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Singularities of a Variational Wave Equation

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We analyze several aspects of the singular behavior of solutions of a variational nonlinear wave equation which models orientation waves in a massive nematic liquid crystal director field. We prove that smooth solutions develop singularities in finite time. We construct exact travelling wave solutions with cusp singularities, and use them to illustrate a phenomena of accumulation and annihilation of oscillations in sequences of solutions with bounded energy. We also prove that constant solutions of the equation are nonlinearly unstable. © 1996 Academic Press, Inc.

1. INTRODUCTION

We consider the nonlinear wave equation

$$u_{tt} - c(u)(c(u)\, u_x)_x = 0, \tag{1.1}$$

where the wave speed c is a given positive function of u. Local existence of smooth solutions to the Cauchy problem for (1.1) follows by standard arguments (see [13], for example). The purpose of this paper is to prove that (1.1) does not have global smooth solutions for general smooth initial data, and that the derivatives u_t, u_x typically become infinite in finite time.

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The singularity formation of smooth solutions of (1.1) is suggested by previous results on singularity formation for the following asymptotic equation [8, 9]

$$(u_t + uu_x)_x = \tfrac{1}{2}u_x^2. \tag{1.2}$$

This equation gives a formal description of weakly nonlinear, unidirectional solutions of (1.1). Although smooth solutions of (1.2) break down, the equation has global continuous weak solutions [9]. This fact suggests that (1.1) also has global continuous weak solutions. However, this question remains open, despite the apparent simplicity of (1.1).

One motivation for studying (1.1) comes from liquid crystals. We give a brief explanation of how the equation arises in that context, and how the liquid crystal problem differs from the related problem of harmonic maps from Minkowski space to the two-sphere. For further details, see [8, 9, 19]. The mean orientation of the molecules in a nematic liquid crystal is described by a director field of unit vectors, $\mathbf{n} \in \mathbb{S}^2$. We consider a regime in which inertia effects dominate viscosity. The propagation of orientation waves in the director field is then modelled by a constrained variational principle

$$\delta \int \{\mathbf{n}_t \cdot \mathbf{n}_t - W(\mathbf{n}, \nabla \mathbf{n})\}\, d\mathbf{x}\, dt = 0, \qquad \mathbf{n} \cdot \mathbf{n} = 1,$$

where W is the Oseen–Franck potential energy density,

$$W(\mathbf{n}, \nabla \mathbf{n}) = \alpha\, |\mathbf{n} \times (\nabla \times \mathbf{n})|^2 + \beta(\nabla \cdot \mathbf{n})^2 + \gamma(\mathbf{n} \cdot \nabla \times \mathbf{n})^2.$$

This potential energy is determined (up to a null Lagrangian) by the requirement that it is invariant under reflections $\mathbf{n} \to -\mathbf{n}$ and under simultaneous rotations O of the spatial variables and the director field, $\mathbf{x} \to O\mathbf{x}$, $\mathbf{n} \to O\mathbf{n}$. The positive constants α, β, γ are elastic constants of the liquid crystal.

A commonly used special case is the one-constant approximation in which $\alpha = \beta = \gamma$. The potential energy density then reduces to

$$W(\mathbf{n}, \nabla \mathbf{n}) = \alpha\, |\nabla \mathbf{n}|^2.$$

The associated variational problem is identical to the variational problem for harmonic maps from $(1+3)$-dimensional Minkowski space into the two sphere, see [21] for example. For harmonic maps, the wave speed $c^2 = \alpha$ is constant, whereas for liquid crystals with distinct values of the elastic constants, the wave speed depends on $\mathbf{n}$. Thus the problem for harmonic maps into the two sphere is a degenerate case of the liquid crystal problem.

The type of singularity we investigate here is subtly different from the type of singularity which arises in harmonic maps. The mechanism of singularity formation depends crucially on the fact that the wave speed is a nonconstant function of the dependent variable. In particular, the critical dimension for the liquid crystal wave equation is $n=1$, whereas the critical dimension for harmonic maps is $n=2$. We give a more detailed discussion of this fact in Section 7.

The simplest class of solutions for orientation waves in a nematic liquid crystal consists of planar deformations depending on a single space variable x. The director field then has the special form

$$\mathbf{n} = \cos u(x,t)\,\mathbf{e}_x + \sin u(x,t)\,\mathbf{e}_y.$$

Here, the dependent variable $u \in \mathbb{S}^1$ measures the angle of the director field to the x-direction, and $\mathbf{e}_x$ and $\mathbf{e}_y$ are the coordinate vectors in the x and y directions, respectively. In this case, the variational principle for $\mathbf{n}$ reduces to

$$\delta \int \{u_t^2 - c^2(u)\,u_x^2\}\,dx\,dt = 0,$$

with the wave speed c given by

$$c^2(u) = \alpha \cos^2 u + \beta \sin^2 u. \tag{1.3}$$

The Euler–Lagrange equation for this variational principle is (1.1). In the harmonic map case we have $\alpha = \beta$, and equation (1.1) reduces to the standard linear wave equation.

We now state our main singularity formation result. The proof is given in Section 2.

THEOREM 1. *Assume that $c(u) \in C^2(\mathbb{R})$ satisfies the following conditions*: (a) *there exist positive constants $0 < c_0 < c_1 < \infty$ such that*

$$c_0 \leqslant c(u) \leqslant c_1$$

for all $u \in \mathbb{R}$; (b) *for some $u_0 \in \mathbb{R}$*,

$$c'(u_0) \neq 0.$$

Suppose that $u(t,x) \in C^1([0,\ T) \times \mathbb{R})$ is a smooth solution of (1.1) *in $0 \leqslant t < T$ with initial data*

$$u(0,x) = u_0 + \varepsilon\phi\left(\frac{x}{\varepsilon}\right) \tag{1.4}$$

$$u_t(0,x) = -\operatorname{sgn}(c'(u_0))\,c(u(0,x))\,u_x(0,x) \tag{1.5}$$

where $\varepsilon > 0$ *is sufficiently small,* $\varphi \in C_c^1(0, 1)$ *with* $\varphi \not\equiv 0$. *Then* $T < \infty$, *so a global smooth solution does not exist.*

To prove this theorem, we show that if u is smooth, then u_t and u_x must blow up in finite time. We use the weakest notion of "smooth solution" compatible with (1.1). That is, we say that u is a smooth solution of (1.1) if it has continuous first order partial derivatives u_t and u_x, and satisfies (1.1) in the sense of distributions in the space $W^{1,\infty}(\mathbb{R}_+^2)$. The energy estimate,

$$E(u) \equiv \int_{-\infty}^{\infty} \{u_t^2 + c^2(u)\, u_x^2\}\, dx = \text{constant}, \tag{1.6}$$

implies that smooth solutions are uniformly Hölder continuous with exponent one half. This fact suggests that solutions remain continuous even after their first derivatives become infinite —a result which has been proved rigorously for the asymptotic equation (1.2).

This result should be contrasted with a result of Lindblad [17], who established the global existence of smooth solutions of the equation

$$u_{tt} - c^2(u)\, \Delta u = 0 \tag{1.7}$$

with smooth, small, and spherically symmetric initial data in $\mathbb{R}^3$. The multidimensional generalization of equation (1.1),

$$u_{tt} - c(u)\, \nabla \cdot (c(u)\, \nabla u) = 0, \tag{1.8}$$

contains a lower order term proportional to $cc'\, |\nabla u|^2$, which (1.7) lacks. This lower order term is responsible for the blow-up in the derivatives of u.

Equation (1.1) also looks very similar to the perturbed wave equation

$$u_{tt} - \Delta u + f(u, Du, D^2u) = 0, \tag{1.9}$$

where $f(u, Du, D^2u)$ satisfies an appropriate convexity condition (for example, $f = u^p$ or $f = au_t^2 + b\, |Du|^2$). Blow-up for (1.9) has been studied extensively by Levine [16], John [10], Glassey [5], Sideris [24, 25], Schaeffer [20], Kato [12], Hanouzet and Joly [7], Balabane [1], and others, using integral methods (see Strauss [26] for detailed references). It is therefore tempting to apply these methods to equation (1.1). However, the possibility of sign changes in $c'(u)$ makes equation (1.1) truly different. The growth of singularities in solutions of (1.1) occurs on a much smaller spatial scale than for (1.9). As a result, the methods developed for (1.9) do not seem to be fine enough to catch the singularities of (1.1).

We therefore use a different approach to prove singularity formation, based on the method of characteristics. (Unlike integral methods, this approach only works in the case of one space dimension.) We first write (1.1) as a system of first order equations, introducing new dependent variables

$$\begin{aligned} R &= u_t + c(u)\, u_x, \\ S &= u_t - c(u)\, u_x. \end{aligned} \tag{1.10}$$

Then, for smooth solutions, equation (1.1) is equivalent to the following system for (R, u, S),

$$\begin{aligned} R_t - cR_x &= \frac{c'}{4c}\,(R^2 - S^2) \\ u_t &= \frac{1}{2}\,(R+S) \\ S_t + cS_x &= \frac{c'}{4c}\,(S^2 - R^2). \end{aligned} \tag{1.11}$$

Any smooth solution of (1.11) satisfying the constraint

$$u_x = \frac{R-S}{2c} \tag{1.12}$$

gives a smooth solution of (1.1), and conversely. The constraint (1.12) is preserved by the first order system, since (1.11) implies that

$$(R - S - 2cu_x)_t = \frac{c'}{2c}\,(R+S)(R - S - 2cu_x).$$

Equation (1.11) is similar to a system of hyperbolic conservation laws with a nonlinear source term and three linearly degenerate characteristics (see [11, 14, 15, 18] for various results on singularity formation for systems of conservation laws in one space dimension). However, (1.11) cannot be written in conservative form.

In systems of conservation laws, the formation of singularities is usually caused by the crossing of characteristics. This crossing does not occur when the characteristics are linearly degenerate, as is the case for (1.11). Instead, it is the nonlinear source terms which drive the singularities in solutions of (1.11). The resulting singularities are indeed different from those of conservation laws: the cusp singularities of solutions of equation (1.1) contrast with the $W^{1,\infty}$-regularity of solutions of the conservation law

$$u_{tt} - (p(u_x))_x = 0 \tag{1.13}$$

considered by Lax [15]. It is interesting to note that solutions of (1.13)—with a "stronger" u_x-dependent nonlinearity—are more regular than solutions of (1.1)—with an apparently "weaker" u-dependent nonlinearity. (This kind of behavior is well-known for nonlinear parabolic partial differential equations.) In each case it appears that singularities develop to the maximum extent permitted by the existence of global weak solutions. Thus, u_x remains bounded for (1.13), but is merely in L^2 for (1.1)

The equations for (R, S) in (1.11) closely resemble the Carleman system, which can be written in the form

$$\begin{aligned} R_t - R_x &= R^2 - S^2 \\ S_t + S_x &= S^2 - R^2. \end{aligned} \tag{1.14}$$

Balbane [2, 3] has studied blow–up for systems like (1.14). The key difference between (1.11) and (1.14) is that in (1.11), the coefficient $c'/4c$ of the quadratically nonlinear term driving the singularity can change sign. If this happens, blow-up may be delayed or even completely prevented.

The reason for introducing a small parameter $\varepsilon > 0$ in the initial data (1.4) is to ensure that $c'(u)$ does not have time to change sign before singularities in u_x or u_t form. The initial data for u_t in (1.5) is chosen to enhance the growth of S and suppress the growth of R. (If the initial data for u_t had the opposite sign, then R would become infinite instead of S.) The energy estimate (1.6) implies that it takes an $O(1/\varepsilon)$-time for u to grow an $O(1)$-distance away from u_0. Therefore, $c'(u)$ does not change sign in $O(1)$ time. We then use another energy estimate over a characteristic cone to show that u_x and u_t become infinite in $O(1)$ time.

For general small initial data, solutions of (1.1) typically do grow large enough that c' changes sign (although they may well lose smoothness before this happens). This fact was shown in [8] by a formal asymptotic argument. In Section 3, we use an averaged integral method to prove rigorously that arbitrarily small smooth disturbances around a constant state u_0 can grow so that $c'(u)$ must change sign—see Theorem 2.

In Section 4, we present some explicit travelling wave solutions of (1.1) which illustrate typical singularities (see Figure 1.1).
This travelling wave has a cusp singularity at its crest of the form

$$u(t, x) - u(t, x^*) \sim A(x - x^*)^{2/3} \qquad \text{as} \quad x \to x^*$$

for some constant A.

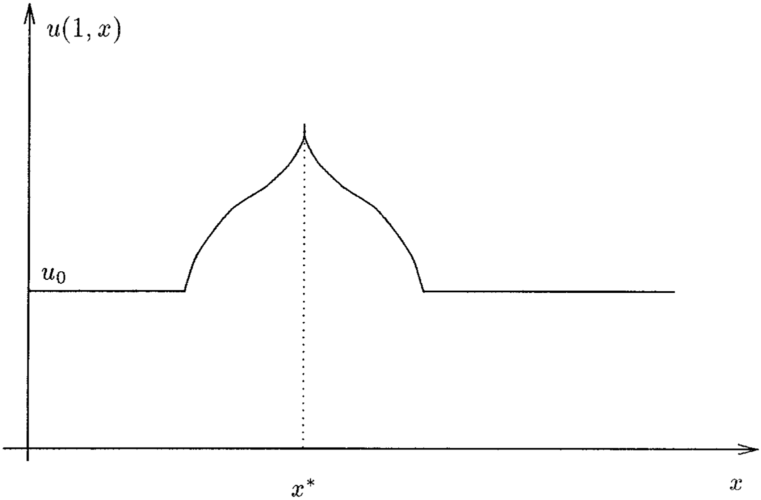

FIG. 1.1. A cusp solution at $t=1$.

In Section 5, we study the weak convergence of a sequence of exact solutions $\{u^n(t,x)\}$ with uniformly bounded energy. These solutions are constructed by patching together n cusp solutions of the type shown in Fig. 1.1. The interesting fact is that there is persistence of oscillations in the term $(u_x^n)^2$ yet annihilation of oscillations in $c'(u^n)(u_x^n)^2$.

In Section 6, we consider a wavefront expansion for (1.1). This expansion illustrates some effects of sign changes in c' on the development of singularities.

In the final section (Section 7), we give a heuristic argument for singularity formation in solutions of equation (1.1), and compare the results with singularity formation in harmonic wave maps.

2. Singularity Formation

We prove Theorem 1 in this section. From equations (1.11) and (1.12), we derive the identity

$$(R^2+S^2)_t+(c(u)(S^2-R^2))_x=0. \tag{2.1}$$

Integrating (2.1) with respect to x over $\mathbb{R}$, we obtain the energy estimate (1.6). We can integrate (2.1) over a backward characteristic cone to obtain another useful energy estimate. Given any point (t_0, x_0) in the upper half plane $t>0$, let $t_{\pm}(x)$ denote the plus and minus characteristics through (t_0, x_0), extended backward in time:

$$\frac{dt_{\pm}(x)}{dx} = \pm\frac{1}{c(u)}, \qquad t_{\pm}(x_0) = t_0, \qquad 0 \leqslant t \leqslant t_0.$$

(See Fig. 2.1). Let x_1 and x_2 denote the intersection points of $t_{\pm}$ with the x–axis. Integrating the identity (2.1) over the region bounded by t_+, t_- and the segment $[x_1, x_2]$, then using the divergence theorem, we obtain

$$\int_{x_1}^{x_0} R^2(t_+(x), x)\, dx + \int_{x_0}^{x_2} S^2(t_-(x), x)\, dx = \tfrac{1}{2} \int_{x_1}^{x_2} (R^2(0, x) + S^2(0, x))\, dx. \tag{2.2}$$

For the data (1.4) and (1.5), we estimate the total energy

$$\begin{aligned} E(u) &= \int_{-\infty}^{\infty} (u_t^2 + c^2 u_x^2)\, dx \\ &= \int_0^{\varepsilon} 2c^2 \varphi' \left(\frac{x}{\varepsilon}\right)^2 dx \\ &\leqslant 2\varepsilon c_1^2 \, \|\varphi'\|_{L^2}^2 \\ &= M\varepsilon \end{aligned} \tag{2.3}$$

for some constant M, where c_1 is an upper bound for $c(u)$.

For smooth solutions, equation (1.1), and therefore the system (1.11), have finite propagation speed. We thus obtain that $u = u_0$ in the regions

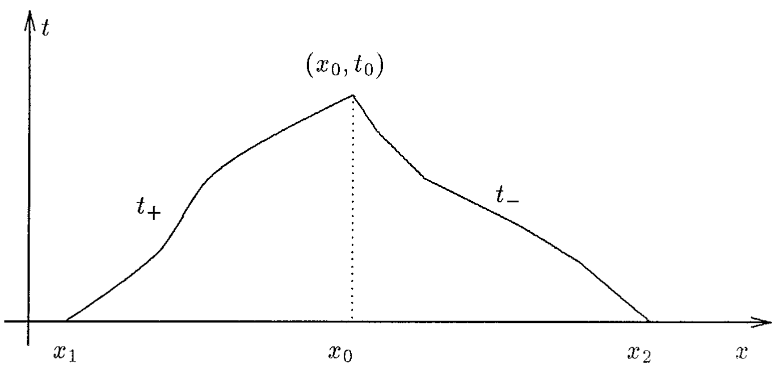

FIG. 2.1. A characteristic region.

$x < -c_1 t$ and $x > \varepsilon + c_1 t$. Using the energy estimate (2.3), we can estimate the deviation of u from u_0 as follows:

$$
\begin{aligned}
|u(t, x) - u_0| &= \left| \int_{-\infty}^{x} u_x(t, x)\, dx \right| \\
&\leqslant \int_{-c_1 t}^{\varepsilon + c_1 t} |u_x(t, x)|\, dx \\
&\leqslant \|u_x\|_{L^2} \sqrt{2c_1 t + \varepsilon} \\
&\leqslant \frac{1}{c_0} (M\varepsilon)^{1/2} \sqrt{2c_1 t + \varepsilon} \\
&= K \sqrt{2c_1 t\varepsilon + \varepsilon^2}
\end{aligned}
\tag{2.4}
$$

for some constant K.

We need to fix the sign of $c'(u_0)$. If $c'(u_0) > 0$, we choose a point $p_0 \in (0, 1)$ such that $\varphi'(p_0) < 0$. If $c'(u_0) < 0$, we choose $p_0 \in (0, 1)$ with $\varphi'(p_0) > 0$. The proofs for the two cases are similar, so we treat only the first case. Thus we assume that $c'(u_0) > 0$ and $\varphi'(p_0) < 0$. We choose $\delta > 0$ small enough that $c'(u)$ does not change sign in the interval $[u_0 - \delta, u_0 + \delta]$. Then

$$
0 < c_0' \leqslant c'(u) \leqslant c_1' \qquad \text{for} \quad u \in (u_0 - \delta, u_0 + \delta) \tag{2.5}
$$

for some positive constants c_0' and c_1'. We also choose sufficiently small numbers ε_0, $\sigma > 0$ so that the term on the right hand side of (2.4) satisfies

$$
K \sqrt{2c_1 t\varepsilon + \varepsilon^2} \leqslant \delta, \qquad \text{for} \quad 0 < \varepsilon \leqslant \varepsilon_0,\, 0 \leqslant t \leqslant \sigma/\varepsilon.
$$

Thus, for each fixed $\varepsilon \in (0, \varepsilon_0]$, the function $c'(u)$ evaluated along the solution u has the upper and lower bounds given in (2.5) for a time interval $0 \leqslant t \leqslant \sigma/\varepsilon$.

Next, we show that R is bounded. More exactly, we show that there is a constant $K > 0$ such that

$$
-K\varepsilon \leqslant R(t, x) \leqslant 0 \qquad \text{for} \quad 0 < \varepsilon \leqslant \varepsilon_0,\, 0 \leqslant t \leqslant \sigma/\varepsilon. \tag{2.6}
$$

The initial data for R is zero. From (1.11), the derivative of R along a minus characteristic $dx/dt = -c(u)$ satisfies

$$
\frac{dR}{dt} = \frac{c'}{4c} (R^2 - S^2) \leqslant \frac{c_1'}{4c_0} R^2
$$

for $0<\varepsilon\leqslant\varepsilon_0$, $0\leqslant t\leqslant\sigma/\varepsilon$. It follows by a standard comparison theorem that $R\leqslant 0$ in the region, since the only solution of the initial value problem

$$\frac{dR}{dt}=\frac{c_1'}{4c_0}R^2, \qquad R(0)=0$$

is the zero solution.

Estimating dR/dt from below gives

$$\frac{dR}{dt}=\frac{c'}{4c}(R^2-S^2)\geqslant -\frac{c_1'}{4c_0}S^2.$$

Therefore, integrating along the minus characteristics we get

$$\begin{aligned}
R(t_0, x_0) &\geqslant -\frac{c_1'}{4c_0}\int_0^{t_0} S^2(t, x_-(t))\,dt \\
&= -\frac{c_1'}{4c_0}\int_{x_0}^{x_2} \frac{S^2(t_-(x), x)}{c(u(t_-(x), x))}\,dx \\
&\geqslant -\frac{c_1'}{4c_0^2}E(u) \\
&= -K\varepsilon
\end{aligned}$$

which proves (2.6).

Now we show that S becomes infinite at an $O(1)$-time. We integrate the equation for S in (1.11) along the plus characteristic $dx/dt=c(u)$ passing through the point $x=\varepsilon p_0$ at $t=0$. Using (2.6), this gives

$$\frac{dS}{dt}=\frac{c'}{4c}(S^2-R^2)\geqslant\frac{c_0'}{4c_1}S^2-K^2\varepsilon^2 \tag{2.7}$$

for some constant K in the region $0<\varepsilon\leqslant\varepsilon_0$ and $0\leqslant t\leqslant\sigma\varepsilon^{-1}$. Now, we note that

$$\begin{aligned}
S(0, \varepsilon p_0) &= -2c(u(0, \varepsilon p_0))\,\varphi'(p_0) \\
&\geqslant 2c_0(-\varphi'(p_0))>0.
\end{aligned}$$

Choosing ε_0 smaller if necessary, we can assume that

$$\left.\frac{dS}{dt}\right|_{t=0}\geqslant\frac{c_0'}{4c_1}[2c_0(-\varphi'(p_0))]^2-K^2\varepsilon^2>0$$

for all $0 < \varepsilon \leqslant \varepsilon_0$. Then S is an increasing function of t along the plus characteristics with positive data, and the quadratic growth in the inequality (2.7) will drive S to infinity. To obtain an upper-bound for the singularity formation time, we integrate the ODE (ordinary differential equation)

$$\frac{dS}{dt} = a^2 S^2 - b^2 \varepsilon^2,$$
$$S(0) = \sigma^2 > 0.$$

The solution is

$$\frac{aS - b\varepsilon}{aS + b\varepsilon} = e^{2abt\varepsilon} \frac{a\sigma^2 - b\varepsilon}{a\sigma^2 + b\varepsilon}.$$

This solution develops a singularity at time

$$t^* = \frac{1}{2ab\varepsilon} \ln \left(\frac{a\sigma^2 + b\varepsilon}{a\sigma^2 - b\varepsilon} \right) \sim \frac{1}{a^2 \sigma^2}$$

which is $O(1)$ as $\varepsilon \to 0$. Therefore S becomes infinite in time $O(1)$. Since R remains of order ε, the derivatives u_x and u_t become infinite simultaneously in an order one time, provided that a smooth solution for u exists up to that time.

3. Instability

In this section, we show that small disturbances around a constant solution $u = u_0$, where $c'(u_0) \neq 0$, can grow until $|u(x, t) - u_0|$ is so large that $c'(u) = 0$. Since smooth solutions may cease to exist before this time, we assume that equation (1.1) has global weak solutions with finite, bounded energy. The existence of finite-energy global weak solutions has been proved for the asymptotic equation (1.2). It is reasonable to assume that the same result is true for the wave equation (1.1), although this has not yet been proved.

Let $u(t, x)$ be a global solution with initial data

$$\begin{aligned} u(0, x) &= u_0 + \varphi_0(x), \\ u_t(0, x) &= \varphi_1(x). \end{aligned} \tag{3.1}$$

Here, $\varphi_0(x)$ and $\varphi_1(x)$ are given functions. We assume that they are sufficiently smooth and that their support is contained in $(-1, 1)$. We also assume that the constant unperturbed state, u_0 is such that $c'(u_0) \neq 0$. For definiteness, we suppose that $c'(u_0) < 0$. If $\varphi_0(x)$ is small enough, then $c'(u(0, x))$ is negative for all $x \in \mathbb{R}$. We will prove that there are arbitrarily small perturbations $\varphi_0(x)$ such that the solution grows so that $c'(u)$ is zero at some point. It follows that the constant state $u = u_0$ is unstable. This instability is a nonlinear instability; the size of the perturbation grows algebraically, rather than exponentially, in time.

THEOREM 2. *Assume that* $c(u) \in C^2(\mathbb{R})$, $0 < c_0 \leqslant c(u) \leqslant c_1 < \infty$, *and* $c'(u_0) < 0$. *If*

$$\int_{-1}^{1} \varphi_0 \, dx > 0 \qquad \textit{and} \qquad \int_{-1}^{1} \varphi_1(x) \, dx > 0, \tag{3.2}$$

then for any finite-energy weak solution of (1.1) *with initial data* (3.1), *either there exists a finite term* t^* *such that* $c'(u(t^*, x^*)) = 0$ *for some* x^*, *or* $c'(u(t, x))$ *is arbitrarily close to zero for large* t *or* x.

Proof. Introduce

$$F(t) = \int_{-\infty}^{\infty} [u(t, x) - u_0] \, dx. \tag{3.3}$$

By the assumption in (3.2), there exists $\delta > 0$ such that

$$\begin{aligned} F(0) &= \int_{-\infty}^{\infty} \varphi_0(x) \, dx > \delta \\ F'(0) &= \int_{-\infty}^{\infty} \varphi_1(x) \, dx > \delta. \end{aligned} \tag{3.4}$$

Differentiating $F(t)$ twice and using (1.1) in the distributional sense, we find for almost all $t > 0$ that

$$F''(t) = -\int_{-\infty}^{\infty} c(u) \, c'(u) \, u_x^2 \, dx. \tag{3.5}$$

We want to obtain a lower bound for the right-hand side of (3.5) in terms of F. Using finite speed of propagation for (1.1), we have

$$\begin{aligned} u(t, x) - u_0 &= \int_{-1-c_1 t}^{x} u_x(t, x) \, dx \\ &\leqslant \left(\int_{-\infty}^{\infty} u_x^2 \, dx \right)^{1/2} (2c_1 t + 2)^{1/2}. \end{aligned} \tag{3.6}$$

Integrating (3.6) with respect to x over the interval $(-1-c_1 t,\ 1+c_1 t)$, we find that

$$\left|\int_{-\infty}^{\infty} (u(t, x)-u_0)\, dx\right| \leqslant \left(\int_{-\infty}^{\infty} u_x^2\, dx\right)^{1/2} (2c_1 t+2)^{3/2}. \tag{3.7}$$

Thus

$$\int_{-\infty}^{\infty} u_x^2\, dx \geqslant \frac{F^2(t)}{(2c_1 t+2)^3}. \tag{3.8}$$

Our goal is to prove that $c'(u)$ changes sign eventually. Suppose, on the contrary, that

$$c'(u(t, x)) \leqslant -c_1' < 0, \qquad t>0,\ x\in\mathbb{R} \tag{3.9}$$

for all time $t>0$ and for some constant $c_1'>0$. From (3.5), (3.7) and (3.8) we deduce that

$$F''(t) \geqslant c_0 c_1' \frac{F^2(t)}{(2c_1 t+2)^3}, \qquad t\geqslant 0. \tag{3.10}$$

The inequality (3.10) together with the data (3.4) enables us to use Sideris' argument [24, p. 313] to show that F becomes infinite in finite time. But this contradicts (3.7) which shows that for any finite-energy solution, the growth rate of $F(t)$ is $(2c_1 t+2)^{3/2}$, at most. So the assumption in (3.9) is not correct, and the proof of the theorem is complete.

4. Exact Solutions with Cusp Singularities

In this section we construct cusped travelling wave solutions of the wave equation (1.1) with the specific wave speed $c(u)$ given in (1.3). We look for solutions of (1.1) of the form

$$u(t, x) = \psi(x-st)$$

where s is a constant. The function $\psi(\xi)$ satisfies the ODE

$$s^2\psi'' - c(\psi)(c(\psi)\psi')' = 0. \tag{4.1}$$

In regions where $(s^2-c^2(\psi))\psi' \neq 0$, we rewrite equation (4.1) as

$$\frac{\psi''}{\psi'} = \frac{cc'\psi'}{s^2-c^2(\psi)}.$$

Simple integration then yields

$$\psi' \sqrt{|s^2 - c^2(\psi)|} = k \tag{4.2}$$

where k is an integration constant. We can express the solution $\psi(\xi)$ implicitly as

$$\int_{\psi_0}^{\psi} \sqrt{|s^2 - c^2(v)|}\, dv = k(\xi - \xi_0) \tag{4.3}$$

where ψ_0 and ξ_0 are arbitrary constants. Since the integrand in (4.3) is strictly positive, except possibly at isolated values of $v \in \mathbb{R}$ such that $c(v) = |s|$, equation (4.3) defines monotonic, continuous solutions $\psi: \mathbb{R} \to \mathbb{R}$.

There are two cases. If $|s| \notin [c_0, c_1]$, where

$$c_0 = \min_{v \in \mathbb{R}} c(v), \qquad c_1 = \max_{v \in \mathbb{R}} c(v),$$

then the solution given in (4.3) is smooth because ψ' is finite from (4.2). Thus, unlike the Carleman equations (1.14) [2, 3], equation (1.1) has no bounded travelling wave solutions with speed $|s|$ greater than max $c(\cdot)$ or less than min $c(\cdot)$. We will not consider these smooth solutions any further here.

The second case is when $|s| \in [c_0, c_1]$. Then there exists $u_0 \in [0, \pi/2]$ such that $|s| = c(u_0)$—see Fig. 4.1. The solution $\psi(\xi)$ given in (4.3) is smooth only in the open ξ-intervals in which $c(\psi(\xi)) \neq c(u_0)$ (see Fig. 4.2). Singularities of the form

$$\psi'(\xi) = \frac{k}{\sqrt{|c^2(u_0) - c^2(\psi)|}} \to \infty \tag{4.4}$$

occur at isolated points of ξ such that $c(\psi(\xi)) = c(u_0)$. These points can be listed as

$$\xi = \xi_n^{\pm} \equiv \psi^{-1}(n\pi \pm u_0), \qquad n = 0, \pm 1, \pm 2, \cdots. \tag{4.5}$$

We claim that the function ψ given by (4.3) is still a weak solution of (4.1) even when it is not smooth.

To prove this fact, we need to estimate the strength of the singularities in ψ. We obtain from (4.2) that

$$\psi'(\xi)|s^2 - c^2(\psi)| = k \sqrt{|s^2 - c^2(\psi)|} = o(1) \tag{4.6}$$

as $\xi \to \xi_n^{\pm}$ so that $c(\psi(\xi)) \to c(u_0)$. However the integrability of ψ' and the Hölder exponent of continuity of ψ differ depending on whether $c'(u_0) = 0$

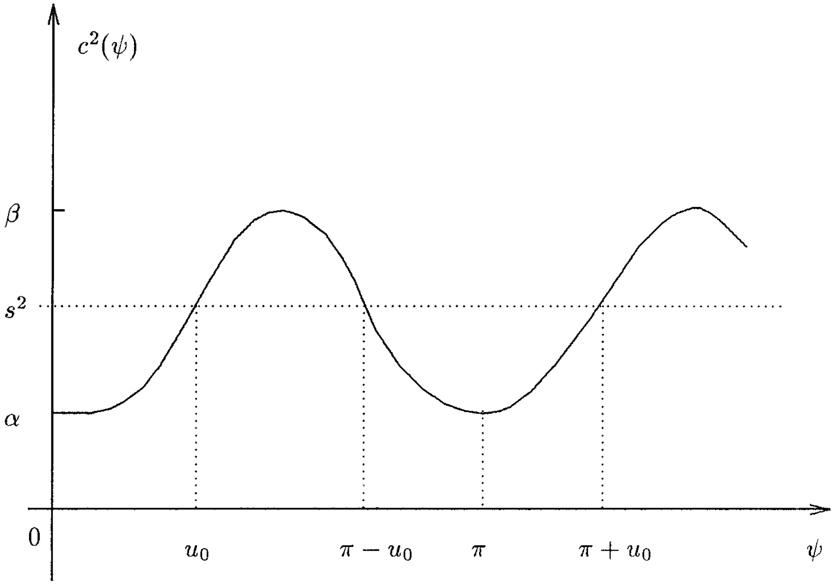

FIG. 4.1. Wave speed $c(\psi)$ and the wave speed s of a singular travelling wave.

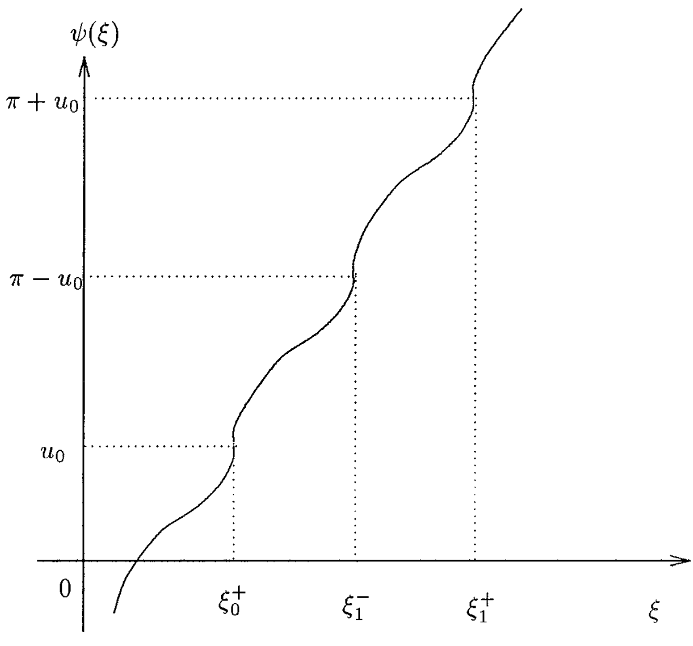

FIG. 4.2. Singularity points of a travelling wave $\psi(\xi)$.

or not. First let us assume $c'(u_0) \neq 0$. We show that $\psi'(\xi)$ is locally integrable in L^2. We calculate the integral of $|\psi'(\xi)|^2$ on a whole interval formed by two neighboring singularity points ξ_0^+ and ξ_1^-. For definiteness, we assume $k > 0$ so that $\xi_0^+ < \xi_1^-$. We then have

$$\begin{aligned}\int_{\xi_0^+}^{\xi_1^-} |\psi'(\xi)|^2 \, d\xi &= \int_{\psi^{-1}(u_0)}^{\psi^{-1}(\pi - u_0)} |\psi'(\xi)|^2 \, d\xi \\ &= \int_{\psi^{-1}(u_0)}^{\psi^{-1}(\pi - u_0)} \psi' \frac{k}{\sqrt{|s^2 - c^2(\psi)|}} \, d\xi \\ &= k \int_{u_0}^{\pi - u_0} \frac{d\psi}{\sqrt{|c^2(u_0) - c^2(\psi)|}} < \infty \end{aligned} \tag{4.7}$$

because $c'(u_0) \neq 0$. Furthermore we find from (4.3)

$$\begin{aligned} |\psi(\xi) - \psi(\xi_n^\pm)| &\sim A(\xi - \xi_n^\pm)^{2/3} \\ |\psi'(\xi)| &\sim \tfrac{2}{3} A \, |\xi - \xi_n^\pm|^{-1/3} \end{aligned} \tag{4.8}$$

as $\xi \to \xi_n^\pm$ for some constant A depending on k, c and u_0. In the case $c'(u_0) = 0$, corresponding to $u_0 = 0$ or $\pi/2$, we find that

$$\begin{aligned} |\psi(\xi) - \psi(\xi_n^\pm)| &\sim B \, |\xi - \xi_n^\pm|^{1/2} \\ |\psi'(\xi)| &\sim \tfrac{1}{2} B \, |\xi - \xi_n^\pm|^{-1/2} \end{aligned} \tag{4.9}$$

as $\xi \to \xi_n^\pm$, for some constant B. But $\psi'(\xi)$ is not in $L^2(\xi_0^+, \xi_1^-)$; that is, the solution has infinite local energy. However, the following weighted energy is finite:

$$\int_{\xi_0^+}^{\xi_1^-} |c'(\psi(\xi))| (\psi'(\xi))^2 \, d\xi < \infty. \tag{4.10}$$

We say $\psi: \mathbb{R} \to \mathbb{R}$, is a *weak solution* of (4.1) if $\psi \in C(\mathbb{R})$, $\psi' \in L^1_{\mathrm{loc}}(\mathbb{R}^1)$, $c'(\psi)\, \psi^2 \in L^1_{\mathrm{loc}}(\mathbb{R})$ and

$$-\int_{-\infty}^{\infty} \varphi'(\xi)(s^2 - c^2(\psi))\, \psi'(\xi) \, d\xi + \int_{-\infty}^{\infty} c(\psi)\, c'(\psi)(\psi')^2 \, \varphi(\xi) \, d\xi = 0 \tag{4.11}$$

for all test functions $\varphi \in C_c^\infty(\mathbb{R})$. We define weak solutions for (1.1) similarly. We claim that the nonsmooth functions $\psi(\xi)$ given in (4.3) implicitly are weak solutions of (4.1) in the case $c_0 \leqslant |s| \leqslant c_1$.

We show first that

$$-\int_{\xi_0^+}^{\xi_1^-} \varphi'(\xi)(s^2 - c^2(\psi))\, \psi'(\xi) \, d\xi + \int_{\xi_0^+}^{\xi_1^-} c(\psi)\, c'(\psi)(\psi')^2 \, \varphi(\xi) \, d\xi = 0 \tag{4.12}$$

for a function ψ given in (4.3) for any test function $\varphi \in C^\infty(\mathbb{R})$ with no restrictions on its support. Because the singularities are similar at ξ_0^+ and ξ_1^-, it is sufficient to assume that the test function φ vanishes near ξ_1^-. We consider a one-parameter family of test functions $\varphi_{1\varepsilon}(\xi) \in C^\infty[\xi_0^+, \xi_1^-]$, parametrized by $\varepsilon > 0$, such that

$$\begin{aligned} &\operatorname{supp} \varphi_{1\varepsilon} \subseteq [\xi_0^+, \xi_0^+ + \varepsilon), \\ &\max |\varphi_{1\varepsilon}| \leqslant M \quad \text{and} \quad \int_{\xi_0^+}^{\xi_1^-} |\varphi'_{1\varepsilon}| \, d\xi \leqslant M \text{ independently of } \varepsilon > 0 \end{aligned} \tag{4.13}$$

and such that the family of test functions $\varphi_{2\varepsilon} \equiv \varphi - \varphi_{1\varepsilon}$ has the property

$$\operatorname{supp} \varphi_{2\varepsilon} \subseteq \left(\xi_0^+ + \frac{\varepsilon}{2}, \xi_1^- - \frac{\varepsilon}{2} \right). \tag{4.14}$$

Since $\psi(\xi)$ is a smooth solution in (ξ_0^+, ξ_1^-), we have by (4.14)

$$-\int_{\xi_0^+}^{\xi_1^-} \varphi'_{2\varepsilon}(s^2 - c^2(\psi)) \, \psi'(\xi) \, d\xi + \int_{\xi_0^+}^{\xi_1^-} c(\psi) \, c'(\psi)(\psi')^2 \, \varphi_{2\varepsilon} \, d\xi = 0, \tag{4.15}$$

$$-\int_{\xi_0^+}^{\xi_1^-} \varphi'_{1\varepsilon}(s^2 - c^2(\psi)) \, \psi'(\xi) \, d\xi = o(1) \int_{\xi_0^+}^{\xi_0^+ + \varepsilon} |\varphi'_{1\varepsilon}| \, d\xi = o(1), \tag{4.16}$$

$$\begin{aligned} \left| \int_{\xi_0^+}^{\xi_1^-} c(\psi) \, c'(\psi)(\psi')^2 \, \varphi_{1\varepsilon} \, d\xi \right| &\leqslant \max |\varphi_{1\varepsilon}| \int_{\xi_0^+}^{\xi_0^+ + \varepsilon} c(\psi) |c'(\psi)| \, (\psi')^2 \, d\xi \\ &= o(1). \end{aligned} \tag{4.17}$$

Combining (4.15), (4.16) and (4.17), we obtain (4.12). Since the integral over R in (4.11) is a finite sum of integrals similar to (4.12), we conclude that (4.11) holds for any $\varphi \in C_c^\infty(\mathbb{R})$. Therefore the proof of the claim is complete. Summarizing these results we obtain the following theorem.

THEOREM 3. *The functions $\psi(\xi)$ defined implicitly for all $\xi \in \mathbb{R}$ in formula* (4.3) *are weak solutions of* (4.1) *for any $s \in \mathbb{R}$. The solutions are always continuous, monotone, and unbounded as $|\xi| \to \infty$ (assuming $k \neq 0$). If $|s| \notin [c_0, c_1]$, the solutions are smooth. If $|s| \in [c_0, c_1]$ with $|s| = c(u_0)$ for some $u_0 \in [0, \pi/2]$, the solutions are not smooth. The derivative $\psi'(\xi)$ has isolated singularities at the points listed in* (4.5). *Furthermore, if $u_0 \in (0, \pi/2)$, so that $c'(u_0) \neq 0$, then the solutions have finite local energy and the cusp singularities have Hölder continuity exponents $\alpha = 2/3$. If $u_0 = 0$ or $u_0 = \pi/2$, then the solutions have an infinite amount of energy between singularities, and their Hölder exponents are equal to* 1/2.

We remark that weak solutions of (4.1) are weak solutions of (1.1). We also observe that other travelling wave solutions can be constructed using (4.12). For each fixed u_0 and a fixed sign of s with $|s| = c(u_0)$, new travelling wave solutions with the same speed s can be formed by continuously patching together two or more smooth pieces of the solutions given in (4.3), including the trivial solutions $\varphi = u_0$. For example, the following function

$$\begin{aligned} \psi(\xi) &= u_0 && \text{for} \quad \xi \leqslant \xi_0 \\ \int_{u_0}^{\psi(\xi)} \sqrt{|c^2(u_0) - c^2(v)|}\, dv &= k_1(\xi - \xi_0) && \text{for} \quad \xi_0 \leqslant \xi < \psi^{-1}(\pi - u_0) \\ \psi(\xi) &= \pi - u_0 && \text{for} \quad \xi \geqslant \psi^{-1}(\pi - u_0) \end{aligned} \tag{4.18}$$

is a weak travelling wave solution for any $u_0 \in [0, \pi/2]$, any $\xi_0 \in \mathbb{R}$, and any $k_1 > 0$ (see Fig 4.3). To construct a second example, we define two functions ψ_1 and ψ_2 as follows

$$\int_{u_0}^{\psi_1} \sqrt{|c^2(u_0) - c^2(v)|}\, dv = k_1^2\,(\xi - \xi_0),$$

$$\int_{\pi - u_0}^{\psi_2} \sqrt{|c^2(u_0) - c^2(v)|}\, dv = -k_2^2(\xi - \xi_1).$$

Here, $u_0 \in [0, \pi/2]$, $k_1 \neq 0$, $k_2 \neq 0$, and $\xi_0 < \xi_1$. If $\psi_1^{-1}(\pi - u_0) \leqslant \xi_1$, then the function

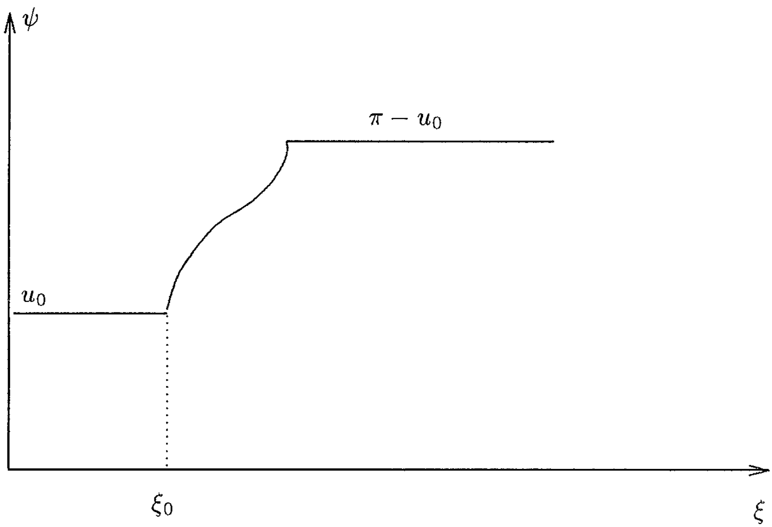

FIG. 4.3. A bounded travelling wave solution.

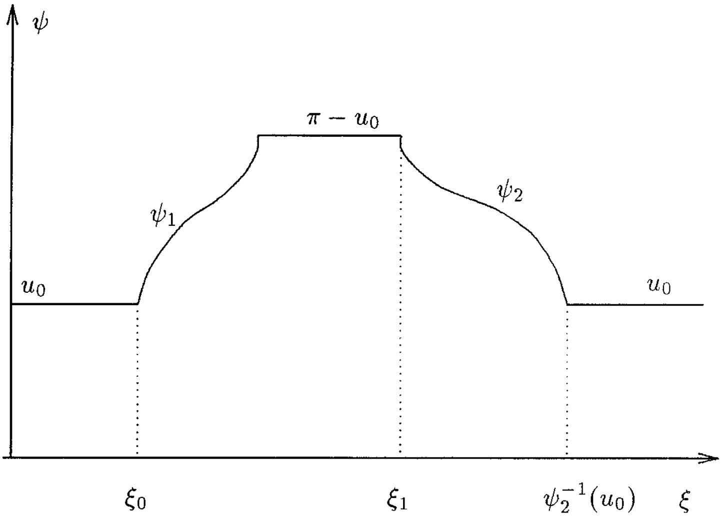

FIG. 4.4. A soliton-like travelling wave solution.

$$\psi(\xi) = \begin{cases} u_0 & \text{for} \quad \xi \leqslant \xi_0 \\ \psi_1(\xi) & \text{for} \quad \xi_0 \leqslant \xi \leqslant \psi_1^{-1}(\pi\text{-}u_0) \\ u_0 & \text{for} \quad \psi_1^{-1}(\pi\text{-}u_0) < \xi < \xi_1 \\ \psi_2(\xi) & \text{for} \quad \xi_1 < \xi \leqslant \psi_2^{-1}(u_0) \\ u_0 & \text{for} \quad \xi > \psi_2^{-1}(u_0) \end{cases}$$

is a weak solution (see Fig 4.4).

We note that the two nonsmooth waves in the second example can travel away from each other if the left wave has $s = -c(u_0)$ and the right wave has $s = c(u_0)$. In this case it is an exact solution, but the overall solution is not a travelling wave. If the travelling directions are reversed, collision of two travelling waves in opposite families will occur. But there is no overtaking of travelling waves in the same family, since all the waves travel at the same speed $s = \pm c(u_0)$ if they can be patched together.

5. PERSISTENCE AND ANNIHILATION OF OSCILLATION.

In this section, we show that striking phenomena can occur in weak limits of sequences of exact solutions of (1.1) with bounded energy. As in Section 4, we assume the specific form of $c(u)$ given in (1.3). To be definite, we assume that $0 < \alpha < \beta$.

We consider a fixed positive integer n, and choose small constants ε, $k>0$ such that

$$\int_{\pi/2-\varepsilon}^{\pi/2+\varepsilon} \sqrt{c^2(u)-c^2\left(\frac{\pi}{2}-\varepsilon\right)}\,du=\frac{k}{2n}.$$

Define $\psi(\xi)$ on $[0, 1/(2n)]$ by

$$\int_{\pi/2-\varepsilon}^{\psi(\xi)} \sqrt{c^2(u)-c^2\left(\frac{\pi}{2}-\varepsilon\right)}\,du=k\xi.$$

Define $\psi(\xi)$ on $[1/(2n), 2/(2n)]$ by

$$\int_{\pi/2+\varepsilon}^{\psi(\xi)} \sqrt{c^2(u)-c^2\left(\frac{\pi}{2}+\varepsilon\right)}\,du=-k\left(\xi-\frac{1}{2n}\right).$$

For $i=1, ..., n-1$, define $\psi(\xi)$ on $[i/n, (i+1)/n]$ by translation,

$$\psi(\xi)=\psi\left(\xi-\frac{i}{n}\right).$$

For $\xi \notin [0, 1]$, we define

$$\psi(\xi)=\frac{\pi}{2}-\varepsilon.$$

The graph of $\psi(\xi)$ consists of n "huts" (see Fig. 5.1). We choose $\varepsilon=k=1/n$ and denote the function $\psi(\xi)$ constructed above by $\psi^n(\xi)$. Let

$$u^n(t, x)=\psi^n(x-s_n t), \qquad s_n=c\left(\frac{\pi}{2}-\frac{1}{n}\right).$$

From the previous section, we know that $u^n(t, x)$ is a weak solution of equation (1.1). The total energy of $u^n(t, x)$ is given by

$$\begin{aligned} E(u^n) &= \int_{-\infty}^{\infty} \{(u_t^n)^2+c^2(u^n)(u_x^n)^2\}\,dx \\ &= \int_0^1 [s_n^2+c^2(\psi^n(\xi))](\psi_\xi^n)^2\,d\xi \\ &\sim 2\beta \int_0^1 (\psi_\xi^n)^2\,d\xi, \end{aligned}$$

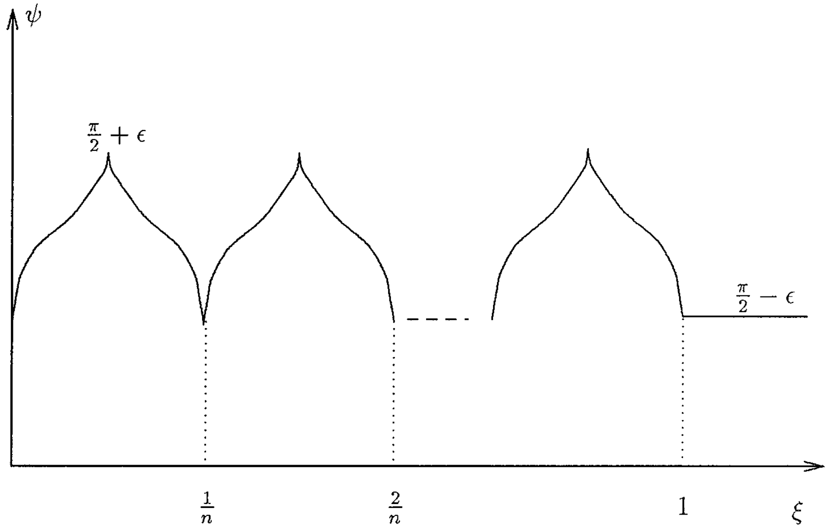

FIG. 5.1. A solution with oscillation.

since $\psi^n \to \pi/2$ as $n \to \infty$. Furthermore,

$$
\begin{aligned}
E(u^n) &\sim 2\beta \cdot 2n \int_0^{1/(2n)} (\psi_\xi^n)^2 \, d\xi \\
&= 4\beta n \int_0^{1/(2n)} \psi_\xi^n \frac{k}{\sqrt{c^2(\psi^n) - s_n^2}} \, d\xi \\
&= 4\beta \int_{\pi/2 - 1/n}^{\pi/2 + 1/n} \frac{du}{\sqrt{c^2(u) - s_n^2}} \\
&= \frac{8\beta\sqrt{2}}{\sqrt{\beta - \alpha}} \int_0^{\varepsilon} \frac{dv}{\sqrt{\cos 2v - \cos 2\xi}} \\
&\sim \frac{8\beta}{\sqrt{\beta - \alpha}} \int_0^1 \frac{dx}{\sqrt{1 - x^2}}
\end{aligned}
\tag{5.1}
$$

so $E(u^n)$ is bounded and has a strictly positive lower bound in the limit $n \to \infty$.

This sequence has the limit $u(t, x) = \pi/2$ which is a trivial solution with zero energy. The energy of $u^n(t, x)$, which tends to a nonzero constant as shown in (5.1), therefore disappears in the limit. This is because of the fact that u_x^n does not converge strongly in L^2 to u_x (which is equal to 0). More precisely, we have

$$u^n(t,x) \to \frac{\pi}{2} \qquad \text{uniformly}$$

$$u^n_x \rightharpoonup 0 \qquad \text{weakly in } L^2$$

$$(u^n_x)^2 \rightharpoonup \frac{4}{\sqrt{\beta-\alpha}} \int_0^1 \frac{dx}{\sqrt{1-x^2}} \chi(t,x) \qquad \text{weakly in} \quad L^1$$

$$c'(u^n)(u^n_x)^2 \rightharpoonup 0 \qquad \text{weakly in} \quad L^1$$

where $\chi(t,x)$ is the characteristic function of the x-interval $[\beta^{1/2}t, \beta^{1/2}t+1]$ for each $t \geqslant 0$. This example shows that oscillations in the derivative u_x can persist in time and accumulate in a sequence of solutions. The lack of strong convergence of u_x in L^2 can be compensated for by the highly oscillatory factor $c'(u)$, so that $c'(u)\, u_x^2$ has better convergence than u_x^2. This phenomena may help in finding a way to establish the existence of a weak solution for the initial value problem of equation (1.1).

6. Wavefront Expansions

In this section, we compute explicit expressions for the singularity formation time of u_x at a wavefront propagating into a uniformly rotating director field. We assume throughout this section that, before the singularity formation time, (1.1) has a piecewise real analytic solution of the form

$$u(t,x) = \begin{cases} u_0(t) & \text{for} \quad x \geqslant \psi(t) \\ u_0(t) + u_1(t)[x-\psi(t)] + 1/2u_2(t)[x-\psi(t)]^2 + \cdots \\ \qquad \text{for} \quad x < \psi(t) \end{cases} \tag{6.1}$$

In this equation,

$$u_0(t) = \omega t + \delta \tag{6.2}$$

is a spatially independent solution of (1.1), which corresponds to a director field rotating with constant angular velocity ω. The wavefront $x = \psi(t)$ is a characteristic curve of (1.1), so that

$$\begin{aligned} \dot{\psi} &= c_0 > 0, \\ c_0(t) &= c(u_0(t)). \end{aligned} \tag{6.3}$$

Here and below, dots denote time derivatives. The solution (6.1) describes a weak discontinuity, carrying a jump in u_x, propagating into a steadily rotating director field.

We will show that the spatial derivative immediately behind the wavefront,

$$u_1(t) = u_x(t, x)|_{x=\psi(t)}, \tag{6.4}$$

satisfies a Ricatti equation. The solution of this equation gives explicit conditions for blow-up of the derivative behind the wavefront. For nonzero ω, the initial slope has to exceed a critical value for blow-up to occur. Wavefront expansions have been used extensively to study derivative blow-up for quasilinear hyperbolic equations. In that case, derivative blow-up usually indicates the formation of shocks. For (1.1), we expect that the solution can be extended past the derivative blow-up time by a Hölder continuous weak solution containing a cusp behind the wavefront (as in Fig. 4.3, for example).

We change variables in (1.1) from (t, x) to (t, ϕ), where

$$\phi = x - \psi(t).$$

Using (6.3), we get

$$[c^2(u) - c_0^2]\, u_{\phi\phi} + 2c_0 \dot{u}_\phi + \dot{c}_0 u_\phi + c(u)\, c'(u)\, u_\phi^2 - \ddot{u} = 0. \tag{6.5}$$

Here, c' denotes the derivative of c with respect to u. We assume that $u(t, \phi)$ has the convergent Taylor expansion behind the wavefront, $\phi \leqslant 0$,

$$\begin{aligned} u(t, \phi) &= \sum_{n=0}^{\infty} u_n(t) \frac{\phi^n}{n!}, \\ u_n(t) &= \partial_\phi^n u(t, \phi)|_{\phi=0}. \end{aligned} \tag{6.6}$$

Setting $\phi = 0$ in (6.5), and using (6.2)–(6.3) and (6.6), we see that

$$2c_0 \dot{u}_1 + \dot{c}_0 u_1 + c_0 c_0' u_1^2 = 0. \tag{6.7}$$

Equations for the higher order Taylor coefficients follow on repeatedly differentiating (6.5) with respect to ϕ and setting $\phi = 0$. The resulting equation for $u_n(t)$ has the form

$$2c_0 \dot{u}_n + (\dot{c}_0 + 4c_0 c_0' u_1)\, u_n = f_n(t), \tag{6.8}$$

where f_n depends only on $\{u_0, u_1, ..., u_{n-1}\}$.

We consider two cases separately. First, suppose $\omega = 0$, so that the state ahead of the wavefront $u = u_0$ is constant. Then (6.7) gives

$$\dot{u}_1 + \tfrac{1}{2} c_0' u_1^2 = 0. \tag{6.9}$$

The solution of (6.9) is

$$u_1(t) = \frac{1}{u_1(0)^{-1} + c_0' t/2}.$$

Hence, assuming that $c_0' = c'(u_0) \neq 0$ and assuming that real analyticity does not break down before blow-up occurs, we conclude that the derivative blows up at time

$$t_* = -\frac{2}{c_0' u_1(0)}.$$

The blow-up time is positive if the initial slope $u_1(0)$ has opposite sign to c_0'.

The second and more interesting case is when $\omega \neq 0$. In this case, the wavefront is propagating into a state where c' is changing sign (if c is given by (1.3), for example). We can therefore observe the competition between the local nonlinear steepening of a wave, whose slope has the opposite sign to c', and the global effect of sign changes of c'. If the initial slope is opposite in sign to c', then this effect tends to oppose blow–up. However, if the initial slope has the same sign as c', then the change in the sign of c' can enhance blow up.

If $\dot{u}_0 = \omega \neq 0$, then differentiation of the second equation in (6.3) with respect to t implies that

$$c_0' = \frac{\dot{c}_0}{\omega}.$$

The use of this result in (6.7) gives

$$2c_0 \dot{u}_1 + \dot{c}_0 u_1 + \frac{1}{\omega} c_0 \dot{c}_0 u_1^2 = 0. \tag{6.10}$$

The solution of this equation is

$$u_1(t) = \left[\frac{\omega}{a(t)}\right] \frac{1}{\omega[a(0)\, u_1(0)]^{-1} + a(t) - a(0)}, \tag{6.11}$$

where we define

$$a(t) = c_0^{1/2}(t).$$

Now, we suppose that c is bounded,

$$0 < m^2 \leqslant c(u) \leqslant M^2,$$

and attains its maximum and minimum wave speeds M^2 and m^2. For instance, if c is given by (1.3) with $\alpha < \beta$, then

$$m = \alpha^{1/4}, \qquad M = \beta^{1/4}.$$

Equation (6.11) shows that the derivative blows up if the initial slope $u_1(0)$ satisfies

$$\begin{aligned} &\frac{a(0)\, u_1(0)}{\omega} \leqslant -\frac{1}{M - a(0)}, \qquad \text{or} \\ &\frac{a(0)\, u_1(0)}{\omega} \geqslant \frac{1}{a(0) - m}, \end{aligned} \tag{6.12}$$

where

$$a(0) = c(u_0(0))^{1/2}.$$

It is interesting to note that either sign of $u_1(0)$ leads to blow up in the forward time direction. If the initial slope is large and has the opposite sign to c_0', then blow up occurs immediately. However, if $u_1(0)$ is large and initially has the same sign as c_0', then blow up only occurs later on, after c_0' has switched sign.

If (6.12) is not satisfied, so that

$$\frac{-1}{M - a(0)} < \frac{a(0)\, u_1(0)}{\omega} < \frac{1}{a(0) - m},$$

then the slope at the wavefront remains finite globally in time. The critical initial slope for blow up increases linearly in ω. This is consistent with the heuristic expectation that the Ricatti-type blow up, which has a time-scale inversely proportional to the initial slope, has to occur before c' changes sign, which has a time scale inversely proportional to ω.

7. Heuristics

Shatah [22] has given a heuristic argument which correctly predicts the fact that $n = 2$ space dimensions is the critical dimension for singularity formation in harmonic maps on $(1 + n)$-dimensional Minkowski spaces [4, 6, 23]. This argument is based on the scaling of the energy under space-time dilations. In this final section we explain why the wave equation for harmonic maps and the nonlinear wave equation (1.8) have *different* critical dimensions, even though their energies have the *same* dilational scaling.

The main point is that equation (1.8) can have non-dilationally invariant singularities.

First, we give a fairly detailed discussion of the heuristic argument as it applies to (1.8). Equation (1.8) is invariant under the dilations

$$\bar{u}(\bar{t}, \bar{\mathbf{x}}) = u(t, \mathbf{x}), \qquad \bar{t} = t/L, \qquad \bar{\mathbf{x}} = \mathbf{x}/L. \tag{7.1}$$

That is, if $u(t, \mathbf{x})$ is a solution of (1.8), then $\bar{u}(\bar{t}, \bar{\mathbf{x}})$ is a solution of

$$\bar{u}_{\bar{t}\bar{t}} - c(\bar{u})\, \bar{\nabla} \cdot (c(\bar{u})\, \bar{\nabla}\bar{u}) = 0,$$

where $\bar{\nabla}$ is the gradient with respect to $\bar{x}$. Conservation of energy for (1.8) implies that

$$E[u(t, \cdot)] \equiv \int_{\mathbb{R}^n} \{u_t^2 + c^2(u)\, |\nabla u|^2\}\, d\mathbf{x} = \text{constant}.$$

From (7.1), the energy of u and $\bar{u}$ are related by

$$E[u(t, \cdot)] = L^{n-2} E[\bar{u}(t/L, \cdot)]. \tag{7.2}$$

More generally, if

$$E_R[u(t, \cdot)] = \int_{|\mathbf{x}| \leqslant R} \{u_t^2 + c^2(u) |\nabla u|^2\}\, d\mathbf{x}$$

is the energy in a spatial ball of radius R about the origin, then

$$E_R[u(t, \cdot)] = L^{n-2} E_{R/L}[\bar{u}(t/L, \cdot)]. \tag{7.3}$$

Now suppose that when $t = 0$ there is a singularity in the solution $u(t, \mathbf{x})$ at $\mathbf{x} = 0$. If the singularity is locally self-similar under the dilations (7.1), then

$$u(t, \mathbf{x}) \sim \bar{u}(t, \mathbf{x}) \qquad \text{as} \quad t, \mathbf{x} \to 0.$$

Using this condition in (7.3), with $L = R/R_0$ and $t = 0$, we find that the asymptotic behavior of the local energy is given by

$$\frac{E_R[u(0, \cdot)]}{E_{R_0}[u(0, \cdot)]} \sim \left(\frac{R}{R_0}\right)^{n-2} \qquad \text{as} \quad R, R_0 \to 0.$$

If the exponent $(n-2)$ of L in the energy scaling law (7.2) is negative, it follows that a dilationally invariant singularity has a locally infinite amount of energy. Consequently, such singularities cannot form from finite energy initial data. If the exponent of L is positive, then the singularity carries no energy and it is therefore possible for such singularities to form. The critical case is when the exponent of L is zero. This means that the singularity carries a nonzero, but finite, amount of energy, and singularities may or may not form. (Below, we give an example of singularity formation in a critical case.)

The two basic ideas in this argument are that: (a) singularities are locally self-similar; (b) singularities with infinite energy cannot occur. In applying the argument it is important to note that singularities may scale according to any local self-similarity of the underlying partial differential equation (including singularities with non-power law similarities). Thus, the argument above prohibits singularities of (1.1) which are locally invariant under the dilations (7.1), but it does *not* prohibit singularities which have different scaling properties. Moreover, if there are other positive conserved quantities in addition to the energy, these may prohibit singularities which are allowed energetically. Globally smooth solutions should only be expected if all relevant local self-similarities lead to a negative exponent when scaling some positive conserved quantity.

An example of a dilationally invariant equation with non-dilationally invariant singularities is provided by the asymptotic equation (1.2). We discuss its relation to the wave equation (1.1) below. Equation (1.2) is invariant under the one-parameter family of scalings

$$\bar{u}(\bar{t}, \bar{x}) = L^{-r} u(t, x), \qquad \bar{x} = L^{-(r+1)} x, \qquad \bar{t} = L^{-1} t, \tag{7.4}$$

A positive conserved quantity for (1.2) is

$$E[u] = \int u_x^2 \, dx.$$

This quantity scales under (7.4) according to

$$E[u] = L^{r-1} E[\bar{u}].$$

Thus, we conclude that smooth solutions can develop singularities in finite time, since we have a positive exponent for $r > 1$. However, dilationally invariant singularities, with $r = 0$, have a negative exponent, so they cannot occur.

The existence of singularities with $r \geqslant 1$ can be verified using the simple explicit solutions constructed in [8, 9]. For example,

$$u = |x|^{2/3},$$

is a time-independent weak solution of (1.2) which has a singularity at $x=0$. This solution is invariant under the scalings (7.4) with $r=2$. Equation (1.2) also has singular solutions in the critical case $r=1$. For example, the following piecewise linear solution,

$$u=\begin{cases}0, & x\leqslant 0,\\ 2x/t, & 0<x<t^2,\\ 2t, & x\geqslant t^2,\end{cases} \tag{7.5}$$

concentrates finite, non-zero energy at a single point $x=0$ and is invariant under the scalings (7.4) with $r=1$.

Finally, we give a scaling argument which suggests that the asymptotic equation (1.2) describes the local structure of singularities for the wave equation (1.1). We consider the scaling transformation $(t, x, u)\mapsto(\bar{t}, \bar{x}, \bar{u})$ given by

$$\begin{aligned}\bar{u}(\bar{t}, \bar{x})&=L^{-r}u(t, x),\\ \bar{x}-c_0\bar{t}&=L^{-(r+1)}(x-c_0t)\\ \bar{t}&=L^{-1}t.\end{aligned} \tag{7.6}$$

where $r>0$. As $L\to 0$, this scaling is appropriate for a singularity located at $t=0$, $x=0$ propagating with instantaneous velocity $c_0=c(0)$, and with the local asymptotic behavior

$$u(t, x)\sim t^rU\left(\frac{x-c_0t}{t^{r+1}}\right) \qquad \text{as} \quad t, x\to 0.$$

Using (7.6) in (1.1) we get

$$\begin{aligned}&\{[c^2(L^r\bar{u})-c_0^2]\,L^{-r}+2c_0^2-c_0^2L^r\}\,\bar{u}_{\bar{x}\bar{x}}\\ &\quad+2c_0(1-L^r)\,\bar{u}_{\bar{x}\bar{t}}-L^r\bar{u}_{\bar{t}\bar{t}}+c(L^r\bar{u})\,c'(L^r\bar{u})\,\bar{u}_{\bar{x}}^2=0.\end{aligned}$$

Expanding $c(u)=c_0+c_0'u+O(u^2)$ and dividing the result by $2c_0$, we find the equation

$$\bar{u}_{\bar{x}\bar{t}}+(c_0+c_0'\bar{u})\,\bar{u}_{\bar{x}\bar{x}}+\tfrac{1}{2}c_0'\bar{u}_{\bar{x}}^2=O(L^r), \qquad \text{as} \quad L\to 0. \tag{7.7}$$

This argument suggests that the leading order terms on the left hand side of (7.7) are the dominant terms in (1.1) near a singularity which is locally invariant under the scaling in (7.6). These leading order terms are equivalent to the asymptotic equation (1.2) after a Galilean transformation and a rescaling (assuming that $c_0'\neq 0$). Therefore, the exact self-similarities of the asymptotic equation in (7.4) correspond to local self-similarities of the original wave equation. The fact that the wave equation has these local self-similarities in addition to its dilational self-similarity allows singularities with finite energy even in one space dimension.

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References

1. M. Balabane, Non-existence of global solutions for some nonlinear wave equations with small Cauchy data, *C. R. Acad. Sc. Paris* **301** (1985), 569–572.
2. M. Balabane, Ondes progressives et résultats d'explosion pour des systèmes non linéaires du premier ordre, *C. R. Acad. Sc. Paris* **302** (1986), 211–214.
3. M. Balabane, A global existence result for the Carleman system, *C. R. Acad. Sc. Paris* **303** (1986), 919–922.
4. D. Christodoulou and A. Tahvildar-Zadeh, On the regularity of spherically symmetric wave maps, *Comm. Pure Appl. Math.* **46** (1993), 1041–1091.
5. R. T. Glassey, Finite-time blow-up for solutions of nonlinear wave equations, *Math. Z.* **177** (1981), 323–340.
6. M. Grillakis, Classical solutions for the equivariant wave map in $1+2$-dimensions, (1991 (preprint)).
7. B. Hanouzet and J. L. Joly, Explosion pour des problèmes hyperboliques semi-linéaires avec second membre non compatible, *C. R. Acad. Sc. Paris* **301** (1985), 581–584.
8. J. K. Hunter and R. H. Saxton, Dynamics of director fields, *SIAM J. Appl. Math.* **51** (1991), 1498–1521.
9. J. K. Hunter and Yuxi Zheng, On a nonlinear hyperbolic variational equation I. and II., *Arch. Rat. Mech. Anal.* **129** (1995), 305–353; 355–383.
10. F. John, Blow-up of solutions of nonlinear wave equations in three space dimensions, *Manuscripta Math.* **28** (1979), 235–268.
11. F. John, Formation of singularities in one-dimensional nonlinear wave propagation, *Comm. Pure Appl. Math.* **27** (1974), 377–405.
12. T. Kato, Blow-up of solutions of some nonlinear hyperbolic equations, *Comm. Pure Appl. Math.* **33** (1980), 501–505.
13. T. Kato, Nonlinear equations of evolution in Banach Spaces, *Proc. Sympos. Pure Math.* **45** (1986), 9–23.
14. S. Klainerman and A. Majda, Formation of singularities for wave equations including the nonlinear vibrating string, *Comm. Pure Appl. Math.* **33** (1980), 241–263.
15. P. Lax, Development of singularities of solutions of nonlinear hyperbolic partial differential equations, *J. Math. Phys.* **5** (1964), 611–613.
16. H. Levine, Instability and non-existence of global solutions to nonlinear wave equations, *Trans. Amer. Math. Soc.* **192** (1974), 1–21.
17. H. Lindblad, Global solutions of nonlinear wave equations, *Comm. Pure Appl. Math.* **45** (1992), 1063–1096.
18. T. P. Liu, Development of singularities in the nonlinear waves for quasi-linear hyperbolic partial differential equations, *J. Diff. Eq.* **33** (1979), 92–111.

19. R. H. Saxton, Dynamic instability of the liquid crystal director, *in* "Contemporary Mathematics Vol. 100: Current Progress in Hyperbolic Systems" (W. B. Lindquist, Ed.), pp. 325–330, AMS, Providence, 1989.
20. J. Schaeffer, The equation $u_{tt} - \Delta u = |u|^p$ for the critical value of p, *Proc. Roy. Soc. Edinburgh Sect. A* **101A** (1985), 31–44.
21. J. Shatah, Weak solutions and development of singularities in the $SU(2)$ σ-model, *Comm. Pure Appl. Math.* **41** (1988), 459–469.
22. J. Shatah, "The evolution of harmonic maps on Lorentz manifolds," Invited Lecture of the ICM94-Zürich (1994).
23. J. Shatah and A. Tahvildar-Zadeh, Regularity of harmonic maps from Minkowski space into Rotationally symmetric manifolds, *Comm. Pure Appl. Math.* **45** (1992), 947–971.
24. T. Sideris, Global behavior of solutions to nonlinear wave equations in three dimensions, *Comm. Partial Diff. Eq.* **8** (1983), 1291–1323.
25. T. Sideris, Nonexistence of global solutions to semilinear wave equations in high dimensions, *J. Diff. Eq.* **52** (1984), 378–406.
26. W. Strauss, Nonlinear wave equations, *CBMS Lectures* **73**, AMS, Providence (1989).