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On Oscillation of Solutions of *n*th-Order Delay Differential Equations

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Oscillatory behavior of the solutions of the nth-order delay differential equation $L_n x(t) + q(t) f(x[g(t)]) = 0$ is discussed. The results obtained are extensions of some of the results by Kim (*Proc. Amer. Math. Soc.* **62** (1977), 77–82) for $y^{(n)} + py = 0$.

The main purpose of this article is to extend some of the results of Kim [1] for

$$x^{(n)} + p(t)\, x = 0$$

to the following nth-order delay differential equation

$$L_n x(t) + q(t) f(x[g(t)]) = 0, \tag{E}$$

where $L_0 x(t) = x(t)$, $L_k x(t) = a_k(t)(L_{k-1} x(t))^{\cdot}$ $(^{\cdot} = d/dt)$, $a_0(t) = a_n(t) = 1$, $k = 1, 2,..., n$.

We shall discuss the following four cases:

(1) n even, $q \geqslant 0$;

(2) n odd, $q \geqslant 0$;

(3) n even, $q \leqslant 0$;

(4) n odd, $q \leqslant 0$.

In the sequel, (E_i), for example, will denote Eq. (E) satisfying condition (i) for $i = 1, 2, 3, 4$.

The conditions we always assume for a_i, q, g, and f are:

(i) g: $[0, \infty) \to [0, \infty)$ is continuous and nondecreasing, $g(t) \leqslant t$, and $\lim_{t\to\infty} g(t) = \infty$;

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(ii) q: $[0, \infty) \to (-\infty, \infty)$ is continuous and q is not eventually identically zero on $R = (-\infty, \infty)$;

(iii) a_i: $[0, \infty) \to (0, \infty)$ is continuous, $\int^\infty (1/a_i(s))\, ds = \infty$, $i = 1, 2, \ldots, n-1$,

and either

(iv) $\lim_{t\to\infty} (1/\alpha_2(t)) \sum_{i=0}^{k} {}_i\alpha_i(t) > 0$, where $\alpha_0(t) = 1$, for every choice of the constants c_i, with $c_k > 0$ for $k = 2, 3, \ldots, n-1$; or

(v) $\lim_{t\to\infty} (1/\alpha_1(t)) \sum_{i=0}^{k} c_i\alpha_i(t) > 0$, where $\alpha_0(t) = 1$, for every choice of the constants c_i, with $c_k > 0$ for $k = 1, 2, \ldots, n-1$;

where

$$\alpha_1(t) = \int_c^t \frac{1}{a_1(s_1)}\, ds_1,$$

$$\alpha_2(t) = \int_c^t \frac{1}{a_1(s_1)} \int_c^{s_1} \frac{1}{a_2(s_2)}\, ds_2\, ds_1,$$

$$\vdots$$

$$\alpha_k(t) = \int_c^t \frac{1}{a_1(s_1)} \int_c^{s_1} \cdots \int_c^{s_{k-1}} \frac{1}{a_k(s_k)}\, ds_k \cdots ds_1,$$

$$\vdots$$

$$\alpha_{n-1}(t) = \int_c^t \frac{1}{a_1(s_1)} \int_c^{s_1} \cdots \int_c^{s_{n-2}} \frac{1}{a_{n-1}(s_{n-1})}\, ds_{n-1} \cdots ds_1;$$

for some $c \geqslant 0$;

(vi) f: $R \to R$ is continuous such that $xf(x) > 0$ for $x \neq 0$.

We also define

$$w_1(t, s) = \int_s^t \frac{1}{a_1(u)}\, du$$

and

$$w_k(t, s) = \int_s^t \frac{1}{a_k(u)} w_{k-1}(u, s)\, du, \qquad k = 2, \ldots, n-1.$$

We restrict our discussion to those solutions x of the above differential equations which exist on some ray $[0, \infty)$. Such a solution is said to be oscillatory if it has arbitrarily large zeros. Otherwise the solution is said to be nonoscillatory.

The oscillatory behavior of solutions of the above equation and/or related equations has recently been studied by many authors, we mention in particular the work of Kim [1], who discussed the monotonicity and the oscillatory behavior of those solutions of (E) with $a_i(t) = 1$, $i = 1, \ldots, n-1$ which have the property that

$$x(t)/t^2 \to 0 \quad \text{as} \quad t \to \infty \qquad \text{or} \qquad x(t)/t \to 0 \quad \text{as} \quad t \to \infty.$$

The following two theorems are extensions of the results by Kim:

THEOREM 1. *Suppose that conditions* (i)–(iv) *and* (vi) *hold. If x is a nontrivial solution of* (E_1) *or* (E_4) *such that $x(t) \geqslant 0$, $x[g(t)] \geqslant 0$, and $x(t)/a_2(t) \to 0$ as $t \to \infty$, then*

$$x(t) \geqslant 0, \quad \dot{x}(t) > 0, \quad (-1)^{k-1} L_k x(t) > 0 \quad \textit{for} \quad t \in [0, \infty),$$

$k = 2, 3, \ldots, n-1$, *and*

$$L_k x(t) \to 0 \text{ monotonically as } t \to \infty, \; k = 2, 3, \ldots, n-1. \tag{5}$$

Proof. Our proof is an adaptation of the argument developed by Kim. Put $y_k = L_{k-1} x$, i.e., $x = y_1$, $\dot{y}_1 = y_2/a_1, \ldots, y_{n-1} = y_n/a_{n-1}$, and let b be an arbitrary point of $[0, \infty)$. Then x satisfies the system

$$y_1(t) = y_1(b) + \int_b^t \frac{y_2(s)}{a_1(s)}\, ds,$$

$$y_2(t) = y_2(b) + \int_b^t \frac{y_3(s)}{a_2(s)}\, ds,$$

$$\vdots$$

$$y_{n-1}(t) = y_{n-1}(b) + \int_b^t \frac{y_n(s)}{a_{n-1}(s)}\, ds,$$

$$y_n(t) = y_n(b) - \int_b^t q(s) f(y_1[g(s))])\, ds.$$

Suppose $x = y_1$ is a solution of (E_1). Then $\int_b^t q(s) f(y_1[g(s)])\, ds$ is a nondecreasing nonnegative function of t and clearly is positive on an interval $[c, \infty)$ for some $c > b$. We claim that $y_n(b) > 0$. To prove this, assume the contrary, that $y_n(b) \leqslant 0$. Then $y_n(t)$ is nonpositive, nonincreasing on $[b, \infty)$ and

$$y_n(c) = y_n(b) - \int_b^c q(s) f(y[g(s)])\, ds < 0,$$

i.e.,

$$y_n(t) \leqslant y_n(c) < 0, \qquad t \in [c, \infty),$$

or

$$\dot{y}_{n-1}(t) \leqslant (1/a_{n-1}(t))\, y_n(c).$$

Integrating the above inequality from b to t, we obtain

$$y_{n-1}(t) \leqslant y_{n-1}(b) + y_n(c) \int_b^t \frac{1}{a_{n-1}(s)}\, ds \to -\infty \qquad \text{as} \quad t \to \infty.$$

This in turn implies that $y_{n-2}(t) \to -\infty$ as $t \to \infty$, and successively $y_k(t) \to -\infty$ as $t \to \infty$, regardless of the values $y_k(b)$, $k = 1,..., n-1$. In particular, $y_1(t) = x(t) \to -\infty$ as $t \to \infty$, contrary to the hypothesis that $x(t) \geqslant 0$ on $[0, \infty)$. This contradiction proves $y_n(b) > 0$.

Since b is arbitrary, we conclude that $y_n(t) > 0$, $t \in [0, \infty)$. It is now easy to see that $y_n(t) \to 0$ as $t \to \infty$ for $n > 2$. If this were not the case, there would exist a constant $C > 0$ such that

$$y_n(t) > C, \qquad t \in [c_1, \infty), \qquad \text{for some } c_1 \geqslant 0.$$

This implies, however, that

$$x(t) = y_1(t) > \sum_{i=0}^{n-2} y_{i+1}(c_1)\, \alpha_{i+1}(t) + C\alpha_{n-1}(t), \qquad \alpha_0(t) = 1.$$

If we divide the above inequality by $\alpha_2(t)$ and take the limit as $t \to \infty$, we get, in view of (iv) with $k = n-1$, a contradiction to the fact that $x(t)/\alpha_2(t) \to 0$ as $t \to \infty$.

Next we shall prove that $y_{n-1}(t) < 0$ if $n > 2$. If $y_{n-1}(b) \geqslant 0$, then $y_{n-1}(t) \geqslant 0$ on $[b, \infty)$ and there would exist constants $C_1 > 0$ and $d > b$ such that

$$y_{n-1}(t) > C_1, \qquad t \in [d, \infty).$$

This would imply

$$x(t) = y_1(t) > \sum_{i=0}^{n-3} y_{i+1}(d)\, \alpha_{i+1}(t) + C_1 \alpha_{n-2}(t),$$

which would again lead to a contradiction. Thus $y_{n-1}(b) < 0$ and hence $y_{n-1}(t) < 0$, since b is arbitrary. Moreover, we must have $y_{n-1}(t) \to 0$ as $t \to \infty$, for otherwise we would again be led to the contradiction that $x(t) \to -\infty$ as $t \to \infty$. In this way, we can successively establish the inequalities $y_n(t) > 0$, $y_{n-1}(t) < 0,..., y_4(t) > 0$, $y_3(t) < 0$, $t \in [0, \infty)$ with the property that $y_k(t) \to 0$ as $t \to \infty$, $k = 3, 4,..., n$. Continuing this process, we deduce $y_2(t) > 0$ and $y_1(t) \geqslant 0$, $t \in [0, \infty)$. This proves the theorem for (E_1). The proof for (E_4) is similar; in this case, we first prove that $y_n(t) < 0$ and $y_n(t) \to 0$ as $t \to \infty$, and continue as in the case of (E_1).

In somewhat similar fashion, we can prove

THEOREM 2. *Let conditions* (i)–(ii), (v) *and* (vi) *hold. If x is a nontrivial solution of* (E_2) *or* (E_2) *such that*

$x(t) \geqslant 0$, $x[g(t)] \geqslant 0$, *and* $x(t)/\alpha_1(t) \to 0$ *as* $t \to \infty$, *then*

$$x(t) > 0,\ \dot{x}(t) < 0,\ (-1)^k L_k x(t) > 0 \text{ for } t \in [0, \infty),\ k = 2, 3, \ldots, n-1 \text{ and } L_k x(t) \to 0 \text{ monotonically as } t \to \infty,\ k = 2, \ldots, n-1. \qquad (6)$$

We now give some illustrative examples.

EXAMPLE 1. Consider the equation

$$\left(\frac{1}{t}\left(t\left(\frac{1}{t}\dot{x}\right)^{\cdot}\right)^{\cdot}\right)^{\cdot} + \frac{15}{16}\frac{1+t^3}{t^5}\frac{x}{1+x^2} = 0, \qquad t \geqslant 1.$$

Thus

$$\alpha_1(t) = \tfrac{1}{2}(t^2 - 1)$$
$$\alpha_2(t) = \tfrac{1}{4}t^2(\log t^2 - 1) + \tfrac{1}{4}$$
$$\alpha_3(t) = \tfrac{1}{4}t^2(\tfrac{1}{4}t^2 - \log t) - \tfrac{1}{16}.$$

Clearly, condition (iv) is satisfied, since t^4 will dominate all other terms when t is sufficiently large. The above equation has a solution $x(t) = t^{3/2}$ satisfying (5). We may note that [1, Theorem 1] is not applicable to this equation since $f(x) \neq x$ and $a_i \neq 1$, $i = 1, 2, 3$.

EXAMPE 2. The equation

$$x^{(4)} + \frac{15\sqrt{2}}{16}\frac{1+t}{t^5}\frac{x^3(t/2)}{1+x^2(t/2)} = 0$$

has a solution $x(t) = \sqrt{t}$, satisfying (5). Again we note that [1, Theorem 1] cannot be applied to this equation.

EXAMPLE 3. The equation

$$\left(\frac{1}{t}(t\dot{x})^{\cdot}\right)^{\cdot} + \tfrac{5}{8}t^{(7-\alpha)/2}|x|^{\alpha}\operatorname{sgn}(x) = 0, \qquad \alpha > 0,$$

has a solution $x(t) = t^{-1/2}$ satisfying (6), while [1, Theorem] cannot be applied.

As was done by Kim, in order to characterize the behavior of solutions of (E_1) or (E_4) we may reformulate Theorem 1 as

THEOREM 3. *Suppose that conditions* (i)–(iv) *and* (vi) *hold. Let x be a nontrivial solution of* (E_1) *or* (E_4) *such that* $x(t)/\alpha_2(t) \to 0$ *as* $t \to \infty$. *Then either*

(a) *x is oscillatory on* $[0, \infty)$, *or*

(b) $x \geqslant 0$ $(x \leqslant 0)$ *on* $[t_0, \infty)$ *for some* $t_0 \geqslant 0$ *and* x $(-x)$ *satisfies inequalities* (5) *of Theorem* 1. *In particular,* x $(-x)$ *increases (decreases) monotonically on* $[t_0, \infty)$.

COROLLARY 1. *If x is a nontrivial solution of* (E_1) *or* (E_4) *such that* $x(t) \to 0$ *as* $t \to \infty$, *then x is oscillatory.*

THEOREM 4. *Suppose that conditions* (i)–(iv) *and* (vi) *hold, and that*

(vii) $f'(x) \geqslant 0$,*for* $x \neq 0$ $(' = d/dx)$.

Let x be a nontrivial solution of (E_1) *or* (E_4) *such that* $x(t)/\alpha_2(t) \to 0$ *as* $t \to \infty$. *Then if*

$$\limsup_{t \to \infty} \frac{1}{\alpha_2(t)} \int_c^t w_{n-1}(s, c)\, q(s)\, ds > 0, \qquad c \geqslant 0,$$

x is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of (E_1) or (E_4). Without loss of generality, we may assume that $x(t)$ and $x[g(t)]$ are positive for $t \geqslant t_0 \geqslant 0$. Now a simple induction argument shows that for $t \geqslant t_0$ and $1 \leqslant k \leqslant n-1$

$$x(t) = x(t_0) + \sum_{j=1}^{k} (-1)^{j-1}\, w_j(t, t_0)\, L_j x(t) + (-1)^k \int_{t_0}^t w_k(s, t_0)(L_k x(s))^{\cdot})\, ds.$$

In particular, if $t \geqslant t_0$,

$$x(t) = x(t_0 + \sum_{j=1}^{n-1} (-1)^{j-1}\, w_j(t, t_0)\, L_j x(t) + \int_{t_0}^t w_{n-1}(s, t_0)\, q(s) f(x[g(s))])\, ds. \tag{7}$$

By Theorem 1, we have

$$\sum_{j=1}^{n-1} (-1)^{j-1} w_j(t, t_0) L_j x(t) \geqslant 0, \qquad t \geqslant t_0.$$

Since $\dot{x}(t) \geqslant 0$ on $[t_0, \infty)$, x is nondecreasing on $[t_0, \infty)$. Thus for $t \geqslant t_0$

$$\begin{aligned} x(t) &\geqslant x(t_0) + \int_{t_0}^{t} w_{n-1}(s, t_0)\, q(s) f(x[g(s)])\, ds \\ &\geqslant x(t_0) + f(x[g(t_0)]) \int_{t_0}^{t} w_{n-1}(s, t_0)\, q(s)\, ds. \end{aligned}$$

Now we divide both sides of the above inequality by $\alpha_2(t)$ and obtain

$$0 = \lim_{t\to\infty} \frac{x(t)}{\alpha_2(t)} \geqslant \limsup_{t\to\infty} \frac{f(x[g(t_0)])}{\alpha_2(t)} \int_{t_0}^{t} w_{n-1}(s, t_0)\, q(s)\, ds > 0,$$

a contradiction. This completes the proof of the theorem.

As an illustration, we consider the equation

$$\left(\frac{1}{t}\left(t\left(\frac{1}{t}\dot{x}\right)^{\cdot}\right)^{\cdot}\right)^{\cdot} + \frac{1}{t^2} f(x[g(t)]) = 0, \qquad t \geqslant 1, \tag{8}$$

where f and g satisfy the conditons in Theorem 4. We let

$$\begin{aligned} w_1(t, 1) &= \int_1^t s\, ds = \tfrac{1}{2}(t^2 - 1), \\ w_2(t, 1) &= \tfrac{1}{4}t^2 - \tfrac{1}{2}\log t - \tfrac{1}{4}, \\ w_3(t, 1) &= \tfrac{1}{4}t^2(\tfrac{1}{4}t^2 - \log t) - \tfrac{1}{16}, \\ \alpha_2(t) &= \tfrac{1}{4}t^2 (\log t^2 - 1) + \tfrac{1}{4}. \end{aligned}$$

Now

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{\alpha_2(t)} \int_1^t w_3(s, 1) \frac{1}{s^2}\, ds \\ &\quad = \limsup_{t\to\infty} \frac{\frac{1}{48}t^3 - \frac{1}{4}t \ln t + \frac{1}{4}t + (1/16t) - \frac{1}{3}}{\frac{1}{4}t^2(\log t^2 - 1) + \frac{1}{4}} \to \infty \qquad \text{as} \quad t \to \infty. \end{aligned}$$

We conclude that if x is a nontrivial solution of (8) such that $\operatorname{lm}_{t\to\infty}(x(t)/\alpha_2(t)) = 0$, then x is oscillatory. We may note that the above conclusion does not appear to be deducible from other known oscillation criteria.

THEOREM 5. *Let conditions* (i)–(iv), (vi), *and* (vii) *hold. If*

$$\int_c^\infty w_{n-1}(s, c)\, q(s)\, ds = \infty, \qquad c \geqslant 0, \tag{9}$$

then every nonoscillatory solution of (E_1) *or* (E_4) *is unbounded on* $[0, \infty)$.

Proof. We only consider Eq. (E_1). Assume the contrary, that there exists a nontrivial solution x of (E_1) which is bounded and positive on $[t_0, \infty)$, $t_0 \geqslant 0$. Since x increases monotonically by Theorem 1, there exist positive constants M_1 and M_2 such that

$$M_1 \leqslant x[g(t)] \leqslant M_2, \qquad t \in [t_0, \infty).$$

Using (7) and Theorem1, we have

$$\sum_{j=1}^{n-1} (-1)^{j-1}\, w_j(t, t_0)\, L_j x(t) - M_2 \leqslant -x(t_0)$$

$$-f(M_1) \int_{t_0}^{t} w_{n-1}(s, t_0)\, q(s)\, ds. \tag{10}$$

The left-hand side of (10) *cannot* tend to $-\infty$ as $t \to \infty$, while the right-hand side *does* tend to $-\infty$ as $t \to \infty$. Therefore, inequality (10) cannot hold throughout $[t_0, \infty)$. This incompatibility proves that the solution x must be unbounded on $[0, \infty)$.

EXAMPLE 5. Consider the equation

$$\begin{aligned} &((1/t)(t((1/t)\,\dot{x})^{\cdot})^{\cdot})^{\cdot} + \tfrac{15}{16}(1/t^5)\, x = 0, \qquad t \geqslant 1, \\ &w_3(t, 1) = \tfrac{1}{4}t^2(\tfrac{1}{24}t^2 - \log t) - \tfrac{1}{16}, \end{aligned} \tag{11}$$

hence

$$\int_1^\infty \frac{1}{s^5}\left(\frac{s^4}{16} - \frac{s^4}{4}\log s - \frac{1}{16}\right) ds = \infty.$$

Thus all nonoscillatory solutions of (11) are unbounded. One such solution is $x(t) = t^{3/2}$.

Remarks. (1) Condition (9) is only a sufficient condition, since the equation

$$((1/t)(t((1/t)\,\dot{x})^{\cdot})^{\cdot})^{\cdot} + \tfrac{15}{16}(1/t^8)\, x^3 = 0, \qquad t \geqslant 1$$

has an unbounded nonoscillatory solution $x(t)=t^{3/2}$, whereas

$$\int_1^\infty w_3(s)\,q(s)\,ds=\int_1^\infty \frac{1}{s^8}\left(\frac{s^4}{16}-\frac{s^2}{4}\log s-\frac{1}{16}\right)ds<\infty.$$

(2) If $a_i(t)=1$, $i=1,\dots,n-1$, $f(x)=x$, and $g(t)=t$, then [1, Theorem 3] and our Theorem 5 are the same.

THEOREM 6. *Let conditions* (i)–(iii) *and* (v)–(vii) *hold. If* $x(t)$ *is a nontrivial solution of* (E_2) *or* (E_3) *such that* $x(t)/\alpha_1(t)\to 0$ *as* $t\to\infty$, *and if*

$$\limsup_{t\to\infty}\int_{g(t)}^t \frac{1}{a_1(s_{n-1})}\int_{s_{n-1}}^t\cdots\int_{s_1}^t q(s)\,ds\cdots ds_{n-1}>\lim_{z\to 0}\sup\frac{z}{f(Z)},\tag{12}$$

then x is oscillatory.

Proof. We only consider (E_2). Let $x(t)$ be a nonoscillatory solution of (E_2) and $x(t)/\alpha_1(t)\to 0$ as $t\to\infty$. Without loss of generality, we may assume that $x(t)$ and $x[g(t)]$ are positive for $t\geqslant t_0$.

Hence (6) of Theorem 2 holds. If $s\leqslant t$, then $g(s)\leqslant g(t)$, and $x[g(s)]\geqslant x[g(t)]$.

Hence we get

$$L_n x(s)+f(x[g(t)])(q(s))\leqslant 0.\tag{13}$$

Integrating (13) n times, we have

$$\begin{aligned}x(t)-x[g(t)]&+(-1)\,L_1x(t)\int_{g(t)}^t\frac{1}{a_1(s_{n-1})}ds_{n-1}+\cdots+(-1)^{n-1}L_{n-1}x(t)\\&\times\int_{g(t)}^t\frac{1}{a_1(s_{n-1})}\int_{s_{n-1}}^t\cdots\int_{s_2}^t\frac{1}{a_{n-1}(s_1)}ds\cdots ds_{n-1}\\&+f(x[g(t)])\int_{g(t)}^t\frac{1}{a_1(s_{n-1})}\int_{s_{n-1}}^t\cdots\int_{s_1}^t q(s)\,ds\cdots ds_{n-1}\leqslant 0,\end{aligned}$$

which implies

$$x(t)+f(x[g(t)])\int_{g(t)}^t\frac{1}{a_1(s_{n-1})}\int_{s_{n-1}}^t\cdots\int_{s_1}^t q(s)\,ds\cdots ds_{n-1}\leqslant x[g(t)].\tag{14}$$

Since $\dot{x}(t)<0$ for $t\geqslant t_0$, $x(t)$ decreases to a limit $c(\geqslant 0)$ as $t\to\infty$. From (14) we obtain $c=0$. By (14)

$$\frac{x[g(t)]}{f(x[g(t)])}\geqslant\int_{g(t)}^t\frac{1}{a_1(s_{n-1})}\int_{s_{n-1}}^t\cdots\int_{s_1}^t q(s)\,ds\,ds_1\cdots ds_{n-1},\qquad t\geqslant t_0.\tag{15}$$

Taking the limit superior of both sides of (15) as $t \to \infty$, we obtain a contradiction to (12). This contradiction completes our proof.

COROLLARY 2. *Assume that $a_k(t) = 1$, $k = 1,..., n-1$. If x is a nontrivial solution of* (E_2) *or* (E_3) *such that $x(t)/t \to 0$ as $t \to \infty$ and*

$$\limsup_{t\to\infty} \int_{g(t)}^{t} [s - g(t)]^{n-1} q(s)\, ds > (n-1)! \limsup_{z\to 0} (z/f(z)), \tag{16}$$

then x is oscillatory.

EXAMPLE 6. Consider the equation

$$((1/t)(t\dot{x})^{\cdot})^{\cdot} + (1/t^3)\,|x[\sqrt{t}]|^{\alpha} \operatorname{sgn} x[\sqrt{t}] = 0, \qquad \alpha \in (0, 1], \quad t > 0.$$

Let $\alpha_1(t) = \log t$. It is easy to verify that

$$\limsup_{t\to\infty} \int_{\sqrt{t}}^{t} \frac{1}{s} \int_{s}^{t} u \int_{u}^{t} \tau^{-3}\, d\tau\, du\, ds$$
$$= \limsup_{t\to\infty} (\log t)^2 \left[\frac{1}{16} - \frac{1}{8 \log t} + \frac{(1/4) - (1/4)\,t}{(\log t)^2}\right] = \infty$$

and

$$\limsup_{z\to 0} (z/f(z)) = \limsup_{z\to 0} z^{1-\alpha} = 0, \qquad \text{if} \quad \alpha \in (0, 1),$$
$$= 1, \qquad \text{if} \quad \alpha = 1.$$

Hence every nontrivial solution x of the above equation such that $x(t)/\log t \to 0$ as $t \to \infty$ is oscillatory.

Remark. Similar oscillation criteria have been obtained by Ladas, Lakshmikantham, and Papadakis [2], Mahfoud [3], and Sficas and Staikos [4]. According to [4, Theorem 1]. all bounded solutions of (E_2) or (E_3) with $a_i = 1$, $i = 1,..., n-1$ are oscillatory if

$$\limsup_{t\to\infty} \int_{g(t)}^{t} [g(t) - g(s)]^{n-1} q(s)\, ds > (n-1)! \limsup_{z\to 0} \frac{z}{f(z)}, \tag{17}$$

where g and f are the same as in Theorem 6. We give an example where Theorem 6 is applicable; however, [4, Theorem 1] cannot be applied.

EXAMPLE 7. Consider the equation

$$\ddot{x} - (1/t^2)\, x[\sqrt{t}] = 0. \tag{18}$$

Now condition (16) implies

$$\limsup_{t\to\infty} \int_{\sqrt{t}}^{t} \frac{1}{s^2}(s-\sqrt{t})\,ds = \limsup_{t\to\infty} \left(\tfrac{1}{2}\log t + \frac{1}{\sqrt{t}} - 1\right) = \infty.$$

Thus every nontrivial solution of the above equation with the prty that $x(t)/t \to 0$ as $t\to\infty$ is oscillatory.

On the other hand

$$\limsup_{t\to\infty} \int_{\sqrt{t}}^{t} \frac{1}{s^2}(\sqrt{t}-\sqrt{s})\,ds = \limsup_{t\to\infty} \left(\frac{1}{\sqrt{t}} - \frac{2}{\sqrt[4]{t}} + 1\right) = 1,$$

so criterion (17) is not applicable to (18).

Remark. The following result provides an oscillation criterion in case either condition (16) or (17) is not satisfied.

THEOREM 7. *Let conditions* (i)–(iii) *and* (v)–(vii) *hold,*

$$\int_{\pm 0}^{\pm 1} \frac{du}{f(u)} < \infty,$$

and

$$\int^{\infty} \frac{1}{a_1(t)}\dot{g}(t) \int_{g(t)}^{t} \frac{1}{a_2(s_{n-2})} \int_{s_{n-2}}^{t} \cdots \int_{s_1}^{t} \times q(s)\,ds\,ds_1 \cdots dt = \infty.$$

If x *is a nontrivial solution of* (E_2) *or* (E_3) *with the property that* $x(t)/a_1(t) \to 0$ *as* $t\to\infty$, *then* x *is oscillatory.*

Proof. As in the proof of Theorem 6, we have inequality (13). Integrating (13) $n-1$ times and using Theorem 2, we obtain

$$\frac{\dot{x}[g(t)]}{f(x[g(t)])} + \frac{1}{a_1(t)} \int_{g(t)}^{t} \frac{1}{a_2(s_{n-2})} \int_{s_{n-2}}^{t} \cdots \int_{s_1}^{t} \times q(s)\,ds\,ds_1 \cdots ds_{n-2} \leqslant 0.$$

Multiplying the above inequality by $\dot{g}(t)$ and integrating, we obtain the desired contradiction.

COROLLARY 3. *Assume that* $a_k(t) = 1$, $k = 1,\dots, n-1$. *If* x *is a nontrivial solution of* (E_2) *or* (E_3) *with the property that* $x(t)/t \to 0$ *as* $t\to\infty$,

$$\int_{\pm 0}^{\pm 1} \frac{du}{f(u)} < \infty,$$

and

$$\int^{\infty} \dot{g}(t) \int_{g(t)}^{t} (s - g(t))^{n-2} q(s)\, ds\, dt = \infty, \tag{19}$$

then x is oscillatory.

For an illustration consider the equation

$$\ddot{x} + |x[t - (1/\sqrt{t})]|^{\alpha} \operatorname{sgn} x[t - (1/\sqrt{t})] = 0, \qquad \alpha \in (0, 1), \quad t > 0. \tag{20}$$

From Corollary 3 it follows that every solution of (20) with the property that$x(t)/t \to 0$ as $t \to \infty$ is oscillatory since

$$\int^{\infty} \dot{g}(t) \int_{g(t)}^{t} (s - g(t))\, ds\, dt = \int^{\infty} \left(1 + \frac{1}{2t\sqrt{t}}\right)\left(\frac{1}{2t}\right) dt = \infty.$$

Condition (16) fails here, however. In fact, it is easily verified that

$$\int_{t-(1/\sqrt{t})}^{t} (s - g(t))^2\, ds = \lim_{t \to \infty} \sup \frac{1}{3t\sqrt{t}} = 0,$$

and $\lim_{z \to 0} \sup z/f(z) = 0$, and hence (16) is not satisfied.

REFERENCES

1. W. J. KIM, Monotone and oscillatory solutions of $y^{(n)} + py = 0$, *Proc. Amer. Math. Soc.* **62** (1977), 77–82.
2. G. LADAS, V. LAKSHMIKANTHAM, AND J. S. PAPADAKIS, Oscillation of higher-order retarded differential equations generated by the retarded argument, *in* "Delay and Functional Differential Equations and Their Applications," pp. 219–231, Academic Press, New York, 1972.
3. W. E. MAHFOUD, Remarks on some oscillation theorems for nth order differential equations with a retarded argument, *J. Math. Anal. Appl.* **62** (1978), 68–80.
4. Y. G. SFICAS AND V. A. STAIKOS, The effect of retarded actions on nonlinear oscillation, *Proc. Amer. Math. Soc.* **46** (1974), 259–264.