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TOPOLOGICAL CATEGORIES*

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This is a survey for the working mathematician of the theory of initially complete categories. These are concrete categories $(\mathscr{A}, T)$ where $T: \mathscr{A} \to \mathscr{X}$ is a topological functor, i.e. $\mathscr{A}$ admits arbitrary T-initial structures. Such categories provide a setting for general topology and topological algebra when $\mathscr{X}$ is the category of sets or a category of algebras. It is characteristic for initially complete categories that increasing richness in structure is the same thing as increasing generality. We end with the core results on initial completions, (E, M)-topological, topologically algebraic and semitopological functors.

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1. What is topology?

When an algebraist, a topologist and an analyst try to converse about the essential message which each man's specialty has for the others, it is likely to be conceded that analysis builds upon intricately intertwined structures that are taken apart, studied in some degree of abstraction and perhaps reassembled, by algebraists, topologists, measure theorists, logicians and others. The question "What is algebra?" has received a great deal of study and on the informal level most mathematicians feel that they have an adequate answer, although it depends on one's point of view whether infinitary operations and complete lattices belong to algebra or to topology.

But what is (general) topology? Is it the study of convergence or of continuity? One can easily see that these two phenomena are really the same. For, on the one hand, we all know how to describe continuity of a mapping between topological

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spaces in terms of convergence of nets or filters, and on the other hand convergence, say of a net, can be described in terms of continuous mappings, as follows. The net $n: D \to X$ is a continuous mapping into the space X if we give the directed set D the discrete topology; now D has a one-point extension $D^* = D \cup \{\infty\}$ with a suitable (no longer discrete) topology such that the net n converges if and only if the continuous mapping n has a continuous extension $n^*: D^* \to X$.

However, general topology has other structural layers than the one which contains the topological spaces and continuous mappings. Take for instance the layer of metric spaces and non-expansive mappings. Forgetting the metrics but retaining the uniform structure, we can picture a lower layer of (metrizable) uniform spaces and uniformly continuous mappings. Applying a further forgetful functor we arrive in the layer of (uniformizable) topological spaces and continuous mappings. This layer, in a sense, does not have enough structure to describe uniform continuity, completeness, precompactness and similar uniform concepts. There are still lower layers, of interest to the analyst and amenable to the general procedures of topology, for instance sequential spaces and (sequentially) continuous mappings, or pre-ordered sets and order-preserving mappings. It is also worth knowing that there is a quite high level intermediate between the metric and uniform layers which is of interest to the analyst: Lipschitz structures [12].

Then again, the structural layers of general topology which we have pictured as vertically arranged are indefinitely extendible in the horizontal direction by a process of generalization: The general topologist, not content with the topological spaces, invents the successively more general closure spaces, pseudo-topological spaces, limit spaces, convergence spaces, etc.; or he generalizes the uniform spaces in different 'horizontal' directions as semi-, quasi-, and semi-quasi-uniform spaces or as uniform limit spaces, etc. We shall however see in section 5 that these two perceptions of, on the one hand, increasing richness in structure and, on the other hand, increasing generality, are only the result of a certain point of view and are in fact one and the same thing; and that this circumstance is one of the surprising distinctions between topology and algebra.

The common and characteristic feature in all the categories of structures with which general topology concerns itself, is to be found in a property of their grounding functors. The grounding functor $T: \mathscr{A} \to \mathscr{X}$ is a forgetful functor which goes from the category $\mathscr{A}$ to some ground or base category $\mathscr{X}$ which is usually the category of sets or some other algebraic category. The property in question is that $\mathscr{A}$ admits initial (or 'weak') structures with respect to T. The functor T is then called *topological* in current terminology [23] (see definition in section 2). The pair $(\mathscr{A}, T)$ is called a *topological category* [23] in case (i) T is topological, (ii) $\mathscr{X} = \mathbf{Set}$, (iii) constant functions lift to $\mathscr{A}$-morphisms, and (iv) the fibres $T^{-1}X$ are small (i.e. they are sets and not proper classes).

Now there are topological categories such as **Ord** (pre-ordered sets and order-preserving functions) and **Meas** (measurable spaces, i.e. sets endowed with σ-algebras of subsets, and measurable functions) which are not ordinarily regarded as 'topological'. Thus there seems to be room for interesting taxonomical studies to determine which topological categories (in the technical sense) are typically 'topological', 'order-theoretic' or 'measure-theoretic'.

This article limits itself to outlining a small but central portion of the theory of topological functors. We shall economize on references, and the reader should find most of the attributions which we omit in Herrlich's comprehensive survey article [23]. For general categorical terminology we refer to the book [25]. The basic notions about reflective and coreflective subcategories are prerequisites at certain points, and are by themselves worth looking up in Chapter X of [25].

We have to mention that the term 'topological category' has been used in completely different senses, e.g. in [9], [32] which we do not consider here.

2. Definitions

We consider a functor $T: \mathscr{A} \to \mathscr{X}$. For objects A, B of $\mathscr{A}$ we say A is *T-finer* than B, or B is *T-coarser* than A, written $A \leq_T B$, if there is an $\mathscr{A}$-morphism $f: A \to B$ such that Tf is an identity. The functor T is called *faithful* if whenever $f, g: A \to B$ are such that $Tf = Tg$, then $f = g$. The functor T is *amnestic* if the preorder $\leq_T$ is a partial order (i.e. equally fine objects are identical). T is *transportable* if for any $\mathscr{A}$-object A and any $\mathscr{X}$-isomorphism $h: TA \to X$ there exists an $\mathscr{A}$-object B and an $\mathscr{A}$-isomorphism $f: A \to B$ with $Tf = h$.

We shall regard *forgetful* functors as being faithful, amnestic and transportable. (Some authors, e.g. [23], require only faithfulness; the other two requirements constitute no essential loss of generality [27] and are in practice usually fulfilled by functors which 'forget' some structure while preserving underlying sets.)

We shall consider indexed families of $\mathscr{A}$-morphisms f_i: $A \to A_i$, i running through an index set or proper class I. In case I is empty we wish to retain the domain object A and therefore we need the notion of an *$\mathscr{A}$-source* which is just the ordered pair $(A, (f_i)_{i \in I})$. Now this $\mathscr{A}$-source is called *T-initial* if for any $\mathscr{A}$-source $(B, (g_i: B \to A_i)_{i \in I})$ and any $v: TB \to TA$ with $Tf_i \circ v = Tg_i$ for all i, there exists unique $u: B \to A$ such that $Tu = v$ and $f_i \circ u = g_i$ for each i.

2.1. Examples. (1) If T: **Top** $\to$ **Set** is the usual grounding functor from topological spaces and continuous mappings, then a source $(A, (f_i))$ in **Top** is T-initial if and only if A has the weak or initial topology determined by the f_i, i.e. the coarsest topology for which the f_i are continuous into the given A_i. (2) If T: **Grp** $\to$ **Set** is the usual grounding functor from the category of groups and homomorphisms, then a source in **Grp** is T-initial if and only if it is a monosource, i.e. distinguishes points [44].

The functor $T: \mathscr{A} \to \mathscr{X}$ is called *topological* if for any indexed collection $(A_i)_{i \in I}$ of $\mathscr{A}$-objects and each $\mathscr{X}$-source of the form $(X, (h_i: X \to TA_i)_{i \in I})$ there exists a unique T-initial source $(A, (f_i: A \to A_i)_{i\in I})$ with $TA = X$ and $Tf_i = h_i$ for each i. The index class I must be allowed to range over all sets, including the empty set, and all proper classes.

The pair $(\mathscr{A}, T)$ is called an *initially complete category* if $T: \mathscr{A} \to \mathscr{X}$ is topological, and a *topological category* under the additional conditions mentioned in Section 1.

2.2. Theorem. *Each topological functor is faithful, amnestic and transportable.*

The faithfulness was first proved by Hoffmann [27] under the restriction that $\mathscr{A}$ has small hom-sets. The amnesticity and transportability are then immediately clear. The restriction to small hom-sets was removed by Börger and Tholen [3] using a combinatorial result with four remarkable corollaries: (i) Cantor's theorem that $2^{|X|} > |X|$; (ii) A strengthening of Freyd's theorem that any small category with products is equivalent to a complete lattice; (iii) Every semitopological functor (see Section 6) is faithful; (iv) In an (E, M) category (see Section 3) $E \subset \{epi\}$.

There are several variants of the concept of topological functor, some of them with the inessential difference of not being amnestic or transportable. We cannot trace out the evolution of the concept here, and the following references have to suffice: [1], [36], [45], [55], [56], [4], [27], [19], [5], [23]; but one must mention that the 'top categories' of Wyler [56], [55] corresponded to his 'topological theories', an analogue of algebraic theories.

2.3. Examples of topological categories (hence with grounding functor to **Set**):

(1) **Top**, closure spaces, pseudotopological spaces, **Lim** (= limit spaces), **Conv** (= convergence spaces) [57]—each a full subcategory of the next;

(2) **Creg** (= completely regular topological spaces), **Zero** (= Alexandroff or zero-set spaces [15]), **Prox** (= proximity spaces), **Unif** (= uniform spaces)—in order of increasing structure;

(3) **Mer** (= merotopic spaces) containing several important topological categories as nicely embedded full subcategories, e.g. **Near** (= nearness spaces [20]), **Unif**, **Prox**, $\mathbf{Top_s}$ (= topological spaces which are symmetric, i.e. satisfy the R_0 axiom), **STop** (= subtopological spaces), $\mathbf{Conv_s}$ (= symmetric convergence spaces), **Grill** (= grill-determined spaces), **Cont** (= contiguity spaces). The embeddings are displayed in Fig. 1 (borrowed from [23]) where R stands for bireflective and C for bicoreflective embedding.

The merotopic spaces are symmetric in a sense illustrated by the R_0 axiom for topological spaces: $x \in \overline{\{y\}} \Leftrightarrow y \in \overline{\{x\}}$. **Mer** and **Near** form a well-behaved setting for a wide range of topological investigations; e.g. if one wishes the product of paracompact topological spaces to be paracompact, one takes the product in **Near**. In fact, the paracompact topological spaces form the intersection of $\mathbf{Top_s}$ and **Unif**—of course, with respect to the particular but very natural way in which $\mathbf{Top_s}$ and **Unif** are embedded in **Near** [21].

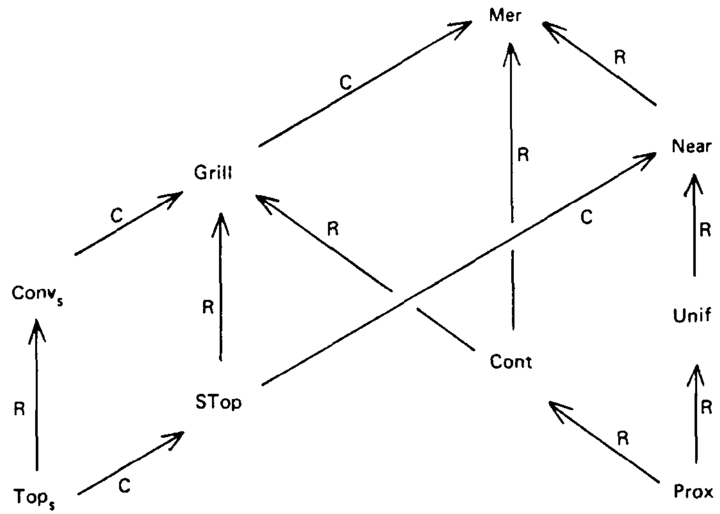

Fig. 1

(4) **Syntop** (the syntopogenous structures of Császár [8] with an inessential modification to render the grounding functor amnestic) contains some topological categories of asymmetric objects as full subcategories, e.g. **Top**, **QProx** (= quasiproximity spaces), **QUnif** (=quasi-uniform spaces).

(5) **Ord**, **Meas** (see Section 1).

(6) Lipschitz structures [12].

(7) Pseudometric spaces of diameter at most 1, with non-expansive mappings, form a topological category. Arbitrary real-valued pseudometrics do not give a topological category, but $[0, \infty]$-valued pseudometrics with non-expansive mappings again give a topological category **Ecart** [28].

2.4. Examples in topological algebra. (1) Let **TopGrp** denote the category of topological groups and continuous homomorphisms. The forgetful functor V: **TopGrp** $\to$ **Grp** is topological. To verify this one considers a source $f: X \to X_i$ in **Grp** and topological groups $(X_i, \mathcal{T}_i)$. One endows X with the initial topology $\mathcal{T}$, guaranteed by the fact that T: **Top** $\to$ **Set** is topological. Then $(X, \mathcal{T})$ is a topological group because the binary operation of multiplication and the unary operation of inversion (and the nullary identity) turn out to be continuous since their composites with the f_i are continuous.

(2) The idea of the above example can be applied to any topological functor $T: \mathcal{A} \to$ **Set** and the grounding functor $U: \mathcal{X} \to$ **Set** of any category $\mathcal{X}$ of universal algebras of fixed type τ (e.g. groups, abelian groups, rings, R-modules for fixed R, vector spaces over a fixed K). One constructs the category $\mathcal{A}\mathcal{X}$ with objects $(A, X)(A \in \mathcal{A}, X \in \mathcal{X})$ subject to $TA = UX$ and the operations lifting to appropriate $\mathcal{A}$-morphisms. The forgetful functor $\mathcal{A}\mathcal{X} \to \mathcal{X}$ is then topological [55], [34], [28], [33].

(3) The above construction can be adapted to situations where $\mathscr{X}$ is a category of partial algebras, e.g. fields.

(4) A fundamental tool of categorical topological algebra is Wyler's Taut Lift Theorem [55], sharpened by Tholen [47], which uses preservation of initial sources to characterize lifting of free functors; a typical instance is the lifting of the free group functor **Set** $\to$ **Grp** to **Top** $\to$ **TopGrp**.

3. Factorization of sources

In this section we let $\mathscr{X}$ be an (E, M)-category in the sense of [23], [26]. Thus E is a class of $\mathscr{X}$-morphisms closed under composition with isomorphisms, M is a conglomerate of $\mathscr{X}$-sources closed under composition with isomorphisms, $\mathscr{X}$ is (E, M)-factorizable (i.e. each $\mathscr{X}$-source decomposes into an E-morphism followed by an M-source), and $\mathscr{X}$ is (E, M)-diagonalizable. The reader unacquainted with these notions will follow the main trend if he confines his attention to the special case which is the origin of the concepts, namely $\mathscr{X}$ = **Set**, E the class of all surjective functions in **Set**, and M consisting of all sources which distinguish points.

Further we consider a forgetful functor $T: \mathscr{A} \to \mathscr{X}$. A morphism $f: A \to B$ in $\mathscr{A}$ is called an *embedding* if f is T-initial and Tf belongs to M (we are abusing the language, regarding f and Tf also as sources). Further, an object A of $\mathscr{A}$ is called *separated* [4] if each T-initial $\mathscr{A}$-morphism with domain A is an embedding. Let us denote by $\mathscr{A}_0$ the full subcategory of $\mathscr{A}$ consisting of all separated objects.

When the ground category is **Set** and (E, M) = (surjective, monosource), the notion of separated object is the appropriate generalization of T_0-space. Thus e.g. **Top$_0$** is the category of T_0-spaces, **Creg$_0$** is the category **Tych** of Tychonoff spaces, **Unif$_0$** is the category of separated uniform spaces, and likewise correct notions of separated quasi-uniform and bitopological spaces are obtained [4]. **Ord$_0$** = **Poset**, the category of partially ordered sets. **Ecart$_0$** consists of those (X, d) for which $d(x, y) = 0$ implies $x = y$ [28]. On the other hand it follows from 2.1(2) that **Grp$_0$** = **Grp**; a general study of sufficient conditions for the equality $\mathscr{A}_0 = \mathscr{A}$ has been given by Pumplün [44].

Now the above examples have the nice properties that $\mathscr{A}_0$ is an (extremal epi)-reflective subcategory of $\mathscr{A}_0$, and that $(\mathscr{A}_0)_0 = \mathscr{A}_0$. These properties may fail even for topological categories. To remedy this, Hoffmann [28] introduced the strong notion of an *(E, M)-universally topological functor* $T: \mathscr{A} \to \mathscr{X}$. For the case that $(\mathscr{A}, T)$ is a topological category such that the empty set and the singleton sets admit unique $\mathscr{A}$-structures, Marny [38], [39] characterizes this nice situation by the condition that $\mathscr{A}$ is the bireflective hull of $\mathscr{A}_0$, and gives these results: $\mathscr{A}_0$ is the largest epireflective subcategory of $\mathscr{A}$ which is not bireflective; each epireflection in $\mathscr{A}$ is either a bireflection or a bireflection followed by the $\mathscr{A}_0$-reflection.

As a rule, if $\mathscr{A}$ is a topological category, then $\mathscr{A}_0$ fails to be one, or more precisely: $\mathscr{A}_0$ fails to be initially complete (with respect to the restricted grounding functor).

Thus **Top₀**, **Creg₀**, **Unif₀** and **Ord₀** are not initially complete. Among the full reflective subcategories of **Top**, precisely those are initially complete which contain all indiscrete spaces [10], cf. [18]. Hence e.g. the categories $\mathbf{Top_n}$(= T_n-spaces) for $n=0,1,2,3$, and the category of compact Hausdorff spaces, fail to be initially complete. This is particularly striking in the case of the T_1-spaces, for there is always a coarsest T_1-topology that will render a given source of functions continuous; the description of an initial source which we gave for the case of **Top** in 2.1(1) will therefore not serve for the general case.

Coseparated objects of $\mathscr{A}$ can be defined in the obvious dual way. In **Top** a space is coseparated iff it is not the coproduct of a singleton and a non-empty space [4]. In **Ecart** an object is coseparated iff no point is at infinite distance from all other points. Hoffmann [31] considers vector spaces equipped with $[0,\infty]$-valued pseudonorms, with ground category the vector spaces over $K(K=\mathbb{R}$ or $K=\mathbb{C})$; here the objects that are both separated and coseparated are precisely the normed vector spaces.

4. Properties of topological functors

4.1. Theorem (Antoine). *The functor $T:\mathscr{A}\to\mathscr{X}$ is topological if and only if it is cotopological.*

Cotopological is the categorical dual of topological; i.e. $\mathscr{A}$ admits T-coinitial cosources (also called *T-final sinks*). The theorem was given by Antoine [1] for the case $\mathscr{X}=\mathbf{Set}$ and then proved for general $\mathscr{X}$ in [45], [4], [27], [19], [5].

The following elementary propositions account for much of the scope and usefulness of topological functors.

4.2. Let $T:\mathscr{A}\to\mathscr{X}$ be a topological functor. Then:

(1) T detects limits and colimits. In particular, T preserves limits and colimits, and when $\mathscr{X}$ is complete or cocomplete, so is $\mathscr{A}$. {The best known special case is that a product in **Top** is formed by equipping the cartesian product of the underlying sets with the initial topology given by the projection mappings.}

(2) T preserves monomorphisms and epimorphisms.

(3) T lifts (E,M)-factorizations appropriately.

(4) The fibers $T^{-1}(X)(X\in\mathscr{X})$ are large-complete lattices under $\leqslant_T$.

(5) The sections (i.e. right inverses) of T form a large-complete lattice when ordered objectwise by $\leqslant_T$. The finest (coarsest) section is left (right) adjoint to T and the objects in its range are called the *T-discrete* (*T-indiscrete*) objects of $\mathscr{A}$. {Example: The sections of the the forgetful functor T:**TopAb**→**Ab**—see 2.4(2) above—are known as functorial topologies for abelian groups ([13], cf. [10]).}

4.3. Let $(\mathscr{A}, T)$ be a topological category. Then:

(1) If $\mathscr{X}$ is well- (cowell-) powered, so is $\mathscr{A}$.

(2) In $\mathscr{A}$, embedding = extremal monomorphism = regular monomorphism, and quotient = extremal epimorphism = regular epimorphism.

(3) Let $\mathscr{S}$ be a full, isomorphism-closed subcategory of $\mathscr{A}$. Then, (see [39] and [10]):

$\mathscr{S}$ is epireflective in $\mathscr{A} \Leftrightarrow \mathscr{S}$ is closed under $\mathscr{A}$-subspaces and $\mathscr{A}$-products;

and:

$\mathscr{S}$ is bireflective in $\mathscr{A} \Leftrightarrow \mathscr{S}$ is epireflective in $\mathscr{A}$ and contains the indiscrete objects

$\Leftrightarrow \mathscr{S}$ is reflective in $\mathscr{A}$ and contains the indiscrete objects.

5. Factorization of functors

5.1. Consider the forgetful functors T: **Unif** $\to$ **Set** and V: **Creg** $\to$ **Set**. It is desired to construct functors F: **Creg** $\to$ **Unif** such that $TF = V$. One construction, already used by Hušek [36], involves the forgetful functor L: **Unif** $\to$ **Creg**; one has $T = VL$. We take any class of **Unif**-objects A_i and form the source of all **Creg**-morphisms $f_{ij}: X \to LA_i$ for fixed X in **Creg**. This gives a source Vf_{ij}: $VX \to TA_i$ in **Set**. Since T is topological, there exists an initial source which we denote $g_{ij}: FX \to A_i$ in **Unif** such that $T(FX) = VX$ and $Tg_{ij} = Vf_{ij}$. Hereby a functor F: **Creg** $\to$ **Unif** is defined (its action on morphisms is seen by using the topologicity of T once more). Properties of F will depend on the choice of the A_i. For instance:

(1) If we choose just one single A_i, namely the interval $[0, 1]$ with its usual uniformity, then F is the functor $\mathscr{C}^*$ which is the coarsest section of L. Thus $\mathscr{C}^*$ sends a completely regular space X to a uniform space $\mathscr{C}^*X$ with $L\mathscr{C}^*X = X$, and $\mathscr{C}^*$ is the coarsest such functor.

(2) If the A_i are all objects of **Unif** then F is the finest section of L, and left adjoint to L.

(3) Every section of L can be obtained by a suitable choice of the class of A_i. In general—e.g. as in the case of the finest section—a set of A_i will not suffice [4]. This special case of the following theorem was the original reason for admitting large index classes in the definition of topological functor.

5.2. Theorem. *Let $T: \mathscr{A} \to \mathscr{X}$ be an amnestic faithful functor. Then the following are equivalent*:

(1) *T is topological*;

(2) *For arbitrary amnestic faithful functors V, W, K such that K is full and $TW = VK$, there exists a functor F such that $TF = V$ and $FK = W$*;

(3) *Same as* (2) *with* $W = \mathrm{id}_{\mathscr{A}}$.

$$\begin{tikzcd} \cdot \arrow[r, "W"] \arrow[d, "K"'] & \mathscr{A} \arrow[d, "T"] \\ \cdot \arrow[r, "V"'] \arrow[ur, dashed, "F"] & \mathscr{X} \end{tikzcd}$$

For the proof of (1)⇔(2) see [5], and for (1)⇔(3) see [6] or [42]. Note that F in the theorem is necessarily faithful and amnestic, and that a full faithful amnestic functor K is the same as a full embedding.

Antoine's theorem (4.1 above) is an immediate corollary because all the terms occurring in 5.2(2) are obviously self-dual.

Form the quasi-category $\mathbf{Cat}_{\mathscr{X}}$ whose objects are all pairs $(\mathscr{A}, T)$ where $T: \mathscr{A} \to \mathscr{X}$ is an amnestic faithful functor and whose morphisms $F:(\mathscr{A}, T) \to (\mathscr{B}, U)$ are the functors $F: \mathscr{A} \to \mathscr{B}$ with $UF = T$. Such F are called *concrete functors*, and the $(\mathscr{A}, T)$ are called *concrete categories over* $\mathscr{X}$. If we have concrete functors $F:(\mathscr{A}, T) \to (\mathscr{B}, U)$ and $L:(\mathscr{B}, U) \to (\mathscr{A}, T)$ with $LF = \mathrm{id}_{\mathscr{A}}$, then we call L a *concrete retraction* and F a *concrete section*; necessarily then F is a full embedding.

Herrlich [22] observed that Theorem 5.2 can immediately be rephrased as follows:

5.3. *For a concrete category* $(\mathscr{A}, T)$ *over* $\mathscr{X}$ *these are equivalent*:
(1) $(\mathscr{A}, T)$ *is initially complete*;
(2) $(\mathscr{A}, T)$ *is an injective object with respect to full embeddings in* $\mathbf{Cat}_{\mathscr{X}}$;
(3) *Every concrete full embedding with domain* $(\mathscr{A}, T)$ *in* $\mathbf{Cat}_{\mathscr{X}}$ *is a concrete section.*

One calls $(\mathscr{A}, T)$ a *(concrete) subcategory* of $(\mathscr{B}, U)$ if $\mathscr{A}$ is a subcategory of $\mathscr{B}$ and the inclusion functor $J: \mathscr{A} \to \mathscr{B}$ is concrete, i.e. $UJ = T$. A standard simple argument for injective objects applied to 5.3 gives the following observation from [5]:

5.4. *Let* $(\mathscr{B}, U)$ *be initially complete, with concrete subcategory* $(\mathscr{A}, T)$. *Then the inclusion* $J:(\mathscr{A}, T) \to (\mathscr{B}, U)$ *is a concrete section if and only if* $(\mathscr{A}, T)$ *is an initially complete full subcategory of* $(\mathscr{B}, U)$.

(Another characterization of initially complete full subcategories, deducible from 5.4 and due to H. Müller, may be found in [22, Theorem 1.4.5 and 23, p. 284].)

For any two concrete categories $(\mathscr{A}, T)$ and $(\mathscr{C}, V)$ over $\mathscr{X}$, the notion of $(\mathscr{C}, V)$ *having more structure than* $(\mathscr{A}, T)$ seems to coincide with $(\mathscr{A}, T)$ being a concrete retract of $(\mathscr{C}, V)$. Granting this definition, we then have from 5.4 the claim made in Section 1:

5.5. *Let $(\mathscr{A}, T)$ be initially complete. Then $(\mathscr{C}, V)$ has more structure than $(\mathscr{A}, T)$ if and only if there is a full concrete embedding of $(\mathscr{A}, T)$ into $(\mathscr{C}, V)$.*

This accounts for the fact that it is a viable undertaking to accommodate various structural levels of general topology in a single setting (e.g. **Mer**, **Near**, **Syntop**). We notice from 5.3 that 5.5 is even characteristic of initially complete categories $(\mathscr{A}, T)$. So this is an essentially topological phenomenon, and it is instructive to compare 5.4 with its algebraic analogue:

5.6. *Let $\mathscr{B}$ be the category of all universal algebras of fixed type τ. Then a full subcategory $\mathscr{A}$ of $\mathscr{B}$ is algebraic in the sense of* [25] *if and only if $\mathscr{A}$ is epireflective in $\mathscr{B}$* ([25], [35]). *The epireflection is a retraction but is in general not concrete.*

Complete lattices can be viewed as initially complete categories, as follows. Take as ground category the category $\mathbb{1}$ of just one object and one morphism. A functor $T: \mathscr{A} \to \mathbb{1}$ is faithful if and only if for any A, B in $\mathscr{A}$ there is at most one morphism $f: A \to B$. There is thus a one-to-one correspondence between concrete categories over $\mathbb{1}$ and partially ordered classes. A source in $(\mathscr{A}, T)$ carries to a lower bound in $(\operatorname{ob}\mathscr{A}, \leq_T)$ and an initial source carries to a greatest lower bound. Initially complete categories over $\mathbb{1}$ thus correspond to large-complete lattices.

Now certain known completions of posets have counterparts for certain concrete categories. Given an amnestic faithful functor $T: \mathscr{A} \to \mathscr{X}$ then by an *initial completion* of $(\mathscr{A}, T)$ we shall understand a full concrete embedding $K: (\mathscr{A}, T) \to (\mathscr{B}, U)$ such that $(\mathscr{B}, U)$ is initially complete (our definition here agrees with [26], though more recently it has become practice to require an initial completion also to be initially dense, cf. [23]). Up to isomorphism there is at most one initial completion K of $(\mathscr{A}, T)$ which is initially and finally dense (K then also preserves initial sources and final sinks [42]). Such K is called the *(Dedekind-) MacNeille completion* of $(\mathscr{A}, T)$ because in the case $\mathscr{X} = \mathbb{1}$ with small $\mathscr{A}$ it corresponds to MacNeille's completion of a poset by Dedekind cuts. The MacNeille completion is the smallest initial completion of $(\mathscr{A}, T)$, and is the injective hull of $(\mathscr{A}, T)$ in $\mathbf{Cat}_{\mathscr{X}}$. The largest initially dense initial completion preserving initial sources of $(\mathscr{A}, T)$ is called the *universal initial completion* of $(\mathscr{A}, T)$.

For a concrete category $(\mathscr{A}, T)$ over $\mathscr{X}$, if $\mathscr{A}$ is small, both the MacNeille completion and the universal initial completion exist [22]. If $\mathscr{A}$ is large, it may happen that the former exists and the latter does not, or that both exist, or that no initial completion of $(\mathscr{A}, T)$ exists at all [22], [26]. (The problem of the existence of initial completions is similar whether $\mathscr{A}$ is small or not, since in both cases the initial completion generally belongs to a higher universe than the one in which $\mathscr{A}$ lies.)

For the category of compact Hausdorff spaces, the MacNeille completion and the universal initial completion both exist and they differ [26], [30]. In the following examples both completions exist and they coincide [26], [30], [31]:

Concrete Category	MacNeille completion = universal initial completion
$\mathbf{Top}_0$ over **Set**	**Top**
$\mathbf{Top}_1$ over **Set**	$\mathbf{Top}_s$
Tych over **Set**	**Creg**
$\mathbf{Unif}_0$ over **Set**	**Unif**
${}_K\mathbf{Ban}_1$ (= Banach spaces and non-expansive linear maps) over ${}_K\mathbf{Vec}$ (= K-vector spaces and K-linear maps, $K = \mathbb{R}$ or $K = \mathbb{C}$)	K-vector spaces with $[0, \infty]$-valued pseudonorms and non-expansive K-linear maps

For the case of small $\mathscr{A}$, the characterizations 5.2 and 5.3 of topologicity follow from the existence of the MacNeille completion. The proofs of 5.2 in [5], [6], [42] are insensitive to whether $\mathscr{A}$ is large or small, because they proceed by constructing only 'localized initial completions', which always exist.

Wolff [54] has given an external characterization of topologicity analogous to 5.2 but purely in terms of functors and natural transformations with no reference to faithfulness or fullness. Wolff's theorem was generalized very neatly in [51] to characterize the semitopological functors (see below).

6. Generalizations

We have seen in Section 3 that imposing a separation axiom in general destroys topologicity. Thus many of the nice categories of general topology fail to be initially complete. We shall see that they are nevertheless nicely embeddable into initially complete categories.

Consider a functor $T: \mathscr{A} \to \mathscr{X}$ where $\mathscr{X}$ is an (E, M)-category. The functor T is called *(E, M)-topological* if for any indexed class $(A_i)_{i\in I}$ of $\mathscr{A}$-objects and any $\mathscr{X}$-source $(X, (h_i: X \to TA_i)_{i\in I})$ which belongs to M, there exists a unique T-initial source $(A, (f_i: A \to A_i)_{i\in I})$ with $TA = X$ and $Tf_i = h_i$ for all i. Thus the definition of topological functor is modified only by requiring the $\mathscr{X}$-sources to belong to M. (The above definition is the amnestic, transportable version of the one introduced in [19].)

Examples of (E, M)-topological functors abound [19]. E.g. if $\mathscr{A}$ is a non-trivial epireflective subcategory of **Top** which contains only T_0-spaces, then the forgetful functor $\mathscr{A} \to \mathbf{Set}$ is (surjective, monosource)-topological but not topological. This includes the examples $\mathbf{Top}_n (n = 0, 1, 2, 3)$ and **Tych**.

6.1. Theorem [19]. *Let $\mathscr{X}$ be an (E, M)-category. The functor $T: \mathscr{A} \to \mathscr{X}$ is (E, M)-topological if and only if there exists an initial completion $K: (\mathscr{A}, T) \to (\mathscr{B}, U)$ which is $U^{-1}E$-reflective.*

In this theorem, moreover, K maps $\mathscr{A}$ to the full subcategory $\mathscr{B}_0$ of separated objects of $(\mathscr{B}, U)$ [28]. See [18] and [28] for interesting related results.

The functor $T: \mathscr{A} \to \mathscr{X}$ is *topologically algebraic* if for every indexed class $(A_i)_{i\in I}$ of A-objects and every $\mathscr{X}$-source $(X, (h_i: X \to TA_i)_{i\in I})$ there exists a T-initial source $(A, (f_i: A \to A_i)_{i\in I})$ and a T-epimorphism $e: X \to TA$ such that $Tf_i \circ e = h_i$ for all i. (One calls e, or more precisely the pair (e, A), a *T-epimorphism* if for any $r, s: A \to B$ with $Tr \circ e = Ts \circ e$ one has $r = s$). Equivalently, T has a left adjoint and each A-source factorizes into an epimorphism followed by a T-initial source [26]. The topologically algebraic functors were introduced by Y.H. Hong [34], [35] as a generalization of topological functors aimed at the study of topological algebras, e.g. in example 2.4(2) above the grounding functor $\mathscr{A}\mathscr{X} \to$ **Set** is topologically algebraic. Other examples are algebraic functors (in the sense of [25]) and monadic functors into **Set** [33].

6.2. Theorem [26]. *The amnestic functor $T: \mathscr{A} \to \mathscr{X}$ is topologically algebraic if and only if $(\mathscr{A}, T)$ has a reflective universal initial completion.*

The composite of two topologically algebraic functors need not be topologically algebraic. The compositive hull of this class of functors was shown in [26] to be the class of *semitopological functors* due to Trnková, Hoffmann, Tholen and Wischnewsky (see [49], [29], [51] and other references in [23]). The following analogue of 6.1 and 6.2 obtains:

6.3. Theorem [49], [29]. *The amnestic functor $T: \mathscr{A} \to \mathscr{X}$ is semitopological if and only if $(\mathscr{A}, T)$ has a reflective Dedekind–MacNeille completion.*

6.4. Theorem. *These strict implications hold:*

T is topological $\Rightarrow$ T is (E, M)-topological

$\Rightarrow$ T is topologically algebraic

$\Rightarrow$ T is semitopological

$\Rightarrow$ T is faithful and has a left adjoint.

For proofs, see [24] and references there.

6.5. Theorem [10]. *For a forgetful functor T these are equivalent:*

(1) *T is topological;*
(2) *T is semitopological and has a full and faithful right adjoint;*
(3) *T is semitopological and has a right adjoint section.*

The fibrations of Gray [14] fit in as follows [56], [49]:

6.6. *For a forgetful functor T these are equivalent*:

(1) *T is topological*;

(2) *T is a fibration and cofibration and all its fibres are large-complete lattices*;

(3) *T is a semitopological and a fibration.*

Among recent generalizations of topological functors we mention:

(1) *structure functors* of Wischnewsky [52], [53];

(2) *locally semitopological functors* of Tholen [50];

(3) *presemitopological functors* of Greve [16], [17];

(4) *discrete functors* of Ohlhoff [40], [41];

(5) *relatively topological functors* of Strecker and Titcomb [46].

Within our topic we have neglected many important results and whole areas, notably cartesian closed and monoidal closed topological categories. The reader can now start out by consulting Herrlich's survey [23], the monographs by Tholen [48] and Porst [43], and the proceedings volumes of the three recent conferences in Ottawa [2], Gummersbach [37] and Cape Town [7].

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