Dirichlet Problem with Nonlinearity Depending only on the Derivative

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Abstract—We study the existence of solutions for the following problem:

\[ u''(t) + u(t) + g(u'(t)) = f(t), \quad t \in (0, \pi), \]
\[ u(0) = u(\pi) = 0, \]

where \( f \in C[0, \pi], g \in C(\mathbb{R}) \) is bounded and has limits \( \lim_{u \to \pm \infty} g(u) \). We also give information on the set of \( f \) for those that solution exists, relating it with the corresponding linear problem. © 2002 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Let us consider the class of Dirichlet problems of form (1) where \( f \in C[0, \pi], g \in C(\mathbb{R}) \) is bounded and there exist the limits \( g(+\infty) = \lim_{u \to +\infty} g(u) \) and \( g(-\infty) = \lim_{u \to -\infty} g(u) \) (we can suppose without lost of generality that \( g(+\infty) = -g(-\infty) \)). The main difficulty for the study of existence of solutions of these problems resides in the following two facts: the problems described by (1)
are resonant, and a Landesman-Lazer type condition for these problems is not known and, in fact, it is hoped that they do not satisfy such a condition anyway (see [1-5]).

The first positive eigenfunction associated to the corresponding linear problem (i.e., \( g = 0 \)) is \( \sin(\cdot) \), so that we can decompose \( f \), in a unique way, as

\[
 f(t) = \tilde{f}(t) + a \sin(t),
\]

where \( \tilde{f} \in \tilde{C}[0,\pi] = \{ \tilde{f} \in C[0,\pi] : \int_{0}^{\pi} \tilde{f}(t) \sin(t) \, dt = 0 \} \) and

\[
a = \frac{2}{\pi} \int_{0}^{\pi} f(t) \sin(t) \, dt.
\]

It is well known (see [1,6]) that for each \( \tilde{f} \in \tilde{C}[0,\pi] \), there exists a nonempty bounded interval \( J_{\tilde{f}} \) such that (1) admits solutions if and only if \( a \in J_{\tilde{f}} \). We will denote by \( a_{1}(\tilde{f}) \) and \( a_{2}(\tilde{f}) \) the end points of the interval \( J_{\tilde{f}} \) (i.e., \( a_{1}(\tilde{f}) := \inf J_{\tilde{f}} \leq \sup J_{\tilde{f}} =: a_{2}(\tilde{f}) \)). In order to unify notation, and to leave patent the new contributions of this work, we state here the following two well-known results (see [3,7]).

**Theorem 1.** (See [3, Theorem 1].) Let \( g : \mathbb{R} \to \mathbb{R} \) be locally Lipschitz continuous and such that the limits \( g(-\infty) \) and \( g(+\infty) \) exist and \( g(-\infty) + g(+\infty) = 0 \). Let \( f \in C[0,\pi] \) split according to (2). Then \( 0 \in [a_{1}(\tilde{f}),a_{2}(\tilde{f})] \) and problem (1) has

(i) at least one solution if \( a \in (a_{1}(\tilde{f}),a_{2}(\tilde{f})) \) or \( a \in \{ a_{1}(\tilde{f}),a_{2}(\tilde{f}) \} \{ 0 \} \);

(ii) at least two solutions if \( a \in (a_{1}(\tilde{f}),a_{2}(\tilde{f})) \{ 0 \} \).

**Theorem 2.** (See [7, Theorem 1.2].) Let \( g : \mathbb{R} \to \mathbb{R} \) be continuous increasing odd and satisfying

\[
g(+\infty) - g(s) \leq \frac{1}{1 + s^\beta},
\]

for all \( s \geq s_{0} \) and \( \beta > 1 \). Then

(i) if \( g(-\infty) < \int_{0}^{\pi/2} \tilde{f}(t) \sin(t) \, dt < g(+\infty) \), then \( a_{1}(\tilde{f}) < 0 < a_{2}(\tilde{f}) \);

(ii) if \( \int_{0}^{\pi/2} \tilde{f}(t) \sin(t) \, dt \leq g(-\infty) \), then \( a_{1}(\tilde{f}) < 0 \leq a_{2}(\tilde{f}) \);

(iii) if \( \int_{0}^{\pi/2} \tilde{f}(t) \sin(t) \, dt \geq g(+\infty) \), then \( a_{1}(\tilde{f}) \leq 0 < a_{2}(\tilde{f}) \).

It follows from the above results that there exists a certain relation between the end points of \( J_{\tilde{f}} \), the nonlinearity \( g \), and the solution of the linear part of (1) (see [8]). On the other hand, these papers do not answer the following questions. If we assume we are in one of the Cases (ii) or (iii) of Theorem 2, does \( 0 \) belong to the interior of \( J_{\tilde{f}} \)? is it possible to find a nonlinear \( g \) such that \( 0 \) belongs to the boundary of \( J_{\tilde{f}} \)?

The main goal of this paper is to answer these two open questions. In Section 2, we state the main results, as well as some remarks and related open problems. Section 3 is devoted to the corresponding proofs.

## 2. MAIN RESULTS

**Theorem 3.** Let \( g \) be as in Theorem 2. Then there are functions \( \tilde{f}_{1}, \tilde{f}_{2} \in \tilde{C}[0,\pi] \) such that

(i) \( a_{1}(\tilde{f}_{1}) < 0 < a_{2}(\tilde{f}_{1}) \) and \( \int_{0}^{\pi/2} \tilde{f}_{1}(t) \sin(t) \, dt \leq g(-\infty) \);

(ii) \( a_{1}(\tilde{f}_{2}) < 0 < a_{2}(\tilde{f}_{2}) \) and \( \int_{0}^{\pi/2} \tilde{f}_{2}(t) \sin(t) \, dt \geq g(+\infty) \).

**Theorem 4.** There exists \( g \) as in Theorem 2, and there are functions \( \tilde{f}_{3}, \tilde{f}_{4} \in \tilde{C}[0,\pi] \) such that

(iii) \( a_{1}(\tilde{f}_{3}) < 0 = a_{2}(\tilde{f}_{3}) \) and \( \int_{0}^{\pi/2} \tilde{f}_{3}(t) \sin(t) \, dt \leq g(-\infty) \);

(iv) \( a_{1}(\tilde{f}_{4}) = 0 < a_{2}(\tilde{f}_{4}) \) and \( \int_{0}^{\pi/2} \tilde{f}_{4}(t) \sin(t) \, dt \geq g(+\infty) \).

**Remark 1.** In [3], it was proved that under the additional hypothesis on \( g, g \in C^{1}(\mathbb{R}), g(0) = 0 \), and \( g'(0) \neq 0 \) for \( \tilde{f} \) being 'sufficiently small' (in the sense that \( ||\tilde{f}||_{\infty} < \varepsilon \) with \( \varepsilon \) small enough), \( 0 \) belongs to the interior of \( J_{\tilde{f}} \), and the authors said nothing for arbitrary choices of \( \tilde{f} \). We will
see in the proof of our results that both cases (i.e., 0 being an end point of $J_f$ and 0 being an interior point of $J_f$) are possible for $g(s) = s^2/(1 + s^2)$ ($s > 0$) extended to the real line as an odd function, and $\tilde{f}$ with uniform norm as large as you want. (Note that $g'(0) = 0$.) On the other hand, it is also clear from the proof of Theorem 3 that there exists functions $g$ as in [3] such that there are functions $\tilde{f}$ with uniform norm as large as you want and 0 being an interior point of $J_f$.

**Remark 2.** Theorem 4 implies that $a_1 < 0 < a_2$ is not always verified, which shows a great difference with the case in that $g$ only depends on $u$ (see [8]), indeed it is proved more that the statement $a_1(f_4) = 0$ and $a_2(f_3) = 0$. That is, zero does not belong to the interval $J_{f_i}$, $i = 3, 4$, i.e., $J_{f_3} = [a_1(f_3), 0]$ and $J_{f_4} = (0, a_2(f_4)]$.

### 3. PROOFS

In this section, we will prove the two theorems of the previous section. With this objective in mind, we would like first to introduce some notation, and to state a technical lemma.

We follow the alternative method of Fredholm, so that we decompose $u$ as $u = \tilde{u} + c\sin(t)$, where $\tilde{u} \in \tilde{C}[0, \pi]$ and $c \in \mathbb{R}$ are uniquely determined. By Schauder fixed-point theorem for each $c \in \mathbb{R}$, there is $\tilde{u} = \tilde{u}_c \in \tilde{C}[0, \pi]$ satisfying the following Dirichlet problem:

\begin{equation}
\tilde{u}''(t) + \tilde{u}(t) + g(\tilde{u}'(t) + c\cos(t)) = \frac{2}{\pi} \int_0^\pi g(\tilde{u}'(s) + c\cos(s)) \sin(s) \, ds \sin(t) = \tilde{f}(t),
\end{equation}

\begin{align*}
\tilde{u}(0) &= \tilde{u}(\pi) = 0,
\end{align*}

and it is well known that if we denote by

\[ \Sigma = \left\{ (c, \tilde{u}) \in \mathbb{R} \times \tilde{C}[0, \pi] : \tilde{u} \text{ satisfies (3)} \right\} \]

and

\[ \Gamma_j : \Sigma \to \mathbb{R}, \]

\[ \Gamma_j(c, \tilde{u}) = \frac{2}{\pi} \int_0^\pi g(\tilde{u}'(t) + c\cos(t)) \sin(t) \, dt, \]

problem (1) has solution if and only if $a \in \Gamma_j(\Sigma) = J_j$ (see [1] for a proof of this last statement).

**Lemma 1.** Let $\tilde{f} \in \tilde{C}[0, \pi]$, if $\tilde{u} \in \tilde{C}[0, \pi]$ is the solution of (3) and $\tilde{w} \in \tilde{C}[0, \pi]$ is a solution of the problem

\begin{equation}
\tilde{w}''(t) + \tilde{w}(t) = \tilde{f}(t),
\end{equation}

\begin{align*}
\tilde{w}(0) &= \tilde{w}(\pi) = 0,
\end{align*}

then

\[ \|\tilde{u}' - \tilde{w}'\|_\infty \leq L, \]

where $L$ is a constant that only depends on $\|g\|_\infty$.

**Proof.** If we use that $\tilde{w} \in \tilde{C}(0, \pi) \cap C^2[0, \pi]$ is the solution of (4), then it is not difficult to prove that its derivative can be written explicitly as

\begin{equation}
\tilde{w}'(t) = \int_0^t \cos(t - s) \tilde{f}(s) \, ds + \frac{2}{\pi} \int_0^\pi \tilde{f}(s) \cos(s)(s - \pi) \, ds \cos(t).
\end{equation}

Now, we take $\tilde{f} = g(\tilde{u}'(\cdot) + c\cos(\cdot)) + 2/\pi \int_0^\pi g(\tilde{u}'(t)c\cos(t)) \sin(t) \, dt \sin(\cdot)$ instead of $\tilde{f}$ in (4) and use a formula like (5) for $\tilde{u}'$ which concludes the proof.
PROOF OF THEOREM 3. Let \( \varepsilon, \gamma \in \mathbb{R} \) such that \( 0 < \varepsilon < \pi/4 \) and \( \varepsilon < \gamma < \pi/2 - \varepsilon \) and set

\[
w_{\varepsilon,\gamma}(t) = (w_{\gamma} \ast \zeta_{\varepsilon})(t) = \int_{0}^{\pi} w_{\gamma}(t - s)\zeta_{\varepsilon}(s) \, ds,
\]

where

\[
w_{\gamma}(t) = t\chi_{[0,\gamma]}(t) + \frac{\gamma}{\gamma - \pi/2} \left( t - \frac{\pi}{2} \right) \chi_{(\gamma,\pi-\gamma)}(t) + (t - \pi)\chi_{[\pi-\gamma,\pi]}(t)
\]

and \( \zeta_{\varepsilon}(t) = (1/\varepsilon)e^{t^2}(e^{\varepsilon^2} - 1)^x(-\varepsilon,x)(t) \). If we define \( \tilde{f}_{\varepsilon,\gamma} := w_{\varepsilon,\gamma}' + w_{\varepsilon,\gamma} \), and we take into account that \( w_{\varepsilon,\gamma}(0) = w_{\varepsilon,\gamma}(\pi) = 0 \), we obtain that

\[
\int_{0}^{\pi/2} \tilde{f}_{\varepsilon,\gamma}(t) \sin(t) \, dt = \frac{\gamma}{\gamma - \pi/2}.
\]

Now we will prove Part (i) of the Theorem 3. To be more precise, we will prove that there exists \( k_1 \) and \( \varepsilon \) with \( 0 < \varepsilon < \pi/4 \) and \( \gamma > \pi/3 + \varepsilon \) such that for all \( k > k_1 \), if we take \( \tilde{f}_1 = k\tilde{f}_{\varepsilon,\gamma}(t) \), then \( \alpha_1(\tilde{f}_1) < 0 < \alpha_2(\tilde{f}_1) \) and \( \int_{0}^{\pi/2} \tilde{f}_1(t) \sin(t) \, dt \leq g(-\infty) \).

It follows from (7) that

\[
\int_{0}^{\pi/2} \tilde{f}_1(t) \sin(t) \, dt = k\frac{\gamma}{\gamma - \pi/2} \leq g(-\infty), \quad \text{for } k \geq g(+\infty)\frac{\pi/2 - \gamma}{\gamma}.
\]

Hence, we are able to use Theorem 2 to claim that \( \alpha_1(\tilde{f}_1) < 0 \leq \alpha_2(\tilde{f}_1) \) whenever \( k \geq g(+\infty) \) \( (\pi/2 - \gamma)/\gamma \), and we only need to prove that \( \alpha_2(\tilde{f}_1) > 0 \). Of course, to this purpose, it is sufficient to check that \( \Gamma_{\tilde{f}_1}(0,\tilde{u}) > 0 \), since \( (0,\tilde{u}) \in \Sigma \) and \( \Gamma_{\tilde{f}_1}(\Sigma) = J_{\tilde{f}_1} \).

Using Lemma 1, we have that \( \tilde{u}'(t) \geq kw_{\varepsilon,\gamma}(t) - L \) for all \( t \in [0,\pi] \), so that

\[
\Gamma_{\tilde{f}_1}(0,\tilde{u}) = 2\pi \int_{0}^{\pi} g(\tilde{u}'(t)) \sin(t) \, dt \geq \frac{2}{\pi} \int_{0}^{\pi} g(kw_{\varepsilon,\gamma}(t) - L) \sin(t) \, dt,
\]

since \( g \) is an increasing function. On the other hand, it is not difficult to check that \( w_{\varepsilon,\gamma}'(t) \geq z(t) \) where \( z \) is defined by

\[
z(t) = \chi_{[0,\gamma-\varepsilon]}(t) + \frac{\gamma}{\gamma - \pi/2} \chi_{[\gamma-\varepsilon,\pi-(\gamma-\varepsilon)]}(t) + \chi_{[\pi-(\gamma-\varepsilon),\pi]}(t)
\]

(see (6)). It follows that

\[
\Gamma_{\tilde{f}_1}(0,\tilde{u}) \geq \frac{2}{\pi} \int_{0}^{\pi - \varepsilon} g(k - L) \sin(t) \, dt + \frac{2}{\pi} \int_{\pi - \varepsilon}^{\pi} g\left(\frac{\gamma}{\gamma - \pi/2}k - L\right) \sin(t) \, dt + \frac{2}{\pi} \int_{\pi - (\gamma-\varepsilon)}^{\pi} g(k - L) \sin(t) \, dt
\]

and

\[
\lim_{k \to +\infty} \left[ \frac{2}{\pi} \int_{0}^{\pi - \varepsilon} g(k - L) \sin(t) \, dt + \frac{2}{\pi} \int_{\pi - \varepsilon}^{\pi} g\left(\frac{\gamma}{\gamma - \pi/2}k - L\right) \sin(t) \, dt \right. \\
\left. + \frac{2}{\pi} \int_{\pi - (\gamma-\varepsilon)}^{\pi} g(k - L) \sin(t) \, dt \right] = \frac{4}{\pi} g(+\infty) \left(1 - 2\cos(\gamma - \varepsilon)\right),
\]

which is a positive real number whenever \( \gamma > \pi/3 + \varepsilon \). Hence, there exist \( k_0 > 0 \) such that for all \( k > k_0 \) and \( \gamma > \pi/3 + \varepsilon \), we have that \( \Gamma_{\tilde{f}_1}(0,\tilde{u}) > 0 \). The result follows taking \( k_1 = \max\{k_0, g(+\infty)(\pi/2 - \gamma)/\gamma\} \).
The proof of (ii) is analogous if we take \( \tilde{f}_2 = -k\tilde{f}_{\varepsilon,\gamma}(t) \) for all \( k > k_2 \) and \( \gamma > \pi/3 + \varepsilon \) and we use that \( \tilde{u}' \leq -kw'_{\varepsilon,\gamma} + L \leq -kz + L \).

**Proof of Theorem 4.** (i) Let us take \( g(s) = s^2/(1+s^2) \) for \( s \geq 0 \) and \( g(s) = -g(-s) \) for \( s < 0 \). We will prove that there exists \( k_3 > 0 \) such that if we take \( f_3 = k\tilde{f}_{\varepsilon,\gamma}(t) \) with \( k > k_3 \), \( \gamma = \pi/6 \) and \( \varepsilon = \pi/12 \), then \( a_1(f_3) < 0 = a_2(f_3) \) and \( \int_0^{\pi/2} \tilde{f}_3(t) \sin(t)\,dt \leq -1 \).

If we take into account (7), we have that \( \int_0^{\pi/2} \tilde{f}_3(t) \sin(t)\,dt = (-1/2)k \leq -1 \) for all \( k \geq 2 \). Hence, we are able to use Theorem 2 to claim that \( a_1(f_3) < 0 \leq a_2(f_3) \), and we only need to prove that

\[
a_2(f_3) = \sup_{(c,\tilde{u}) \in \Sigma} \frac{2}{\pi} \int_0^\pi g(\tilde{u}'(t) + c\cos(t))\sin(t)\,dt = 0.
\]

Clearly, it is enough to prove that

\[
\frac{2}{\pi} \int_0^\pi g(\tilde{u}'(t) + c\cos(t))\sin(t)\,dt < 0
\]

for all \( (c, \tilde{u}) \in \Sigma \).

Using Lemma 1, we have that \( \tilde{u}'(t) \leq kw'_{\pi/12,\pi/6}(t) + L \) for all \( t \in [0, \pi] \), so that

\[
\frac{2}{\pi} \int_0^\pi g(\tilde{u}'(t) + c\cos(t))\sin(t)\,dt \leq \frac{2}{\pi} \int_0^\pi g\left(kw'_{\pi/12,\pi/6}(t) + L + c\cos(t)\right)\sin(t)\,dt.
\]

On the other hand, it is not difficult to check that \( w'_{\pi/12,\pi/6}(t) \leq \rho(t) \), where \( \rho \) is defined by

\[
r(t) = \chi(0,\pi/4)(t) - \frac{1}{2}\chi[\pi/4,3\pi/4](t) + \chi(3\pi/4,\pi](t).
\]

Hence,

\[
\frac{2}{\pi} \int_0^\pi g\left(kw'_{\pi/12,\pi/6}(t) + L + c\cos(t)\right)\sin(t)\,dt \leq \frac{2}{\pi} \int_0^\pi g\left(k\rho(t) + L + c\cos(t)\right)\sin(t)\,dt
\]

and, if we denote by \( G \) the unique primitive of \( g \) satisfying \( G(0) = 0 \), we have that

\[
\int_0^\pi g\left(k\rho(t) + L + c\cos(t)\right)\sin(t)\,dt = \int_0^{\pi/4} g(k + L + c\cos(t))\sin(t)\,dt + \int_{\pi/4}^{3\pi/4} g(-k + L + c\cos(t))\sin(t)\,dt + \int_{3\pi/4}^\pi g(k + L + c\cos(t))\sin(t)\,dt
\]

\[
= -\frac{1}{c} \left[ G\left(k + L + \frac{c}{\sqrt{2}}\right) - G\left(k + L - \frac{c}{\sqrt{2}}\right) + G\left(-\frac{k}{2} + L - \frac{c}{\sqrt{2}}\right) - G\left(\frac{k}{2} - L + \frac{c}{\sqrt{2}}\right) - G\left(k + L - c\right) + G\left(k + L + c\right) \right].
\]

Using that \( G \) is even and \( k > 0 \) is fixed, then \( \lim_{c \to +\infty} \left(-c \int_0^\pi g\left(k\rho(t) + L + c\cos(t)\right)\sin(t)\,dt\right) = k - 2L \), so that there exists \( c_0 > 0 \) such that \( \int_0^\pi g\left(k\rho(t) + L + c\cos(t)\right)\sin(t)\,dt < 0 \), for all \( c > c_0 \) and \( k > 2L \).

Now, it is clear that \( \int_0^\pi g\left(k\rho(t) + L + c\cos(t)\right)\sin(t)\,dt \) is an even function if we only consider it as depending on the parameter \( c \). Hence,

\[
\int_0^\pi g\left(k\rho(t) + L + c\cos(t)\right)\sin(t)\,dt < 0, \quad \text{for all } |c| > c_0.
\]
On the other hand, the limit
\[
\lim_{k \to +\infty} -\frac{1}{c} \left[ G \left( k + L + \frac{c}{\sqrt{2}} \right) - G \left( k + L - \frac{c}{\sqrt{2}} \right) + G \left( \frac{k}{2} + L - \frac{c}{\sqrt{2}} \right) - G \left( \frac{k}{2} + L + \frac{c}{\sqrt{2}} \right) + G (k + L - c) - G (k + L + c) \right] = 2 - 2\sqrt{2} < 0
\]
exists uniformly on \( c \in [-c_0, c_0] \).

Hence, there exists \( k_0 > 0 \) such that for all \( k > k_0 \) and for all \( c \in [-c_0, c_0] \),
\[
\frac{2}{\pi} \int_0^\pi g (\tilde{u}'(t) + c \cos(t)) \sin(t) \, dt < 0.
\]
The result follows taking \( k_3 = \max\{k_0, 2, 2L\} \).

The proof of (iv) is analogous if we take \( f_4 = -k f_{\epsilon,\gamma}(t) \) with \( k > k_4, \gamma = \pi/6, \epsilon = \pi/12 \) and we use the inequality \( \tilde{u}' \leq -kw_{\epsilon,\gamma}' + L \leq -kr + L \).

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