv 0 \pmod{p^{2u_1}}.
\]

This is impossible, since $p \nmid D_2$. Thus, $t_2$ is an odd prime too.

By (7), we have

\[
\varepsilon = p^{u_1/2} \phi \sqrt{-1}, \quad \bar{\varepsilon} = p^{u_1/2} e^{-\phi} \sqrt{-1},
\]

where $\phi$ is a real number satisfying,

\[
\sin \phi = \frac{\varepsilon - \bar{\varepsilon}}{2p^{u_1/2} \sqrt{-1}} = (D_2/4p^{u_1})^{1/2}.
\]
(17)

Similarly, we see from (11) that

\[
|\sin n_1 \phi| = (D_2/4p^{n_1})^{1/2}, \quad |\sin n_2 \phi| = (D_2/4p^{n_2})^{1/2}.
\]
(19)

By (19), there exist suitable nonnegative integers $k_1$ and $k_2$ such that

\[
|k_1 \pi - n_1 \phi| = \text{Arc sin}(D_2/4p^{n_1})^{1/2},
\]

\[
|k_2 \pi - n_2 \phi| = \text{Arc sin}(D_2/4p^{n_2})^{1/2}.
\]
(20)

Since $n_1 < n_2$ and $\text{Arc sin}(D_2/4p^{n_1})^{1/2} > \text{Arc sin}(D_2/4p^{n_2})^{1/2}$, we get from (20) that

\[
0 < |k_1/n_1 - k_2/n_2| \pi < \frac{1}{n_1} \text{Arc sin} \left( D_2/4p^{n_1} \right)^{1/2} + \frac{1}{n_2} \text{Arc sin} \left( D_2/4p^{n_2} \right)^{1/2}.
\]
(21)

Since $|k_1/n_1 - k_2/n_2| \geq Z_1/n_1 n_2$ if $k_1/n_1 \neq k_2/n_2$ by (3), we see from (21) that

\[
t_2 > \frac{\pi}{2} \text{Arc sin}(D_2/4p^{n_1})^{1/2}.
\]
(22)
On the other hand, by Lemma 2, if \( Y_t = |(e'^2 - e'^3)/(e - \bar{e})| = 1 \), then we have

\[ t_1 < 8.5 \times 10^6. \] (23)

The combination of (22) and (23) yields (4). The lemma is proved.

3. PROOF OF THEOREM

If \( D_1 = 3 \) and \( D_2 = 4p' - 3 \) for some \( r \in \mathbb{N} \), then Eq. (2) has solutions \((X, Y, Z)\) and its least solution is \((X_1, Y_1, Z_1) = (1, 1, r)\). By Lemma 4, if \( N(D_1, D_2, p) > 2 \), then Eq. (1) has two solutions \((x_1, n_1)\) and \((x_2, n_2)\) such that

\[ n_1 = rt_1, \quad n_2 = rt_2, \quad 3 < t_1 < t_2, \] (24)

where \( t_1 \) and \( t_2 \) are odd primes satisfying

\[ \frac{1}{2} \text{Arc sin}((4p' - 3)/4p'^{rt_1})^{1/2} < t_2 < 8.5 \times 10^6. \] (25)

Let

\[ e = \frac{\sqrt{3} + \sqrt{4p' - 3}}{2}, \quad \bar{e} = \frac{\sqrt{3} - \sqrt{4p' - 3}}{2}. \] (26)

and let \( Y_t = |(e'^2 - e'^3)/(e - \bar{e})| \) for any \( t \in \mathbb{N} \) with \( \gcd(6, t) = 1 \). By the proof of Lemma 4, we have \( Y_{t_1} = 1 \) and

\[ \frac{e'^{t_1} - \bar{e}^{t_1}}{e - \bar{e}} = \sum_{i=0}^{(t_1 - 1)/2} \left( t_1 \right)_i (3 - 4p')^{(t_1 - 1)/2 - i} p'^i = \pm 1. \] (27)

If \( t_1 = 5 \), then we get from (27) that \( p^{2r} - 9p' + 9 = \pm 1 \) and \( p' = 8 \), a contradiction. If \( t_1 = 7 \), then we have \( p^{2r} + 18p'^9 - 45p' + 27 = \pm 1 \), which is impossible. Therefore, we obtain \( t_1 \geq 11 \). Then we have \((4p' - 3) > p^{10}\). Since \( p \neq 3 \) and \( \text{Arc sin } z < \pi z/2 \) if \( 0 < z < 1 \), we get from (25) that

\[ 8.5 \times 10^6 > \frac{\pi}{2} \text{Arc sin}(1/p^{10}) > p \]

whence we conclude that \( p' \leq 23 \).
On the other hand, we can conclude that (27) is false for any odd prime power \( p^r \) with \( p^r < 23 \) and any odd prime \( t_1 \) with \( 11 < t_1 < 19 \). Therefore, if \( p^r < 23 \), then we have \( t_1 \geq 23 \), and by (25),

\[
8.5 \times 10^6 \geq \frac{2}{\pi} \arcsin((4p^r - 3)/4p^r)^{1/2} > 2^{11r} \geq 2^{11} > 4 \times 10^7,
\]
a contradiction. Thus, we deduce that \( N(3, 4p^r - 3, p) < 2 \). The theorem is proved.

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