Appl. Comput. Harmon. Anal. 30 (2011) 214–230
Contents lists available at ScienceDirect
Applied and Computational Harmonic Analysis
www.elsevier.com/locate/acha

Uncertainty constants and quasispline wavelets ☆

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A R T I C L E I N F O

Article history:
Received 14 July 2009
Revised 29 July 2010
Accepted 13 August 2010
Available online 19 August 2010
Communicated by Charles K. Chui

Keywords:
Wavelet function
Scaling function
Mask
The Meyer wavelet
A linear method of summation
De la Vallee Poussin mean
Uncertainty constant

A B S T R A C T

In 1996 Chui and Wang proved that the uncertainty constants of scaling and wavelet functions tend to infinity as smoothness of the wavelets grows for a broad class of wavelets such as Daubechies wavelets and spline wavelets. We construct a class of new families of wavelets (quasispline wavelets) whose uncertainty constants tend to those of the Meyer wavelet function used in construction.

1. Introduction

One of the main advantages of wavelet systems is a good time-frequency localization. Smoothness of wavelets is also a useful and desired property. So

to construct orthonormal wavelets that preserve time-frequency localization as their orders of smoothness increase (1)

is a very attractive and interesting problem. In the sequel, by a wavelet we mean a function generating an orthonormal basis of $L_2(\mathbb{R})$ (see the definition in Section 2). The measure of the time-frequency localization is an uncertainty constant (see the definition in Section 2). So we are interested in wavelet families such that their uncertainty constants are bounded. It is well known that the main classical families of wavelets contain wavelet functions with arbitrarily large finite smoothness. Thus, one can investigate how a functional defined on a family of wavelets depends on smoothness of the wavelets. Let the functional be the uncertainty constant. Unfortunately, the main classical families of wavelets lose the time-frequency localization as smoothness of chosen wavelet function grows. More precisely, Chui and Wang in [1] show that the uncertainty constants of scaling and wavelet functions tend to infinity as smoothness of the wavelets grows for a broad class of wavelets such as, for example, Daubechies wavelets and spline wavelets. So Daubechies wavelets and spline wavelets don't settle (1).

Later Chui and Wang in [2] and Goodman and Lee in [3] construct families of nonorthogonal scaling functions and semi-orthogonal wavelet functions. These functions have an optimal uncertainty constants (in the sense of Heisenberg uncertainty principle) as smoothness parameter tends to infinity. But there is no information about orthogonal scaling and wavelet functions in [2] and [3].

☆ The work is supported by RFBR (grant 09-01-00162) and by grant MK-7413.2010.1.
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1063-5203/$ – see front matter
doi:10.1016/j.acha.2010.08.005

Trying to solve problem (1), Novikov in [4,5] constructs a family of modified Daubechies wavelets. The wavelet functions are compactly supported. The squared modulus of the modified Daubechies mask is the Bernstein polynomial interpolating a piecewise linear function (in the case of the classical Daubechies wavelet, the characteristic function is interpolated). Smoothness of the modified Daubechies wavelet grows as the order of the Bernstein polynomial increases. The time-frequency localization of the autocorrelation function constructed for the scaling function of this family is preserved with respect to the smoothness parameter. But whether the modified Daubechies scaling and wavelet functions preserve the time-frequency localization as smoothness grows is a still open question.

In [6], the author constructs a new wavelet family solving problem (1) for scaling functions. New scaling functions decay exponentially and their Fourier transforms decay as $O(\omega^{-l})$, like spline wavelets; the uncertainty constants of the scaling functions are uniformly bounded with respect to the smoothness parameter l. The construction is based on the de la Vallee Poussin means of a function closely connected with a Meyer mask.

In the present paper, we construct a wide class of such wavelets (see Theorem 1). A new wavelet function also decays exponentially at infinity and its Fourier transform decays as $O(\omega^{-l})$, like spline wavelet; that is why it is named a quasispline wavelet function (see Definition 1). The construction is based on a linear method of summation satisfying some weak, easily satisfied conditions (see Theorem 2). The wavelet system constructed in [6] is an example of the quasispline wavelets. It is proven that the quasispline wavelets solve problem (1) for scaling and wavelet functions. Moreover, since the uncertainty constant for Meyer scaling and wavelet functions is bounded, a stronger than the boundedness property for the quasispline wavelets is proven (see Theorem 1, item 3). Namely, we establish the convergence of the uncertainty constants defined for the new scaling (wavelet) functions to those of the Meyer scaling (wavelet) function used in construction with respect to the smoothness parameter l. It is well known that the Meyer scaling functions and wavelets decay faster then any polynomial $O(t^{-n})$, $t \to \infty$, $n \in \mathbb{N}$, but slower then exponent (see, for example, [7] the end of Section 5.4). So the spline wavelets and the quasispline wavelets are better then the Meyer ones in the time domain (but, of course, not in the frequency domain). Moreover, there is no infinitely smooth wavelet ($\psi \in C^\infty$) decaying like exponent [7, Corollary 5.5.3]. The above result (see Theorem 1, item 3) also means that the uncertainty constant is a continuous functional of a nonorthogonal mask m_l. It is necessary to note that the construction of quasispline wavelets can be based not only on the Meyer mask but also on any smooth orthogonal mask m such that $m(\omega)=1$ if $|\omega|<a$ and $m(\omega)=0$ if $b<|\omega|<\pi$ for some $\pi/3 \leqslant a < b \leqslant 2\pi/3$. Also we estimate the rate of the convergence.

2. Notations and auxiliary results

By $[x]$ denote an integer part of a real number x. By $C^k[a,b]$ denote the space of all k times continuously differentiable functions defined on the interval $[a,b]$. This is a Banach space with respect to the norm $\|f\|_{W^k_\infty} := \sum_{j=0}^k \max_{x\in[a,b]} |f^{(j)}(x)|$. Be definition, put $C^0[a,b]=C[a,b]$ and $C[-\pi,\pi]=C$.

We choose the Fourier transform and the reconstruction formula as

$$\hat{g}(\omega) := \int_{\mathbb{R}} g(t) e^{-it\omega}\, dt, \qquad g(t) := \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(\omega) e^{it\omega}\, d\omega$$

respectively. For the Fourier series $f \sim \frac{a_0}{2} + \sum_{n\in\mathbb{N}} a_n \cos n\omega + b_n \sin n\omega$ the sequence $(\lambda_{n,k})$, $k=1,\dots,n$, $n \in \mathbb{N}$ defines a linear method of summation

$$u_n(f,\omega) := \frac{a_0}{2} + \sum_{k=1}^n \lambda_{n,k}(a_k \cos k\omega + b_k \sin k\omega) = \int_{-\pi}^{\pi} f(x) U_n(x,\omega)\, dx,$$

where $U_n(x,\omega) := 1/2 + \sum_{k=1}^n \lambda_{n,k} \cos k(x-\omega)$ and the terms

$$a_n := \frac{1}{\pi} \int_{-\pi}^{\pi} f(\omega) \cos n\omega \, d\omega, \qquad b_n := \frac{1}{\pi} \int_{-\pi}^{\pi} f(\omega) \sin n\omega \, d\omega$$

are the Fourier coefficients. The following property holds true

$$u_n(f',\omega) = \big(u_n(f,\omega)\big)'_\omega. \tag{2}$$

A function ψ is called a wavelet function if the functions $2^{j/2}\psi(2^j\cdot -k)$, $j,k \in \mathbb{Z}$ form an orthonormal basis of $L_2(\mathbb{R})$.

By $\theta(\omega)$ denote an odd function equal to $\pi/4$ for $\omega > \pi/3$. We assume henceforth that $\theta(\omega)$ is a nondecreasing twice continuously differentiable function. By ω_0 denote a parameter such that $\pi/3 \leqslant \omega_0 < \pi/2$ and put $\omega_1 := \pi - \omega_0$. In the sequel, the notations of ω_0 and ω_1 will be frequently employed. A Meyer scaling function φ^M is defined by

$$\widehat{\varphi^M}(\omega) := \begin{cases} 1, & |\omega| \leqslant 2\omega_0, \\ \cos(\frac{\pi}{4} + \theta(\frac{\pi}{3(\pi-2\omega_0)}(|\omega| - \pi))), & 2\omega_0 < |\omega| \leqslant 2\pi - 2\omega_0, \\ 0, & |\omega| > 2\pi - 2\omega_0. \end{cases}$$

A Meyer mask is a 2π-periodic function defined on $[-\pi, \pi]$ as follows $m^M(\omega) := \widehat{\varphi^M}(2\omega)$. It is well known (see, for example [7]) that under the above restrictions on the function θ the constant of uncertainty for the Meyer scaling and wavelet function is bounded.

The uncertainty constant of f is the functional $\Delta_f \Delta_{\hat{f}}$ such that

$$\Delta_f^2 := \|f\|_{L^2(\mathbb{R})}^{-2} \int_{\mathbb{R}} (t - t_{0f})^2 \big|f(t)\big|^2 \, dt, \qquad \Delta_{\hat{f}}^2 := \|\hat{f}\|_{L^2(\mathbb{R})}^{-2} \int_{\mathbb{R}} (\omega - \omega_{0\hat{f}})^2 \big|\hat{f}(\omega)\big|^2 \, d\omega,$$

$$t_{0f} := \|f\|_{L^2(\mathbb{R})}^{-2} \int_{\mathbb{R}} t\big|f(t)\big|^2 \, dt, \qquad \omega_{0\hat{f}} := \|\hat{f}\|_{L^2(\mathbb{R})}^{-2} \int_{\mathbb{R}} \omega\big|\hat{f}(\omega)\big|^2 \, d\omega.$$

The terms Δ_f, $\Delta_{\hat{f}}$, t_{0f}, and $\omega_{0\hat{f}}$ are called a time radius, a frequency radius, a time centre, and a frequency centre of the function f respectively.

The numbers $\pm e^{i\bar{\omega}}$ are called a pair of symmetric roots of a mask m if $m(\bar{\omega}) = m(\bar{\omega} + \pi) = 0$. A set $B := \{b_1, \ldots, b_n\}$ of distinct complex numbers is called cyclic if $b_{j+1} = b_j^2$ for $j = 1, \ldots, n$ and $b_{n+1} = b_1$. A cyclic set B is called a cycle of a mask m if $m(\omega + \pi) = 0$ for all ω such that $\exp(i\omega) = b_j$ for some $j = 1, \ldots, n$. A trivial cycle is the set $\{1\}$. A mask is called pure if it has neither pairs of symmetric zeros nor cycles. The following result gives a necessary and sufficient condition for integer shifts $\varphi(\cdot + k)$, $k \in \mathbb{Z}$ of a scaling function φ to be stable (i.e., to form a Riesz basis).

Proposition 1. *(See [8, Corollary 3.4.15].) Integer shifts of a scaling function are stable (i.e., form a Riesz basis) iff its corresponding mask has neither pairs of symmetric zeros nor nontrivial cycles.*

The Hölder exponent α_f of a function f defined on some closed interval $[a, b]$ is

$$\alpha_f := k + \sup_{\beta \in \mathbb{R}} \big\{\beta \in \mathbb{R} \;\big|\; \big|f^{(k)}(x_1) - f^{(k)}(x_2)\big| \leqslant C_\beta |x_1 - x_2|^\beta, \; x_1, x_2 \in [a, b]\big\},$$

where $k := \max_{h \in \mathbb{Z}} \{h \mid f \in C^h[a, b]\}$. Another characteristic of smoothness of f is

$$\theta_{\hat{f}} := \sup_{\beta \in \mathbb{R}} \big\{\beta \in \mathbb{R} \;\big|\; \big|\hat{f}(\omega)\big| \leqslant C\big(|\omega| + 1\big)^{-\beta}\big\}.$$

Smoothness characteristics we introduced are known to satisfy the inequality $\theta_{\hat{f}} - 1 \leqslant \alpha_f \leqslant \theta_{\hat{f}}$. By $\theta(m)$ we mean $\theta_{\hat{\varphi}}$, where φ is the scaling function corresponding to the mask m. The following result can be used for finding $\theta(m)$.

Proposition 2. *(See [8, Lemma 7.4.2 and Proposition 7.4.4].) Suppose that some mask m is represented as $m(\omega) = (\cos\frac{\omega}{2})^{L+1} m_c(\omega)$, where m_c is a pure mask; then $\theta(m) = L + 1 + \theta(m_c)$ and $\theta(m_c) = \lim_{k\to\infty} \theta_k$, where*

$$\theta_k := -\frac{1}{k} \log_2 \big\|m_c(\omega) \cdots m_c\big(2^{k-1}\omega\big)\big\|_\infty. \tag{3}$$

3. Basic construction and conditions for a linear method of summation

Let us introduce a nonorthogonal mask of a new wavelet function. It is defined as the following 2π-periodic trigonometric polynomial

$$m_l(\omega) := \left(\cos\frac{\omega}{2}\right)^{2l} \frac{u_{n(l)}(m_l^M, \omega)}{u_{n(l)}(m_l^M, 0)}, \tag{4}$$

where

$$m_l^M(\omega) := \frac{m^M(\omega)}{(\cos\frac{\omega}{2})^{2l}}, \quad l \in \mathbb{N}, \tag{5}$$

m^M is a fixed Meyer mask, and the trigonometric polynomial $u_{n(l)}(m_l^M, \cdot)$ is defined by a fixed linear method of summation for the function m_l^M.

Since m_l is a trigonometric polynomial and $m_l(0) = 1$, we see that the infinite product $\prod_{j=1}^\infty m_l(\frac{\omega}{2^j})$ converges absolutely and uniformly on an arbitrary compact set. (If an infinite product is equal to zero, we assume that it converges.) Thus the function m_l is a mask for a stable, but not orthogonal scaling function φ_l, where the Fourier transform of φ_l is determined by the equality

$$\widehat{\varphi_l(\omega)} = \prod_{j=1}^{\infty} m_l\left(\frac{\omega}{2^j}\right). \tag{6}$$

The functions $\varphi_l(\cdot+k)$ for $k \in \mathbb{Z}$ form a Riesz basis in the closure of their linear span; this claim is a straight corollary of the subsequent Lemma 6 and Proposition 1. From the estimate (17) to be established later it follows that the orthogonalizing factor

$$\Phi_l(\omega) := \sum_{k\in\mathbb{Z}} \left|\widehat{\varphi}_l(\omega+2\pi k)\right|^2 \tag{7}$$

is well defined. Using the function Φ_l we define the Fourier transform of an orthogonal scaling function

$$\widehat{\varphi_l^{\perp}}(\omega) := \widehat{\varphi}_l(\omega)\Phi_l^{-0.5}(\omega), \tag{8}$$

an orthogonal mask

$$m_l^{\perp}(\omega) := m_l(\omega)\Phi_l^{0.5}(\omega)\Phi_l^{-0.5}(2\omega), \tag{9}$$

and, finally, the Fourier transform of a wavelet function

$$\widehat{\psi_l^{\perp}}(\omega) := e^{\frac{-i\omega}{2}}\overline{m_l^{\perp}}\left(\frac{\omega}{2}+\pi\right)\widehat{\varphi_l^{\perp}}\left(\frac{\omega}{2}\right). \tag{10}$$

Definition 1. By `a quasispline wavelet function` we mean the function $\psi_l^{\perp}$, where the Fourier transform $\widehat{\psi_l^{\perp}}$ is defined by (10) and a nonorthogonal mask is defined by (4). The functions $\varphi_l^{\perp}$, $m_l^{\perp}$, φ_l, m_l defined by (8), (9), (6), and (4) respectively are called `a quasispline scaling function`, `a quasispline mask`, `a nonorthogonal quasispline scaling function`, and `a nonorthogonal quasispline mask` respectively.

So for any fixed Meyer mask and for any fixed linear method of summation we get the sequence $(\psi_l^{\perp})_{l\in\mathbb{N}}$ of quasispline wavelet functions, and the symbol l is a smoothness parameter (see Theorem 4).

In the remaining part of the article the following main theorem will be proven.

Theorem 1. *Suppose that $\psi_l^{\perp}$ ($\varphi_l^{\perp}$) is a quasispline wavelet (scaling) function (see Definition 1). Let us introduce the following notations*

$$\begin{aligned}&\mu(l) := l\alpha(l)+\gamma(l), \qquad \varepsilon(l) := \alpha(l)/\left\|m_l^M\right\|_C, \qquad C_0 := 32\pi^2 e^{2\omega_0}/27,\\ &u_l := u_{n(l)}\left(m_l^M,\cdot\right), \qquad u_{1,l} := u_{n(l)}\left(\left(m_l^M\right)',\cdot\right), \qquad u_{0,l} := u_l/u_l(0), \qquad c := \inf_{l\geqslant l_0}\left|u_l(0)\right|,\end{aligned} \tag{11}$$

where $\alpha(l)$, $\gamma(l)$ are defined by (12) and (13) respectively in the subsequent Theorem 2, the parameter l_0 is defined in the proof of the subsequent Lemma 1, the term ω_0, $\pi/3 \leqslant \omega_0 < \pi/2$ is the parameter of the Meyer mask, the function m_l^M is defined by (5); then

1. *The functions $\varphi_l^{\perp}$ and $\psi_l^{\perp}$ decay exponentially at infinity (Theorem 5).*
2. *The functions $\widehat{\varphi_l^{\perp}}$ and $\widehat{\psi_l^{\perp}}$ decay as $O(\omega^{-l})$ at infinity, namely the Hölder exponents $\alpha_{\varphi_l^{\perp}}$ and $\alpha_{\psi_l^{\perp}}$ of the functions satisfy the inequalities*
$$2l-1+\log_2\left(\frac{c}{1+\varepsilon(l)}\right) \leqslant \alpha_{\varphi_l^{\perp}} \leqslant 2l, \qquad 2l-1+\log_2\left(\frac{c}{1+\varepsilon(l)}\right) \leqslant \alpha_{\psi_l^{\perp}} \leqslant 2l$$
for sufficiently large $l \in \mathbb{N}$ (Theorem 4).
3. *The uncertainty constants $\Delta^2_{\varphi_l^{\perp}}\Delta^2_{\widehat{\varphi_l^{\perp}}}$ ($\Delta^2_{\psi_l^{\perp}}\Delta^2_{\widehat{\psi_l^{\perp}}}$) of the quasispline scaling (wavelet) functions $\varphi_l^{\perp}$ ($\psi_l^{\perp}$) tend to those of the Meyer scaling (wavelet) function, namely*
$$\begin{aligned}&\left|\Delta^2_{\widehat{\varphi_l^{\perp}}} - \Delta^2_{\widehat{\varphi^M}}\right| = O\left(\max\left\{\mu(l), \left(4e^{2\omega_0}\right)^{-2l+4\log_2\frac{1+\varepsilon(l)}{c}}\right\}\right),\\ &\left|\Delta^2_{\varphi_l^{\perp}} - \Delta^2_{\varphi^M}\right| = O\left(\max\left\{\mu(l), lC_0^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}\right\}\right),\\ &\left|\Delta^2_{\widehat{\psi_l^{\perp}}} - \Delta^2_{\widehat{\psi^M}}\right| = O\left(\max\left\{\mu(l), lC_0^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}\right\}\right),\\ &\left|\Delta^2_{\psi_l^{\perp}} - \Delta^2_{\psi^M}\right| = O\left(\max\left\{\mu(l), \left(4e^{2\omega_0}\right)^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}\right\}\right)\end{aligned}$$
as $l \to \infty$ (Theorems 3, 6, and 7).

For the quasispline wavelet function $\psi_l^{\perp}$ to satisfy Theorem 1 and therefore to solve problem (1) it is sufficient to have the following three conditions for the polynomials u_l (see (11)).

Theorem 2. *Suppose that there exists a sequence $n(l)$ for $l \in \mathbb{N}$ such that*

$$\|u_l - m_l^M\|_C =: \alpha(l) = o(l^{-1}) \quad \text{as } l \to \infty, \tag{12}$$

$$\|u_{1,l} - (m_l^M)'\|_C =: \gamma(l) = o(1) \quad \text{as } l \to \infty, \tag{13}$$

$$u_{n(l)}(m_l^M, \pi) \neq 0, \tag{14}$$

where u_l and $u_{1,l}$ are defined by (11); *then the corresponding quasispline scaling* (8) *and wavelet* (10) *functions satisfy the conditions of Theorem* 1.

De la Vallee Poussin means satisfy these conditions (for the proof see [6, p. 460, p. 465, and p. 461] respectively). Conditions (12)–(14) define a very wide class of linear methods of summation. It follows from Proposition 1 in [6] that for a linear method of summation to satisfy the assumptions (12)–(14) it is sufficient to have $\|u_{n(l)}(f,\cdot) - f\|_C \leqslant A\omega(f,(n(l))^{-\alpha})$ for any $f \in C(-\pi,\pi)$, where $\alpha > 0$, A is an absolute constant, and $\omega(f,\cdot)$ is a modulus of continuity. The inequality is satisfied for many of the famous classical means defined for example by operators such as Fejer, Rogosinski, monotonous de la Vallee Poussin, Abel–Poisson operators.

4. Convergence of frequency radii for the scaling functions

Lemma 1. $\|m_l - m^M\|_C \leqslant K\alpha(l) = o(l^{-1})$ *as $l \to \infty$, where $K := \frac{\|u_l\|_C}{\inf_{k \geqslant l}|u_k(0)|} + 1$ is bounded.*

Proof. Combining (4) and (12), we get

$$\begin{aligned}
\|m_l - m^M\|_C &= \left\| (\cos(\omega/2))^{2l} \frac{u_l}{u_l(0)} - m^M \right\|_C \leqslant \left\| \frac{u_l}{u_l(0)} - m_l^M \right\|_C \\
&\leqslant \left\| \frac{u_l}{u_l(0)} - u_l \right\|_C + \|u_l - m_l^M\|_C \leqslant \frac{\|u_l\|_C}{\inf_{k \geqslant l}|u_k(0)|}\left|u_l(0) - 1\right| + \alpha(l) \\
&\leqslant \left(\frac{\|u_l\|_C}{\inf_{k \geqslant l}|u_k(0)|} + 1 \right)\alpha(l).
\end{aligned}$$

Since $u_l(0) \to m_l^M(0) = 1$ as $l \to \infty$, it follows that $\inf_{k \geqslant l_0}|u_k(0)| \geqslant c_0 > 0$ for some $l_0 \in \mathbb{N}$ and some positive constant c_0, therefore $\frac{\|u_l\|_C}{\inf_{k \geqslant l}|u_k(0)|}$ is bounded. □

From here, we suppose that $l \geqslant l_0$.

Lemma 2. $\|m_l' - (m^M)'\|_C = O(\mu(l))$ *as $l \to \infty$. The parameter $\mu(l)$ are defined by* (11).

Proof. Using Lemma 1, (2), and (13), we get

$$\begin{aligned}
&\left| \left(\left(\cos\frac{\omega}{2} \right)^{2l} u_l(\omega) \right)' - (m^M)'(\omega) \right| = \left| -l \left(\cos\frac{\omega}{2} \right)^{2l-1} \sin\frac{\omega}{2} u_l(\omega) + \left(\cos\frac{\omega}{2} \right)^{2l} u_l'(\omega) - (m^M)'(\omega) \right| \\
&\quad = \left| -l \left(\cos\frac{\omega}{2} \right)^{2l-1} \sin\frac{\omega}{2} \left(m_l^M(\omega) + u_l(\omega) - m_l^M(\omega) \right) \right. \\
&\qquad \left. + \left(\cos\frac{\omega}{2} \right)^{2l} \left((m_l^M)'(\omega) + u_{1,l}(\omega) - (m_l^M)'(\omega) \right) - (m^M)'(\omega) \right| \\
&\quad = \left| -l \tan\frac{\omega}{2} m^M(\omega) - l \left(\cos\frac{\omega}{2} \right)^{2l-1} \sin\frac{\omega}{2} \left(u_l(\omega) - m_l^M(\omega) \right) \right. \\
&\qquad + \left(\cos\frac{\omega}{2} \right)^{2l} \cdot \frac{(m^M)'(\omega)(\cos\frac{\omega}{2})^{2l} + l(\cos\frac{\omega}{2})^{2l-1}\sin\frac{\omega}{2} m^M(\omega)}{(\cos\frac{\omega}{2})^{4l}} \\
&\qquad \left. + \left(\cos\frac{\omega}{2} \right)^{2l} \left(u_{1,l}(\omega) - (m_l^M)'(\omega) \right) - (m^M)'(\omega) \right|
\end{aligned}$$

$$= \left| -l\left(\cos\frac{\omega}{2}\right)^{2l-1} \sin\frac{\omega}{2}\big(u_l(\omega) - m_l^M(\omega)\big) + \left(\cos\frac{\omega}{2}\right)^{2l}\big(u_{1,l}(\omega) - \big(m_l^M\big)'(\omega)\big)\right| = O\big(l\alpha(l) + \gamma(l)\big).$$

Hence for m_l, we have

$$\begin{aligned}\big|m_l'(\omega) - \big(m^M\big)'(\omega)\big| &= \left|\frac{((\cos\frac{\omega}{2})^{2l}u_l(\omega))'}{u_l(0)} - \big(m^M\big)'(\omega)\right| \\ &\leqslant \left|\left(\left(\cos\frac{\omega}{2}\right)^{2l}u_l(\omega)\right)'\right|\left|u_l^{-1}(0) - 1\right| + \left|\left(\left(\cos\frac{\omega}{2}\right)^{2l}u_l(\omega)\right)' - \big(m^M\big)'(\omega)\right| \\ &= \big(\big\|\big(m^M\big)'(\omega)\big\|_C + O\big(l\alpha(l) + \gamma(l)\big)\big)\frac{O(\alpha(l))}{c} + O\big(l\alpha(l) + \gamma(l)\big) \\ &= O\big(l\alpha(l) + \gamma(l)\big). \qquad \square\end{aligned}$$

Lemma 3. *$\|\widehat{\varphi_l} - \widehat{\varphi^M}\|_{C[a,b]} = O(\mu(l))$ as $l \to \infty$ for any $a < b$, $a, b \in \mathbb{R}$. The parameter $\mu(l)$ is defined by* (11)*.*

Proof. One can rewrite the proof of the lemma from [6, Lemma 1]. It is sufficient to change the notation ν_l by u_l and so on and to use the conditions (12), (13) instead of the property of the de la Vallee Poussin mean (see the formulas (4)–(7), (11), (12) in [6]). □

Lemma 4. *$\|\widehat{\varphi_l} - \widehat{\varphi^M}\|_{L^2(\mathbb{R})} = O(\max\{\mu(l), (4e^{2\omega_0})^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}\})$ as $l \to \infty$. The parameters are defined by* (11)*.*

Proof. We claim that there exists a function ξ such that $\xi \in L^2(\mathbb{R})$ and $|\widehat{\varphi_l}(\omega)| \leqslant \xi(\omega)$. The construction of the majorant can be rewritten with an inessential change of notation from [6, Lemma 2]. So let us write the results. By definition, put

$$\widehat{\varphi_{0,l}}(\omega) := \prod_{j=1}^{\infty}\frac{u_l(\omega 2^{-j})}{u_l(0)}. \tag{15}$$

Then under the assumption $|\omega| \geqslant 1$ we have

$$\big|\widehat{\varphi_{0,l}}(\omega)\big| \leqslant |\omega|^{-2\theta(u_{0,l})}e^{2\omega_0(l+O(\mu(l)))} \leqslant |\omega|^{2\log_2\frac{1+\varepsilon(l)}{c}}e^{2\omega_0(l+O(\mu(l)))}. \tag{16}$$

So $|\widehat{\varphi_l}(\omega)|$ are majorized by the functions

$$\big|\widehat{\varphi_l}(\omega)\big| \leqslant \xi_l(\omega) := \begin{cases} |\widehat{\varphi^M}(\omega)| + O(\mu(l)), & |\omega| \leqslant 4e^{2\omega_0}, \\ e^{O(\mu(l))}|\omega|^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}, & |\omega| > 4e^{2\omega_0}. \end{cases} \tag{17}$$

Thus the function ξ may be defined as

$$\xi(\omega) := \begin{cases} \nu_1, & |\omega| \leqslant 4e^{2\omega_0}, \\ \nu_2|\omega|^{-l_1+2\log_2\frac{1+\varepsilon(l)}{c}}, & |\omega| > 4e^{2\omega_0}, \end{cases}$$

where ν_1 and ν_2 are constants, $\nu_1, \nu_2 > 0$, $l_1 := \max\{l_0, 2\log_2\frac{1+\varepsilon(l)}{c} + 2\}$. Then the convergence follows from the Lebesgue dominated convergence theorem and Lemma 3.

Let us estimate the rate of the convergence. If $|\omega| \geqslant 4e^{2\omega_0}$, then $\widehat{\varphi^M}(\omega) = 0$, so

$$\begin{aligned}\|\widehat{\varphi_l} - \widehat{\varphi^M}\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}}\big|\widehat{\varphi_l}(\omega) - \widehat{\varphi^M}(\omega)\big|^2\,d\omega = \int_{|\omega|<4e^{2\omega_0}} + \int_{|\omega|\geqslant 4e^{2\omega_0}} \\ &\leqslant 8e^{2\omega_0}\big\|\widehat{\varphi_l} - \widehat{\varphi^M}\big\|^2_{C[-4e^{2\omega_0},4e^{2\omega_0}]} + e^{O(\mu(l))}\int_{|\omega|\geqslant 4e^{2\omega_0}}|\omega|^{-2l+4\log_2\frac{1+\varepsilon(l)}{c}}\,d\omega \\ &= 8e^{2\omega_0}\big\|\widehat{\varphi_l} - \widehat{\varphi^M}\big\|^2_{C[-4e^{2\omega_0},4e^{2\omega_0}]} + \frac{e^{O(\mu(l))}(4e^{2\omega_0})^{-2l+4\log_2\frac{1+\varepsilon(l)}{c}+1}}{2l - 4\log_2\frac{1+\varepsilon(l)}{c} - 1}.\end{aligned}$$

This completes the proof of Lemma 4. □

Remark 1. If we combine Lemmas 3 and 4, we get $\|\widehat{\varphi_l}-\widehat{\varphi^M}\|_{C(\mathbb{R})}=O(\max\{\mu(l),(4e^{2\omega_0})^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}\})$. Indeed, we have $\|\widehat{\varphi_l}-\widehat{\varphi^M}\|_{C(\mathbb{R})}=\max\{\sup_{|\omega|\leqslant 4e^{2\omega_0}}|\widehat{\varphi_l}(\omega)-\widehat{\varphi^M}(\omega)|,\ \sup_{|\omega|>4e^{2\omega_0}}|\widehat{\varphi_l}(\omega)-\widehat{\varphi^M}(\omega)|\}$. Using for the first item Lemma 3 and for the second one the definition of $\widehat{\varphi^M}$ and (17), we obtain $\sup_{|\omega|\leqslant 4e^{2\omega_0}}|\widehat{\varphi_l}(\omega)-\widehat{\varphi^M}(\omega)|=O(\mu(l))$ and $\sup_{|\omega|>4e^{2\omega_0}}|\widehat{\varphi_l}(\omega)-\widehat{\varphi^M}(\omega)|=\sup_{|\omega|>4e^{2\omega_0}}|\widehat{\varphi_l}(\omega)|=(4e^{2\omega_0})^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}$.

Lemma 5. *$\|\Phi_l-1\|_C=O(\max\{\mu(l),(4e^{2\omega_0})^{-2l+4\log_2\frac{1+\varepsilon(l)}{c}}\})$ as $l\to\infty$. The parameters are defined by* (11)*.*

Proof. Suppose $\omega\in[-\pi,\pi]$. Since φ^M is an orthogonal scaling function, we see that $\sum_{k\in\mathbb{Z}}|\widehat{\varphi^M}(\omega+2\pi k)|^2=1$. Taking into account (17), we define $k_0:=[2e^{2\omega_0}/\pi+1/2]$. Hence

$$\begin{aligned}\left|\Phi_l(\omega)-1\right|&=\left|\sum_{k\in\mathbb{Z}}\left|\widehat{\varphi_l}(\omega+2\pi k)\right|^2-\sum_{k\in\mathbb{Z}}\left|\widehat{\varphi^M}(\omega+2\pi k)\right|^2\right|\\&\leqslant\sum_{k\in\mathbb{Z}}\left|\left(\widehat{\varphi_l}(\omega+2\pi k)\right)^2-\left(\widehat{\varphi^M}(\omega+2\pi k)\right)^2\right|=\sum_{|k|\leqslant k_0}+\sum_{|k|>k_0}.\end{aligned}$$

Using Lemma 3, we get

$$\sum_{|k|\leqslant k_0}\leqslant(2k_0+1)\Big(\sup_{|\omega|\leqslant 4e^{2\omega_0}}\left|\widehat{\varphi_l}(\omega)-\widehat{\varphi^M}(\omega)\right|+2\sup_{|\omega|\leqslant 4e^{2\omega_0}}\left|\widehat{\varphi^M}(\omega)\right|\Big)\sup_{|\omega|\leqslant 4e^{2\omega_0}}\left|\widehat{\varphi_l}(\omega)-\widehat{\varphi^M}(\omega)\right|\leqslant O\big(\mu(l)\big).$$

Since $\widehat{\varphi^M}(\omega)=0$ as $|\omega|\geqslant 4e^{2\omega_0}$, using (17) and the definition of k_0, we obtain

$$\sum_{|k|>k_0}\leqslant\sum_{|k|>k_0}e^{O(\mu(l))}|\omega+2\pi k|^{-2l+4\log_2\frac{1+\varepsilon(l)}{c}}=O\Big(\big(4e^{2\omega_0}\big)^{-2l+4\log_2\frac{1+\varepsilon(l)}{c}}\Big).$$

Therefore,

$$\|\Phi_l-1\|_C=O\Big(\max\Big\{\mu(l),\big(4e^{2\omega_0}\big)^{-2l+4\log_2\frac{1+\varepsilon(l)}{c}}\Big\}\Big).\qquad\square$$

Now let us prove the convergence of the frequency radii for the scaling function.

Theorem 3. *$|\Delta^2_{\widehat{\varphi_l^\perp}}-\Delta^2_{\widehat{\varphi^M}}|=O(\max\{\mu(l),(4e^{2\omega_0})^{-2l+4\log_2\frac{1+\varepsilon(l)}{c}}\})$ as $l\to\infty$. The parameters are defined by* (11)*.*

Proof. Since the functions $\widehat{\varphi_l^\perp}$ and $\widehat{\varphi^M}$ are even, we see that $\omega_{0\widehat{\varphi_l^\perp}}=\omega_{0\widehat{\varphi^M}}=0$, where $\omega_{0\widehat{\varphi_l^\perp}}$, $\omega_{0\widehat{\varphi^M}}$ are the frequency centers (see Section 2).

Taking into account Lemma 3, the proof of Lemma 5 ($\widehat{\varphi^M}(\omega)=0$ as $|\omega|\geqslant 4e^{2\omega_0}$ and so on), and the estimate (17), we have

$$\begin{aligned}\left|\Delta^2_{\widehat{\varphi_l^\perp}}-\Delta^2_{\widehat{\varphi^M}}\right|&=\left|\int_{\mathbb{R}}\omega^2\Big(\big(\widehat{\varphi_l^\perp}\big)^2(\omega)-\big(\widehat{\varphi^M}\big)^2(\omega)\Big)\,d\omega\right|\\&\leqslant\int_{\mathbb{R}}\omega^2\left|\frac{(\widehat{\varphi_l})^2(\omega)}{\Phi_l(\omega)}-\big(\widehat{\varphi^M}\big)^2(\omega)\right|d\omega\leqslant\int_{|\omega|<4e^{2\omega_0}}+\int_{|\omega|\geqslant 4e^{2\omega_0}}\\&\leqslant 16e^{4\omega_0}\int_{|\omega|<4e^{2\omega_0}}\left((\widehat{\varphi_l})^2(\omega)\left|\frac{1}{\Phi_l(\omega)}-1\right|+\left|(\widehat{\varphi_l})^2(\omega)-\big(\widehat{\varphi^M}\big)^2(\omega)\right|\right)d\omega\\&\quad+\int_{|\omega|\geqslant 4e^{2\omega_0}}\omega^2(\widehat{\varphi_l})^2(\omega)\frac{1}{\Phi_l(\omega)}\,d\omega\leqslant 16e^{4\omega_0}\left(\|\Phi_l-1\|_C\int_{|\omega|<4e^{2\omega_0}}\frac{(\widehat{\varphi_l})^2(\omega)}{\Phi_l(\omega)}\,d\omega\right.\\&\quad\left.+\left\|\widehat{\varphi_l}-\widehat{\varphi^M}\right\|_{C[-4e^{2\omega_0},4e^{2\omega_0}]}\int_{|\omega|<4e^{2\omega_0}}\left|\widehat{\varphi_l}(\omega)+\widehat{\varphi^M}(\omega)\right|d\omega\right)\\&\quad+\frac{2e^{O(\mu(l))}(4e^{2\omega_0})^{-2l+4\log_2\frac{1+\varepsilon(l)}{c}+3}}{\inf_{\omega,l}\Phi_l(\omega)(2l-4\log_2\frac{1+\varepsilon(l)}{c}-3)}.\end{aligned}$$

From Lemmas 3 and 5 it follows that the integrals

$$\int\limits_{|\omega|<4e^{2\omega_0}} \frac{(\widehat{\varphi}_l)^2(\omega)}{\Phi_l(\omega)}\,d\omega, \qquad \int\limits_{|\omega|<4e^{2\omega_0}} \left|\widehat{\varphi}_l(\omega)+\widehat{\varphi^M}(\omega)\right|d\omega$$

are bounded. Hence

$$\begin{aligned}\left|\Delta^2_{\widehat{\varphi_l^\perp}}-\Delta^2_{\widehat{\varphi^M}}\right| &= O\left(\max\left\{\mu(l),\left(4e^{2\omega_0}\right)^{-2l+4\log_2\frac{1+\varepsilon(l)}{c}}\right\}\right)+O\left(\mu(l)\right)+O\left(\left(4e^{2\omega_0}\right)^{-2l+4\log_2\frac{1+\varepsilon(l)}{c}}\right)\\ &= O\left(\max\left\{\mu(l),\left(4e^{2\omega_0}\right)^{-2l+4\log_2\frac{1+\varepsilon(l)}{c}}\right\}\right). \qquad \square\end{aligned}$$

5. The growth of the smoothness and the exponential decaying

Lemma 6. *The polynomial $m_l = (\cos\frac{\cdot}{2})^{2l}u_{0,l}$ is a pure mask.*

Proof. Let us use Proposition 1. Recall that $u_{0,l}=u_l/u_l(0)$. Since $(\cos\omega/2)^{2l}=0$ iff $\omega=\pi+2\pi k$, $k\in\mathbb{Z}$, we apply Proposition 1 for the polynomial u_l. By the condition (12), $\sup_{[-\pi/2,\pi/2]}|u_l-m_l^M| = O(\alpha(l)) = o(l^{-1})$ as $l\to\infty$. Recall that $\pi/3\leqslant\omega_0<\pi/2$, where ω_0 is a parameter of the Meyer mask. Combining this with the definition (5), we get

$$\inf_{|\omega|\leqslant\omega_0}\left|m_l(\omega)\right| = \inf_{|\omega|\leqslant\omega_0}(\cos\omega/2)^{-2l}=1,$$

$$\inf_{\omega_0<|\omega|\leqslant\pi/2}\left|m_l(\omega)\right| = \inf_{\omega_0<|\omega|\leqslant\pi/2}\left|\cos\left(\frac{\pi}{4}+\theta\left(\frac{\pi(2|\omega|-\pi)}{3(\pi-2\omega_0)}\right)\right)\right| \geqslant \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right)^{-2l}.$$

Therefore for sufficiently large l, $u_l(\omega)\neq 0$ on the interval $[-\pi/2,\pi/2]$. Hence the polynomial u_l has no pair of symmetric zeros.

If $B:=\{b_1,\dots,b_n\}$ is a cyclic set and $b_1=re^{i\xi}$, then $r=1$, $\xi=\frac{2\pi k}{2^n-1}$. If B is a nontrivial cycle of the mask m_l, then any number $\pi+\frac{2\pi k}{2^n-1}$ is a root of u_l. But we have just proven that $u_l(\omega)\neq 0$ on the interval $\omega\in[-\pi/2,\pi/2]$. So the mask m_l has no a nontrivial cycle. Finally, the condition $u_l(\pi)\neq 0$ is postulated in (14). Therefore u_l has no a trivial cycle. $\square$

Using Lemma 6, one can apply Proposition 2 to estimate smoothness of the nonorthogonal quasispline scaling function φ_l.

Lemma 7. *The following inequality holds true $2l-1+\log_2(\frac{c}{1+\varepsilon(l)})\leqslant\alpha_{\varphi_l}\leqslant 2l$ for sufficiently large $l\geqslant l_0$. The parameters are defined by* (11)*.*

Proof. If we recall (12) and the notation $c=\inf_{l\geqslant l_0}u_l(0)$, then we get

$$\sup_\omega\left|u_{0,l}(\omega)\right| \leqslant \frac{\sup_\omega|u_l(\omega)|}{c} \leqslant \frac{(1+\varepsilon(l))\sup_\omega|m_l^M(\omega)|}{c} \leqslant \sup_\omega\left|f_{0,l}(\omega)\right|$$

for $\varepsilon(l):=\alpha(l)/\|m_l^M\|_C\to 0$ as $l\to\infty$, where $f_{0,l}$ is an even 2π-periodic function and $f_{0,l}(\omega):=(1+\varepsilon(l))(\cos\omega/2)^{-2l}/c$ for $0\leqslant\omega\leqslant\omega_1$ and $f_{0,l}(\omega):=0$ for $\omega_1<\omega\leqslant\pi$. So we get $\theta_k(u_{0,l})\geqslant\theta_k(f_{0,l})$.

The definition of $f_{0,l}$ yields

$$\left\|f_{0,l}(\omega)\cdots f_{0,l}\left(2^{k-1}\omega\right)\right\|_\infty = f_{0,l}(\omega_1)\cdots f_{0,l}\left(2^{-k+1}\omega_1\right) = \left(\cos\frac{\omega_1}{2}\cdots\cos\frac{\omega_1}{2^k}\right)^{-2l}\left(\frac{1+\varepsilon(l)}{c}\right)^k.$$

Then using Proposition 2, we have

$$\theta_k(f_{0,l}) = -\frac{1}{k}\log_2\left(\frac{1+\varepsilon(l)}{c}\right)^k - 2l\log_2\left|\cos\frac{\omega_1}{2}\cdots\cos\frac{\omega_1}{2^k}\right|^{-\frac{1}{k}} \to \log_2\left(\frac{c}{1+\varepsilon(l)}\right)$$

as $k\to\infty$. Passing to the limit, we use the identity $\prod_{j=1}^\infty\cos\frac{\omega}{2^j}=\frac{\sin\omega}{\omega}$. Therefore $\theta(u_{0,l})\geqslant\log_2(\frac{c}{1+\varepsilon(l)})$. For $u_{0,l}$, the multiplicity of the trivial cycle is equal to $2l$. Hence $2l-1+\log_2(\frac{c}{1+\varepsilon(l)})\leqslant\alpha_{\varphi_l}$. By the definition of the norm $\|\cdot\|_\infty$ we have $\|u_{0,l}(\omega)\dots u_{0,l}(2^{k-1}\omega)\|_\infty\geqslant u_{0,l}(0)\dots u_{0,l}(2^{k-1}\cdot 0)=1$. Therefore Proposition 2 yields $\theta_k(u_{0,l})\leqslant 0$, then $\theta(u_{0,l})\leqslant 0$, thus $\alpha_{\varphi_l}\leqslant 2l$. Finally, we obtain $2l-1+\log_2(\frac{c}{1+\varepsilon(l)})\leqslant\alpha_{\varphi_l}\leqslant 2l$. $\square$

Lemma 5 allows to estimate smoothness of the orthogonal scaling and wavelet functions.

Theorem 4. *The following inequalities hold true*

$$2l-1+\log_2\left(\frac{c}{1+\varepsilon(l)}\right)\leqslant\alpha_{\varphi_l^\perp}\leqslant 2l,\qquad 2l-1+\log_2\left(\frac{c}{1+\varepsilon(l)}\right)\leqslant\alpha_{\psi_l^\perp}\leqslant 2l$$

for sufficiently large $l\geqslant l_0$. The parameters are defined by (11)*.*

Proof. It is sufficient to prove $\theta_{\widehat{\varphi_l}}=\theta_{\widehat{\varphi_l^\perp}}=\theta_{\widehat{\psi_l^\perp}}$. Using Lemma 5, we get $0<c_1\leqslant\Phi_l(\omega)\leqslant c_2<\infty$. Therefore $c_2^{-0.5}|\widehat{\varphi_l}|\leqslant|\widehat{\varphi_l^\perp}|\leqslant c_1^{-0.5}|\widehat{\varphi_l}|$. Thus taking into account the definition of $\theta_{\widehat{f}}$, we get $\theta_{\widehat{\varphi_l}}=\theta_{\widehat{\varphi_l^\perp}}$.

Then the application of (10) yields

$$\left|\widehat{\psi_l^\perp}(\omega)\right|=\left|m_l\left(\frac{\omega}{2}+\pi\right)\Phi_l^{0.5}\left(\frac{\omega}{2}+\pi\right)\Phi_l^{-0.5}(\omega+2\pi)\widehat{\varphi_l}\left(\frac{\omega}{2}\right)\Phi_l^{-0.5}\left(\frac{\omega}{2}\right)\right|.$$

There exists an arbitrary large ω (for example, $\omega\in[-2\omega_0+2\pi(2k-1),2\omega_0+2\pi(2k-1)]$, $k\in\mathbb{Z}$) such that $1-\alpha(l)\leqslant m_l(\omega/2+\pi)\leqslant 1+\alpha(l)$. Therefore for such ω we have $(1-\alpha(l))c_1^{0.5}c_2^{-1}|\widehat{\varphi_l}(\omega/2)|\leqslant|\widehat{\psi_l^\perp}(\omega)|\leqslant(1+\alpha(l))c_2^{0.5}c_1^{-1}|\widehat{\varphi_l}(\omega/2)|$. Finally, again taking into account the definition of $\theta_{\widehat{f}}$, we get $\theta_{\widehat{\varphi_l}}=\theta_{\widehat{\psi_l^\perp}}$. □

Lemma 5 also allows to deduce exponential decay of the orthogonal scaling function $\varphi_l^\perp$ and the wavelet function $\psi_l^\perp$.

Theorem 5. *The functions $\varphi_l^\perp$ and $\psi_l^\perp$ decay exponentially at infinity.*

Proof. Let us fix sufficiently large $l\in\mathbb{N}$. Consider the function $\Phi_l^{-0.5}(\omega)$. Let a_k, $k\in\mathbb{Z}$ be its Fourier coefficients. First, we claim that $a_k=O(e^{-\beta_1|k|})$, $\beta_1>0$, $k\in\mathbb{Z}$. Indeed, since m_l is a trigonometric polynomial, we see that φ_l is compactly supported. Therefore its orthogonalizing factor $\Phi_l(\omega)$ is a trigonometric polynomial too. By (7) and Lemma 5, $\Phi_l>A>0$ for some absolute constant A, so that $\Phi_l(\omega)\neq 0$ for the band $|\operatorname{Im}\omega|<\beta$, $\beta>0$. Using the substitution $z=e^{i\omega}$, we deduce that the Laurent series $\sum_{k\in\mathbb{Z}}a_kz^k$ converges for z such that $e^{-\beta}<|z|<e^{\beta}$. Taking $0<\beta_1<\beta$ and applying the Cauchy inequality, we get $|h_k|\leqslant M(\beta_1)e^{-\beta_1|k|}\leqslant Me^{-\beta_1|k|}$, where $M(\beta_1):=\max_{|z|=e^{\beta_1}}|\sum_{k\in\mathbb{Z}}a_kz^k|$. Since $\sum_{k\in\mathbb{Z}}a_kz^k$ is an analytical function on the ring $e^{-\beta}<|z|<e^{\beta}$, we obtain $M(\beta_1)\leqslant M$ for some absolute constant M. Thus we have $a_k=O(e^{-\beta_1|k|})$, $\beta_1>0$.

Now, by (8), $\varphi_l^\perp(t)=\sum_{k\in\mathbb{Z}}a_k\varphi_l(t-k)$. Since φ_l is compactly supported, it follows that $\varphi_l^\perp(t)=O(e^{-\beta_1|t|})$, $t\to\infty$, $\beta_1>0$.

Then by (10), it follows that $\psi_l^\perp(t)=\sum_{k\in\mathbb{Z}}(-1)^kh_{-k+1}\varphi_l^\perp(2t-k)$, where h_k are the Fourier coefficients of the function $m_l^\perp$. In the same way as for a_k, one can show that $h_k=O(e^{-\beta_2|k|})$, $\beta_2>0$. Therefore, we have

$$\left|\psi_l^\perp(t)\right|=\left|\sum_{k\in\mathbb{Z}}(-1)^kh_{-k+1}\varphi_l^\perp(2t-k)\right|\leqslant\sum_{k\in\mathbb{Z}}\left|h_{-k+1}\varphi_l^\perp(2t-k)\right|\leqslant A\sum_{k\in\mathbb{Z}}e^{-\beta_1|2t-k|-\beta_1|-k+1|},$$

where A is a constant. The application of the property of modulus and geometric series yields

$$\sum_{k\in\mathbb{Z}}e^{-\beta_1|2t-k|-\beta_2|-k+1|}=\frac{e^{\pm\beta_1-2\beta_1|t|}}{1-e^{-\beta_2-\beta_1}}+\frac{e^{\pm\beta_1}}{e^{\beta_1-\beta_2}-1}\left(e^{\beta_1|-2t+[2t]|-\beta_2|[2t]|}-e^{-2\beta_1|t|}\right)+\frac{e^{\kappa+\beta_1|2t-[2t]|-\beta_2|[2t]|}}{1-e^{-\beta_2-\beta_1}},$$

where $\kappa=-\beta_1$ as $t\geqslant 0$ and $\kappa=-\beta_2$ as $t<0$. Therefore $\psi_l^\perp=O(e^{-\max\{\beta_2,\beta_1\}|2t|})$. □

6. Convergence of time radii for the scaling functions

First, let us establish an auxiliary technical result.

Lemma 8. *For any a,b such that $-\infty<a<b<\infty$, we have*

$$\left\|\left(\prod_{j=1}^{\infty}m_l\left(\omega 2^{-j}\right)\right)'-\left(\prod_{j=1}^{n}m_l\left(\omega 2^{-j}\right)\right)'\right\|_{C[a,b]}\longrightarrow 0$$

as $n\to\infty$, where $(\prod_{j=1}^{\infty}m_l(\omega 2^{-j}))'$ is the notation for the series $\sum_{j_0=1}^{\infty}2^{-j_0}m_l'(\omega 2^{-j_0})\prod_{j=1,j\neq j_0}^{\infty}m_l(\omega 2^{-j})$.

Proof. Using the introduced notation we have

$$\left|\left(\prod_{j=1}^{\infty} m_l\left(\frac{\omega}{2^j}\right)\right)' - \left(\prod_{j=1}^{n} m_l\left(\frac{\omega}{2^j}\right)\right)'\right|$$
$$= \left|\sum_{j_0=1}^{\infty} 2^{-j_0} m_l'\left(\frac{\omega}{2^{j_0}}\right) \prod_{j=1, j\neq j_0}^{\infty} m_l\left(\frac{\omega}{2^j}\right) - \sum_{j_0=1}^{n} 2^{-j_0} m_l'\left(\frac{\omega}{2^{j_0}}\right) \prod_{j=1, j\neq j_0}^{n} m_l\left(\frac{\omega}{2^j}\right)\right|$$
$$\leqslant \left|\sum_{j_0=n+1}^{\infty} 2^{-j_0} m_l'\left(\frac{\omega}{2^{j_0}}\right) \prod_{j=1, j\neq j_0}^{\infty} m_l\left(\frac{\omega}{2^j}\right)\right| + \left|\sum_{j_0=1}^{n} 2^{-j_0} m_l'\left(\frac{\omega}{2^{j_0}}\right)\left(\prod_{j=1, j\neq j_0}^{\infty} m_l\left(\frac{\omega}{2^j}\right) - \prod_{j=1, j\neq j_0}^{n} m_l\left(\frac{\omega}{2^j}\right)\right)\right|$$
$$=: J_{1,n}(\omega) + J_{2,n}(\omega).$$

The application of the Lagrange Theorem and Lemma 2 yields $|m_l(\omega)| \leqslant 1 + A|\omega|$, where A is a constant. Hence

$$\left|\prod_{j=1}^{j_0-1} m_l\left(\frac{\omega}{2^j}\right)\right| \leqslant \prod_{j=1}^{j_0-1}\left(1 + A|\omega|2^{-j}\right) = e^{\sum_{j=1}^{j_0-1}\ln(2^j+A|\omega|)-\ln 2^j} \leqslant e^{A|\omega|\sum_{j=1}^{j_0-1}\frac{1}{2^j}} \leqslant e^{A|\omega|}.$$

So using additionally Lemmas 2 and 3 for the first sum, we get

$$J_{1,n}(\omega) \leqslant \sum_{j_0=n+1}^{\infty} 2^{-j_0}\left|m_l'\left(\frac{\omega}{2^j}\right)\prod_{j=1}^{j_0-1} m_l\left(\frac{\omega}{2^j}\right)\widehat{\varphi}_l\left(\frac{\omega}{2^{j_0}}\right)\right|$$
$$\leqslant \left(\left\|\left(m^M\right)'\right\|_C + \left\|m_l' - \left(m^M\right)'\right\|_C\right)e^{A|\omega|}\left(\left\|\widehat{\varphi^M}\right\|_{C(\mathbb{R})} + \left\|\widehat{\varphi^M} - \widehat{\varphi}_l\right\|_{C(\mathbb{R})}\right)2^{-n},$$

where all factors are bounded as $a \leqslant \omega \leqslant b$, $l \in \mathbb{N}$. Thus $J_{1,n}(\omega) \to 0$ as $n \to \infty$.

For the second sum $J_{2,n}(\omega)$, we obtain

$$J_{2,n}(\omega) = \left|\sum_{j_0=1}^{n} 2^{-j_0} m_l'\left(\frac{\omega}{2^{j_0}}\right)\prod_{j=1, j\neq j_0}^{n} m_l\left(\frac{\omega}{2^j}\right)\left(\widehat{\varphi}_l\left(\frac{\omega}{2^n}\right) - 1\right)\right|.$$

Since the function $\widehat{\varphi}_l$ is continuous, $\widehat{\varphi}_l(0) = 1$, and $a \leqslant |\omega| \leqslant b$, it follows that $|\widehat{\varphi}_l(\frac{\omega}{2^n}) - 1| = \varepsilon_1(n)$, $\varepsilon_1(n) \to 0$ as $n \to \infty$. So we get

$$J_{2,n}(\omega) \leqslant \left(1 - \frac{1}{2^{n+1}}\right)\left(\left\|\left(m^M\right)'\right\|_C + \left\|m_l' - \left(m^M\right)'\right\|_C\right)e^{A|\omega|}\varepsilon_1(n),$$

where all factors are bounded as $a \leqslant \omega \leqslant b$, $l \in \mathbb{N}$. Thus $J_{1,n}(\omega) \to 0$ as $n \to \infty$. □

Lemma 9. *$\|\widehat{\varphi}_l{}' - \widehat{\varphi^M}{}'\|_{C[a,b]} = O(\mu(l))$ as $l \to \infty$, where $-\infty < a < b < \infty$. The parameter $\mu(l)$ is defined by* (11)*.*

Proof. Using the definition of $\widehat{\varphi}_l$ and Lemma 8, we get

$$\left|\widehat{\varphi}_l{}'(\omega) - \widehat{\varphi^M}{}'(\omega)\right| = \left|\left(\prod_{j=1}^{\infty} m_l\left(\frac{\omega}{2^j}\right)\right)' - \left(\prod_{j=1}^{\infty} m^M\left(\frac{\omega}{2^j}\right)\right)'\right|$$
$$= \left|\sum_{j_0=1}^{\infty} 2^{-j_0}\left(m_l'\left(\frac{\omega}{2^{j_0}}\right)\prod_{j=1, j\neq j_0}^{\infty} m_l\left(\frac{\omega}{2^j}\right) - \left(m^M\right)'\left(\frac{\omega}{2^{j_0}}\right)\prod_{j=1, j\neq j_0}^{\infty} m^M\left(\frac{\omega}{2^j}\right)\right)\right|$$
$$\leqslant \sum_{j_0=1}^{\infty} 2^{-j_0}\left(\left|m_l'\left(\frac{\omega}{2^{j_0}}\right) - \left(m^M\right)'\left(\frac{\omega}{2^{j_0}}\right)\right|\prod_{j=1, j\neq j_0}^{\infty}\left|m^M\left(\frac{\omega}{2^j}\right)\right|\right.$$
$$\left. + \left|m_l'\left(\frac{\omega}{2^{j_0}}\right)\right|\left|\prod_{j=1, j\neq j_0}^{\infty} m_l\left(\frac{\omega}{2^j}\right) - \prod_{j=1, j\neq j_0}^{\infty} m^M\left(\frac{\omega}{2^j}\right)\right|\right).$$

From Lemma 2 it follows that $|m_l'(\frac{\omega}{2^{j_0}}) - (m^M)'(\frac{\omega}{2^{j_0}})| = O(\mu(l))$ and $|m_l'(\frac{\omega}{2^{j_0}})| = \overline{M} + O(\mu(l))$, where $\overline{M} := \|(m^M)'\|_C$. Since $|m^M| \leqslant 1$, we have $|\prod_{j=1, j\neq j_0}^{\infty} m^M(\frac{\omega}{2^j})| \leqslant 1$.

Taking into account Lemma 1 and the definition of $\widehat{\varphi_l}$ (6), we obtain

$$\left|\prod_{j=1, j\neq j_0}^{\infty} m_l\left(\frac{\omega}{2^j}\right) - \prod_{j=1, j\neq j_0}^{\infty} m^M\left(\frac{\omega}{2^j}\right)\right| \leqslant \left|\prod_{j=1}^{j_0-1} m_l\left(\frac{\omega}{2^j}\right) - \prod_{j=1}^{j_0-1} m^M\left(\frac{\omega}{2^j}\right)\right|\left|\widehat{\varphi_l}\left(\frac{\omega}{2^{j_0}}\right)\right| + \left|\prod_{j=1}^{j_0-1} m^M\left(\frac{\omega}{2^j}\right)\right|\left|\widehat{\varphi_l}\left(\frac{\omega}{2^{j_0}}\right) - \widehat{\varphi^M}\left(\frac{\omega}{2^{j_0}}\right)\right|.$$

Using (12) and the property of the Meyer mask $m^M \leqslant 1$, we get

$$\begin{aligned}
&\left|\prod_{j=1}^{j_0-1} m_l\left(\frac{\omega}{2^j}\right) - \prod_{j=1}^{j_0-1} m^M\left(\frac{\omega}{2^j}\right)\right| \\
&\quad \leqslant \left|m_l\left(\frac{\omega}{2}\right) - m^M\left(\frac{\omega}{2}\right)\right|\prod_{j=2}^{j_0-1} m^M\left(\frac{\omega}{2^j}\right) + \left|m_l\left(\frac{\omega}{2}\right)\right|\left|\prod_{j=2}^{j_0-1} m_l\left(\frac{\omega}{2^j}\right) - \prod_{j=2}^{j_0-1} m^M\left(\frac{\omega}{2^j}\right)\right| \\
&\quad \leqslant \|m_l - m^M\|_C + \left(1 + \|m_l - m^M\|_C\right)\left|\prod_{j=2}^{j_0-1} m_l\left(\frac{\omega}{2^j}\right) - \prod_{j=2}^{j_0-1} m^M\left(\frac{\omega}{2^j}\right)\right|.
\end{aligned}$$

Reiterating the procedure $j_0 - 2$ times, we obtain

$$\left|\prod_{j=1}^{j_0-1} m_l\left(\frac{\omega}{2^j}\right) - \prod_{j=1}^{j_0-1} m^M\left(\frac{\omega}{2^j}\right)\right| = \left(1 + O\left(\alpha(l)\right)\right)^{j_0-1} - 1.$$

From Lemma 3 and the definition of the Meyer scaling function it follows that $|\widehat{\varphi_l}(\frac{\omega}{2^{j_0}})| = 1 + O(\mu(l))$ and $|\widehat{\varphi_l}(\frac{\omega}{2^{j_0}}) - \widehat{\varphi^M}(\frac{\omega}{2^{j_0}})| = O(\mu(l))$. Finally, we note that $|m^M| \leqslant 1$, therefore $|\prod_{j=1}^{j_0-1} m^M(\frac{\omega}{2^j})| \leqslant 1$.

Combining all the estimates together, we obtain

$$\begin{aligned}
\left|\widehat{\varphi_l}'(\omega) - \widehat{\varphi^M}'(\omega)\right| &\leqslant O\left(\mu(l)\right)\sum_{j_0=1}^{\infty} 2^{-j_0} + \left(\overline{M} + O\left(\mu(l)\right)\right) \\
&\quad \times \left(O\left(\mu(l)\right)\sum_{j_0=1}^{\infty} 2^{-j_0} + \left(1 + O\left(\mu(l)\right)\right)\sum_{j_0=1}^{\infty} 2^{-j_0}\left(\left(1 + O\left(\alpha(l)\right)\right)^{j_0-1} - 1\right)\right) \\
&= O\left(\mu(l)\right) + \frac{O(\alpha(l))}{1 - O(\alpha(l))} = O\left(\mu(l)\right).
\end{aligned}$$

The next to the last equality follows from the identity

$$\sum_{j_0=1}^{\infty} 2^{-j_0}\left(\left(1 + O\left(\alpha(l)\right)\right)^{j_0-1} - 1\right) = \sum_{j_0=1}^{\infty}\left(\frac{1}{2}\left(\frac{1 + O(\alpha(l))}{2}\right)^{j_0-1} - \frac{1}{2^{j_0}}\right) = \frac{O(\alpha(l))}{1 - O(\alpha(l))}. \qquad \square$$

Lemma 10. *$\|\widehat{\varphi_l}' - \widehat{\varphi^M}'\|_{L_2(\mathbb{R})} = O(\max\{\mu(l), l^{0.5}C_0^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}\})$ as $l \to \infty$. The parameters are defined by* (11).

Proof. We prove the Lemma in a similar manner as Lemma 4. Let us find a majorant $\xi_1 \in L_2(\mathbb{R})$ for the function $\widehat{\varphi_l}'$. From the definition of $\widehat{\varphi_l}$, Lemma 8, (15), and the identity $\sum_{j_0=1}^{\infty} 2^{-j_0}\tan\frac{\omega}{2^{j_0+1}} = \frac{2}{\omega} - \cot\frac{\omega}{2}$ it follows that

$$\begin{aligned}
(\widehat{\varphi_l})'(\omega) &= \left(\prod_{j=1}^{\infty} m_l\left(\frac{\omega}{2^j}\right)\right)' = \sum_{j_0=1}^{\infty} 2^{-j_0} m_l'\left(\frac{\omega}{2^{j_0}}\right)\prod_{j=1, j\neq j_0}^{\infty} m_l\left(\frac{\omega}{2^j}\right) \\
&= \sum_{j_0=1}^{\infty} 2^{-j_0}\left(l\left(\cos\frac{\omega}{2^{j_0+1}}\right)^{2l-1}\left(-\sin\frac{\omega}{2^{j_0+1}}\right)\frac{u_l(\frac{\omega}{2^{j_0}})}{u_l(0)}\prod_{j=1, j\neq j_0}^{\infty}\left(\cos\frac{\omega}{2^{j+1}}\right)^{2l}\frac{u_l(\frac{\omega}{2^j})}{u_l(0)}\right. \\
&\quad \left. + \left(\cos\frac{\omega}{2^{j_0+1}}\right)^{2l}\frac{u_{1,l}(\frac{\omega}{2^{j_0}})}{u_l(0)}\prod_{j=1, j\neq j_0}^{\infty}\left(\cos\frac{\omega}{2^{j+1}}\right)^{2l}\frac{u_l(\frac{\omega}{2^j})}{u_l(0)}\right)
\end{aligned}$$

$$= \sum_{j_0=1}^{\infty} 2^{-j_0} l\left(-\tan\frac{\omega}{2^{j_0+1}}\right)\prod_{j=1}^{\infty}\left(\cos\frac{\omega}{2^{j+1}}\right)^{2l}\prod_{j=1}^{\infty}\frac{u_l(\frac{\omega}{2^j})}{u_l(0)}$$

$$+ \sum_{j_0=1}^{\infty} 2^{-j_0}\frac{u_{1,l}(\frac{\omega}{2^{j_0}})}{u_l(0)}\prod_{j=1}^{\infty}\left(\cos\frac{\omega}{2^{j+1}}\right)^{2l}\prod_{j=1, j\neq j_0}^{\infty}\frac{u_l(\frac{\omega}{2^j})}{u_l(0)}$$

$$= l\left(\cot\frac{\omega}{2}-\frac{2}{\omega}\right)\left(\frac{\sin\omega/2}{\omega/2}\right)^{2l}\widehat{\varphi_{l,0}}(\omega) + \left(\frac{\sin\omega/2}{\omega/2}\right)^{2l}\sum_{j_0=1}^{\infty}2^{-j_0}\frac{u_{1,l}(\frac{\omega}{2^{j_0}})}{u_l(0)}\prod_{j=1, j\neq j_0}^{\infty}\frac{u_l(\frac{\omega}{2^j})}{u_l(0)}$$

$$=: I_{1,l}(\omega) + \left(\frac{\sin\omega/2}{\omega/2}\right)^{2l} I_{2,l}(\omega).$$

If $|\omega| > 4e^{2\omega_0}$, then applying (16) and (17) for the first item, we have

$$\left|I_{1,l}(\omega)\right| = \left|l\left(\cos\frac{\omega}{2}-\frac{2\sin\omega/2}{\omega}\right)\left(\frac{2}{\omega}\right)^{2l}(\sin\omega/2)^{2l-1}\widehat{\varphi_{l,0}}(\omega)\right| \leqslant Cle^{O(\mu(l))}|\omega|^{-l+2\log_2\frac{1+\varepsilon(l)}{2}}.$$

Let us estimate the second item

$$I_{2,l}(\omega) = \sum_{j_0=1}^{\infty}2^{-j_0}\frac{u_{1,l}(\frac{\omega}{2^{j_0}})}{u_l(0)}\prod_{j=1}^{j_0-1}\frac{u_l(\frac{\omega}{2^j})}{u_l(0)}\widehat{\varphi_{l,0}}\left(\frac{\omega}{2^{j_0}}\right).$$

Using (16) for $|\omega| > 4e^{2\omega_0}$, we get $|\widehat{\varphi_{l,0}}(\frac{\omega}{2^{j_0}})| \leqslant |\omega 2^{-j_0}|^{-2\theta(u_{0,l})}e^{2\omega_0(l+O(\mu(l)))}$. Using the condition (13), the definition of the function m_l^M (5), and the inequality $\pi/2 < \omega_1 \leqslant 2\pi/3$, we obtain

$$\left|u_{1,l}\left(\frac{\omega}{2^{j_0}}\right)\right| \leqslant \left|\left(m_l^M\right)'\left(\frac{\omega}{2^{j_0}}\right)\right| + O\big(\gamma(l)\big)$$

$$\leqslant \left(m^M\right)'\left(\frac{\omega}{2^{j_0}}\right)\left(\cos\frac{\omega_1}{2^{j_0+1}}\right)^{-2l} + lm^M\left(\frac{\omega}{2^{j_0+1}}\right)\sin\frac{\omega_1}{2^{j_0+1}}\left(\cos\frac{\omega_1}{2^{j_0+1}}\right)^{-2l-1}$$

$$+ O\big(\gamma(l)\big) \leqslant \left(\cos\frac{\omega_1}{2^{j_0+1}}\right)^{-2l}\left(\overline{M} + l\tan\frac{\omega_1}{2^{j_0+1}}\right) + O\big(\gamma(l)\big)$$

$$\leqslant (4/3)^l(\overline{M}+l\sqrt{3}) + O\big(\gamma(l)\big).$$

Then taking into account condition (12), the properties of the Meyer mask $|m^M| \leqslant 1$, $m^M(\omega) = 0$ as $\omega_1 \leqslant |\omega| \leqslant \pi$, and the inequality $\pi/2 < \omega_1 \leqslant 2\pi/3$, we have

$$\left|\prod_{j=1}^{j_0-1}\frac{u_l(\frac{\omega}{2^j})}{u_l(0)}\right| \leqslant \prod_{j=1}^{j_0-1}\left(\frac{m^M(\omega 2^{-j})}{(\cos\omega 2^{-j-1})^{2l}} + \alpha(l)\right) \leqslant \prod_{j=1}^{j_0-1}\left(\frac{1}{(\cos\omega_1 2^{-j-1})^{2l}} + \alpha(l)\right)$$

$$\leqslant \prod_{j=1}^{j_0-1}\frac{a}{(\cos\omega_1 2^{-j-1})^{2l}} = a^{j_0-1}\prod_{j=1}^{\infty}\left(\cos\omega_1 2^{-j-1}\right)^{-2l}$$

$$= a^{j_0-1}\left(\frac{\omega_1/2}{\sin\omega_1/2}\right)^{2l} \leqslant a^{j_0-1}\left(\frac{\omega_1}{\sqrt{2}}\right)^{2l} \leqslant a^{j_0-1}\left(\frac{2\pi}{3\sqrt{2}}\right)^{2l},$$

where a is a majorant of the expression $1+\alpha(l)(\cos\omega_1 2^{-j-1})^{2l}$, so it can be chosen $a < 1.5$.

Collecting the estimates, we obtain

$$I_{2,l}(\omega) \leqslant \sum_{j_0=1}^{\infty}2^{-j_0}\frac{(\frac{4}{3})^l(\overline{M}+l\sqrt{3}) + O(\gamma(l))}{1-\alpha(l)}a^{j_0-1}\left(\frac{2\pi}{3\sqrt{2}}\right)^{2l}\left|\frac{\omega}{2^{j_0}}\right|^{-2\theta(u_{0,l})}e^{2\omega_0(l+O(\mu(l)))}.$$

Since $\log_2(\frac{c}{1+\varepsilon(l)}) \leqslant \theta(u_{0,l}) \leqslant 0$ and $a < 1.5$, we get $|\omega|^{-2\theta(u_{0,l})} \leqslant |\omega|^{2\log_2\frac{1+\varepsilon(l)}{c}}$ as $|\omega| \geqslant 1$, $2^{j_0\theta(u_{0,l})} \leqslant 1$, and $\sum_{j_0=1}^{\infty}2^{-j_0}a^{j_0-1} = (2-a)^{-1}$. So

$$I_{2,l}(\omega) \leqslant \frac{e^{O(\mu(l))}(\overline{M}+l\sqrt{3}+(3/4)^l O(\gamma(l)))}{(1-\alpha(l))(2-a)}\left(\frac{8\pi^2 e^{2\omega_0}}{27}\right)^l|\omega|^{2\log_2\frac{1+\varepsilon(l)}{c}}.$$

Thus we have for $|\omega| > \frac{32\pi^2 e^{2\omega_0}}{27}$

$$\left(\frac{\sin\omega/2}{\omega/2}\right)^{2l} I_{2,l}(\omega) \leqslant (\sin\omega/2)^{2l} \frac{e^{O(\mu(l))}(\overline{M} + l\sqrt{3} + (3/4)^l O(\gamma(l)))}{(1-\alpha(l))(2-a)} \left(\frac{32\pi^2 e^{2\omega_0}}{27}\right)^l |\omega|^{-2l+2\log_2 \frac{1+\varepsilon(l)}{c}}$$
$$\leqslant C(l,\omega_0) l |\omega|^{-l+2\log_2 \frac{1+\varepsilon(l)}{c}},$$

where $C(l,\omega_0) := e^{O(\mu(l))}(\overline{M}/l + \sqrt{3} + l^{-1}(3/4)^l O(\gamma(l)))(1-\alpha(l))^{-1}(2-a)^{-1}$ is bounded with respect to the parameters l and ω_0. Put $C(l,\omega_0) \leqslant A$, A is a constant.

So if $|\omega| > C_0 := \frac{32\pi^2 e^{2\omega_0}}{27}$, we have the following estimate $|(\widehat{\varphi_l})'(\omega)| \leqslant Al|\omega|^{-l+2\log_2 \frac{1+\varepsilon(l)}{c}}$.

Finally, using Lemma 9 one can define the functions $\xi_{1,l}$ such that

$$\left|(\widehat{\varphi_l})'(\omega)\right| \leqslant \xi_{1,l}(\omega) := \begin{cases} (\widehat{\varphi^M})'(\omega) + O(\mu(l)), & |\omega| \leqslant \frac{32\pi^2 e^{2\omega_0}}{27}, \\ Al|\omega|^{-l+2\log_2 \frac{1+\varepsilon(l)}{c}}, & |\omega| \geqslant \frac{32\pi^2 e^{2\omega_0}}{27}. \end{cases} \tag{18}$$

So the majorant ξ_1 is defined in the following way

$$\xi_1(\omega) := \begin{cases} \nu_1', & |\omega| \leqslant \frac{32\pi^2 e^{2\omega_0}}{27}, \\ \nu_2' l|\omega|^{-l_1+2\log_2 \frac{1+\varepsilon(l)}{c}}, & |\omega| \geqslant \frac{32\pi^2 e^{2\omega_0}}{27}, \end{cases}$$

where ν_1' and ν_2' are constants, $\nu_1', \nu_2' > 0$, $l_1 = \max\{l_0, 2\log_2 \frac{1+\varepsilon(l)}{c} + 2\}$ is defined in the proof of Lemma 4. Then the convergence follows from the Lebesgue dominated convergence theorem and Lemma 9.

Let us estimate the rate of the convergence. If $|\omega| \geqslant C_0$, then $\widehat{\varphi^M}(\omega) = 0$, so

$$\left\|\widehat{\varphi_l}' - \widehat{\varphi^M}'\right\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \left|\widehat{\varphi_l}'(\omega) - \widehat{\varphi^M}'(\omega)\right|^2 d\omega = \int_{|\omega|<C_0} + \int_{|\omega|\geqslant C_0}$$
$$\leqslant 2C_0 \left\|\widehat{\varphi_l}' - \widehat{\varphi^M}'\right\|^2_{C[-C_0,C_0]} + A^2 l^2 \int_{|\omega|\geqslant C_0} |\omega|^{-2l+4\log_2 \frac{1+\varepsilon(l)}{c}} d\omega$$
$$= 2C_0 \left\|\widehat{\varphi_l}' - \widehat{\varphi^M}'\right\|^2_{C[-C_0,C_0]} + \frac{A^2 l^2 (C_0)^{-2l+4\log_2 \frac{1+\varepsilon(l)}{c}+1}}{2l - 4\log_2 \frac{1+\varepsilon(l)}{c} - 1}$$
$$= O\left(\max\left\{\mu^2(l), lC_0^{-2l+4\log_2 \frac{1+\varepsilon(l)}{c}}\right\}\right).$$

This completes the proof of Lemma 10. □

Remark 2. Using Lemma 9 and the estimate (18) (similarly to Remark 1) we get

$$\left\|\widehat{\varphi_l}' - \widehat{\varphi^M}'\right\|_{C(\mathbb{R})} = O\left(\max\left\{\mu(l), l^{0.5} C_0^{-l+2\log_2 \frac{1+\varepsilon(l)}{c}}\right\}\right).$$

Lemma 11. $\|\Phi_l'\|_C = O(\max\{\mu(l), l^{0.5}(4C_0 e^{2\omega_0})^{-l+2\log_2 \frac{1+\varepsilon(l)}{c}}\})$ *as $l \to \infty$. The parameters are defined by* (11).

Proof. Taking into account the definition Φ_l and the estimate (17) one can termwise differentiate the series, i.e.,

$$\Phi_l'(\omega) = \left(\sum_{k\in\mathbb{Z}} \left|\widehat{\varphi_l}(\omega + 2\pi k)\right|^2\right)' = \sum_{k\in\mathbb{Z}} \left(\left|\widehat{\varphi_l}(\omega + 2\pi k)\right|^2\right)'.$$

Since the Fourier transform of the Meyer scaling function is compactly supported and satisfies the property $\sum_{k\in\mathbb{Z}} |\widehat{\varphi^M}(\omega + 2\pi k)|^2 \equiv 1$, we get

$$\sum_{k\in\mathbb{Z}} \left(\left|\widehat{\varphi^M}(\omega + 2\pi k)\right|^2\right)' = \left(\sum_{k\in\mathbb{Z}} \left|\widehat{\varphi^M}(\omega + 2\pi k)\right|^2\right)' = (1)' = 0.$$

Suppose $|\omega| \leqslant \pi$, then we obtain

$$\begin{aligned}\left|\Phi_l'(\omega)\right| &\leqslant 2\sum_{k\in\mathbb{Z}}\left|\widehat{\varphi}_l(\omega+2\pi k)\widehat{\varphi}_l{}'(\omega+2\pi k)-\widehat{\varphi^M}(\omega+2\pi k)\widehat{\varphi^M}{}'(\omega+2\pi k)\right| \\ &\leqslant 2\sum_{k\in\mathbb{Z}}\left|\widehat{\varphi}_l(\omega+2\pi k)\right|\left|\widehat{\varphi}_l{}'(\omega+2\pi k)-\widehat{\varphi^M}{}'(\omega+2\pi k)\right| \\ &\quad + 2\sum_{k\in\mathbb{Z}}\left|\widehat{\varphi^M}{}'(\omega+2\pi k)\right|\left|\widehat{\varphi}_l(\omega+2\pi k)-\widehat{\varphi^M}(\omega+2\pi k)\right| =: 2I_{3,l}(\omega)+2I_{4,l}(\omega).\end{aligned}$$

Using the parameter $k_0 = [2e^{2\omega_0}/\pi + 1/2]$ defined in the proof of Lemma 5, we get

$$I_{3,l}(\omega) = \sum_{|k|\leqslant k_0} + \sum_{|k|>k_0}.$$

Taking into account Lemma 9, we have for the first sum

$$\sum_{|k|\leqslant k_0} \leqslant \left\|\widehat{\varphi}_l{}' - \widehat{\varphi^M}{}'\right\|_{C[-e^{2\omega_0}, e^{2\omega_0}]} \sum_{|k|\leqslant k_0}\left|\widehat{\varphi}_l(\omega+2\pi k)\right| = O\bigl(\mu(l)\bigr).$$

If we combine Remark 2 and (17), then we have for the second sum

$$\begin{aligned}\sum_{|k|>k_0} &\leqslant O\Bigl(\max\bigl\{\mu(l), l^{0.5}C_0^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}\bigr\}\Bigr)\sum_{|k|>k_0}|\omega+2\pi k|^{-l+2\log_2\frac{1+\varepsilon(l)}{c}} \\ &= O\Bigl(\max\bigl\{\mu(l), l^{0.5}C_0^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}\bigr\}\Bigr)\bigl(4e^{2\omega_0}\bigr)^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}.\end{aligned}$$

Next, we estimate $I_{4,l}(\omega)$. Since $\operatorname{supp}\widehat{\varphi^M} = [-2\omega_1, -2\omega_0]\cup[2\omega_0, 2\omega_1]$, then $\widehat{\varphi^M}(\omega+2\pi k) = 0$ as $k > 1$. So for the sum $I_{4,l}(\omega)$ we have

$$I_{4,l}(\omega) = \sum_{|k|\leqslant 1}\left|\widehat{\varphi^M}{}'(\omega+2\pi k)\right|\left|\widehat{\varphi}_l(\omega+2\pi k)-\widehat{\varphi^M}(\omega+2\pi k)\right|.$$

Thus the application of Lemma 3 yields $I_{4,l}(\omega) = O(\mu(l))$. Finally, for Φ_l' we get

$$\left|\Phi_l'(\omega)\right| \leqslant 2\bigl(I_{3,l}(\omega)+I_{4,l}(\omega)\bigr) = O\Bigl(\max\bigl\{\mu(l), l^{0.5}\bigl(4C_0e^{2\omega_0}\bigr)^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}\bigr\}\Bigr). \qquad \square$$

Now let us prove the convergence of the time radii for the scaling functions.

Theorem 6. $|\Delta^2_{\varphi_l^\perp} - \Delta^2_{\varphi^M}| = O(\max\{\mu(l), lC_0^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}\})$ *as* $l\to\infty$. *The parameters are defined by* (11).

Proof. If the function $\hat{\varphi}$ is real-valued, then $\overline{\varphi}(t) = \varphi(-t)$. Hence the function $|\varphi|^2$ is even. So the time centre $t_{0\varphi} = 0$. Then the square of the time radius $\Delta^2_\varphi = \int_{\mathbb{R}} t^2|\varphi(t)|^2\,dt$. Using the property of the Fourier transform $\widehat{\varphi'}(\omega) = i\omega\hat{\varphi}(\omega)$, we obtain $\Delta^2_\varphi = (2\pi)^{-1}\int_{\mathbb{R}}|(\widehat{\varphi})'(\omega)|^2\,d\omega$.

Since the functions $\widehat{\varphi_l^\perp}$, $\widehat{\varphi^M}$ are real-valued, we have $t_{0\varphi_l^\perp} = t_{0\varphi^M} = 0$. So for the squares of the time radii we get

$$\Delta^2_{\varphi_l^\perp} = \frac{1}{2\pi}\int_{\mathbb{R}}\left|\bigl(\widehat{\varphi_l^\perp}\bigr)'(\omega)\right|^2 d\omega \quad \text{and} \quad \Delta^2_{\varphi^M} = \frac{1}{2\pi}\int_{\mathbb{R}}\left|\bigl(\widehat{\varphi^M}\bigr)'(\omega)\right|^2 d\omega.$$

Then we have

$$\begin{aligned}\left|\Delta^2_{\varphi_l^\perp} - \Delta^2_{\varphi^M}\right| &\leqslant \frac{1}{2\pi}\int_{\mathbb{R}}\left|\bigl(\bigl(\widehat{\varphi_l^\perp}\bigr)'(\omega)\bigr)^2 - \bigl(\bigl(\widehat{\varphi^M}\bigr)'(\omega)\bigr)^2\right| d\omega \\ &\leqslant \frac{1}{2\pi}\sup_{\omega\in\mathbb{R}}\left|\bigl(\widehat{\varphi_l^\perp}\bigr)'(\omega) + \bigl(\widehat{\varphi^M}\bigr)'(\omega)\right|\int_{\mathbb{R}}\left|\bigl(\widehat{\varphi_l^\perp}\bigr)'(\omega) - \bigl(\widehat{\varphi^M}\bigr)'(\omega)\right| d\omega.\end{aligned}$$

Applying the definition of $\widehat{\varphi_l^{\perp}}$ (8), Lemmas 5, 11, Remarks 1, 2, and the triangle inequality, we establish the boundedness of the supremum factor

$$\begin{aligned}\sup_{\omega\in\mathbb{R}}\bigl|\widehat{\varphi_l^{\perp}}'(\omega)+\widehat{\varphi^M}'(\omega)\bigr| &\leqslant \left\|\frac{\widehat{\varphi}_l'}{\Phi_l}\right\|_{C(\mathbb{R})}+\left\|\frac{\Phi_l'\widehat{\varphi}_l}{\Phi_l^2}\right\|_{C(\mathbb{R})}+\bigl\|\widehat{\varphi^M}'\bigr\|_{C(\mathbb{R})}\\ &\leqslant \frac{\|\widehat{\varphi^M}'\|_{C(\mathbb{R})}+\|\widehat{\varphi^M}'-\widehat{\varphi}_l'\|_{C(\mathbb{R})}}{1-\|\Phi_l-1\|_C}+\frac{\|\Phi_l'\|_C(\|\widehat{\varphi^M}\|_{C(\mathbb{R})}+\|\widehat{\varphi^M}-\widehat{\varphi}_l\|_{C(\mathbb{R})})}{(1-\|\Phi_l-1\|_C)^2}+\bigl\|\widehat{\varphi^M}'\bigr\|_{C(\mathbb{R})}\\ &=O\bigl(\bigl\|\widehat{\varphi^M}'\bigr\|_{C(\mathbb{R})}\bigr).\end{aligned}$$

Let us show that $\int_{\mathbb{R}}|\widehat{\varphi}_l(\omega)|\,d\omega$ is bounded. In fact, the application of Lemma 3 and (17) yields

$$\begin{aligned}\int_{\mathbb{R}}\bigl|\widehat{\varphi}_l(\omega)\bigr|\,d\omega &\leqslant \int_{|\omega|\leqslant 4e^{2\omega_0}}\bigl|\widehat{\varphi}_l(\omega)-\widehat{\varphi^M}(\omega)\bigr|\,d\omega+\int_{|\omega|\leqslant 4e^{2\omega_0}}\bigl|\widehat{\varphi^M}(\omega)\bigr|\,d\omega+\int_{|\omega|>4e^{2\omega_0}}\bigl|\widehat{\varphi}_l(\omega)\bigr|\,d\omega\\ &\leqslant 8e^{2\omega_0}\bigl\|\widehat{\varphi}_l-\widehat{\varphi^M}\bigr\|_{C[-4e^{2\omega_0},4e^{2\omega_0}]}+8e^{2\omega_0}\bigl\|\widehat{\varphi^M}\bigr\|_{C[-4e^{2\omega_0},4e^{2\omega_0}]}+\frac{2e^{O(\mu)}(4e^{2\omega_0})^{-l+2\log_2\frac{1+\varepsilon(l)}{c}+1}}{l-2\log_2\frac{1+\varepsilon(l)}{c}-1}\\ &=O(1).\end{aligned}$$

Applying this, Lemmas 5, 11, and Remarks 1, 2 we get the convergence to 0 of the integral

$$\begin{aligned}\int_{\mathbb{R}}\bigl|\widehat{\varphi_l^{\perp}}'(\omega)-\widehat{\varphi^M}'(\omega)\bigr|\,d\omega &\leqslant \int_{\mathbb{R}}\left|\frac{\widehat{\varphi}_l'(\omega)-\widehat{\varphi^M}'(\omega)\Phi_l(\omega)}{\Phi_l(\omega)}\right|d\omega+\int_{\mathbb{R}}\left|\frac{\widehat{\varphi}_l(\omega)\Phi_l'(\omega)}{\Phi_l^2(\omega)}\right|d\omega\\ &\leqslant \frac{1}{1-\|\Phi_l-1\|_C}\int_{\mathbb{R}}\bigl|\widehat{\varphi}_l'(\omega)-\widehat{\varphi^M}'(\omega)\bigr|+\bigl|\widehat{\varphi^M}'(\omega)\bigr|\bigl|1-\Phi_l(\omega)\bigr|\,d\omega\\ &\quad+\frac{\|\Phi_l'\|_C}{(1-\|\Phi_l-1\|_C)^2}\int_{\mathbb{R}}\bigl|\widehat{\varphi}_l(\omega)\bigr|\,d\omega\\ &\leqslant \frac{1}{1-\|\Phi_l-1\|_C}\left(\int_{|\omega|\leqslant C_0}\bigl|\widehat{\varphi}_l'(\omega)-\widehat{\varphi^M}'(\omega)\bigr|\,d\omega+\int_{|\omega|>C_0}\bigl|\widehat{\varphi}_l'(\omega)-\widehat{\varphi^M}'(\omega)\bigr|\,d\omega\right)\\ &\quad+\frac{\|1-\Phi_l\|_C}{1-\|\Phi_l-1\|_C}\int_{\mathbb{R}}\bigl|\widehat{\varphi^M}'(\omega)\bigr|\,d\omega+O\bigl(\bigl\|\Phi_l'\bigr\|_C\bigr)\\ &\leqslant \frac{\|\widehat{\varphi}_l'-\widehat{\varphi^M}'\|_{C[-C_0,C_0]}}{1-\|\Phi_l-1\|_C}+\frac{2AlC_0^{-l+2\log_2\frac{1+\varepsilon(l)}{c}+1}}{(1-\|\Phi_l-1\|_C)(l-2\log_2\frac{1+\varepsilon(l)}{c}-1)}\\ &\quad+O\bigl(\|\Phi_l-1\|_C\bigr)+O\bigl(\bigl\|\Phi_l'\bigr\|_C\bigr)=O\bigl(\max\bigl\{\mu(l),lC_0^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}\bigr\}\bigr).\qquad\Box\end{aligned}$$

7. Convergence of time and frequency radii for the wavelet functions

Theorem 7. $|\Delta^2_{\widehat{\psi_l^{\perp}}}-\Delta^2_{\widehat{\psi^M}}|=O(\max\{\mu(l),lC_0^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}\})$, $|\Delta^2_{\psi_l^{\perp}}-\Delta^2_{\psi^M}|=O(\max\{\mu(l),(4e^{2\omega_0})^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}\})$ *as* $l\to\infty$. *The parameters are defined by* (11).

Proof. The equality (10) shows that $\widehat{\psi_l^{\perp}}$ is even. The function $\widehat{\psi^M}$ is also even. Therefore $\omega_{0\widehat{\psi_l^{\perp}}}=\omega_{0\widehat{\psi^M}}=0$ and $t_{0\psi_l^{\perp}}=t_{0\psi^M}=1/2$. The mask $m_l^{\perp}$ is a real-valued function. Therefore, using the structure of (10) and applying Lemmas 1, 5, and Theorem 3 we get for the frequency radii

$$\begin{aligned}\bigl|\Delta^2_{\widehat{\psi_l^{\perp}}}-\Delta^2_{\widehat{\psi^M}}\bigr| &=\left|\int_{\mathbb{R}}\omega^2\left(\bigl(m_l^{\perp}\bigr)^2\left(\frac{\omega}{2}+\pi\right)\bigl(\widehat{\varphi_l^{\perp}}\bigr)^2\left(\frac{\omega}{2}\right)-\bigl(m^M\bigr)^2\left(\frac{\omega}{2}+\pi\right)\bigl(\widehat{\varphi^M}\bigr)^2\left(\frac{\omega}{2}\right)\right)d\omega\right|\\ &\leqslant \bigl\|\bigl(m_l^{\perp}\bigr)^2\bigr\|_C\int_{\mathbb{R}}\omega^2\left|\bigl(\widehat{\varphi_l^{\perp}}\bigr)^2\left(\frac{\omega}{2}\right)-\bigl(\widehat{\varphi^M}\bigr)^2\left(\frac{\omega}{2}\right)\right|d\omega+\bigl\|\bigl(m_l^{\perp}\bigr)^2-\bigl(m^M\bigr)^2\bigr\|_C\int_{\mathbb{R}}\omega^2\bigl(\widehat{\varphi^M}\bigr)^2\left(\frac{\omega}{2}\right)d\omega\end{aligned}$$

$$= O\left(\left|\Delta^2_{\widehat{\varphi_l^\perp}} - \Delta^2_{\widehat{\varphi^M}}\right|\right) + O\left(\|\Phi_l - 1\|_C\right) + O\left(\left\|m_l - m^M\right\|_C\right)$$
$$= O\left(\max\left\{\mu(l), lC_0^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}\right\}\right).$$

Going on to the time radii we use the identity $\Delta_f^2 = \int_{\mathbb{R}} t^2 |f(t)|^2\,dt - t_{0f}^2$, and the following elementary formulas $(abc)' = a'bc + ab'c + abc'$, $|e^{i\omega}| = 1$. In the sequel, we omit the argument ω.

$$2\pi\left|\Delta^2_{\psi_l^\perp} - \Delta^2_{\psi^M}\right| = \left|\int_{\mathbb{R}} \left(\widehat{\psi_l^\perp}'\right)^2 - \left(\widehat{\psi^M}'\right)^2\right| \leqslant \left\|\widehat{\psi_l^\perp}' + \widehat{\psi^M}'\right\|_{C(\mathbb{R})} \int_{\mathbb{R}} \left|\widehat{\psi_l^\perp}' - \widehat{\psi^M}'\right|$$
$$\leqslant \frac{\|\widehat{\psi_l^\perp}' + \widehat{\psi^M}'\|_{C(\mathbb{R})}}{4} \int_{\mathbb{R}} \left|m_l^{\perp\prime}\widehat{\varphi_l^\perp} + m_l^\perp\widehat{\varphi_l^\perp}' - im_l^\perp\widehat{\varphi_l^\perp} - m^{M\prime}\widehat{\varphi^M} - m^M\widehat{\varphi^M}' + im^M\widehat{\varphi^M}\right|$$
$$\leqslant \frac{\|\widehat{\psi_l^\perp}' + \widehat{\psi^M}'\|_{C(\mathbb{R})}}{4}\Bigg(\left\|m_l^{\perp\prime} - im_l^\perp\right\|_C \int_{\mathbb{R}} \left|\widehat{\varphi_l^\perp} - \widehat{\varphi^M}\right| + \left\|m_l^{\perp\prime} - m^{M\prime}\right\|_C \int_{\mathbb{R}} \left|\widehat{\varphi^M}\right|$$
$$+ \left\|m_l^\perp\right\|_C \int_{\mathbb{R}} \left|\widehat{\varphi_l^\perp}' - \widehat{\varphi^M}'\right| + \left\|m_l^\perp - m^M\right\|_C \int_{\mathbb{R}} \left|\widehat{\varphi^M}' - i\widehat{\varphi^M}\right|\Bigg)$$
$$=: \frac{\|\widehat{\psi_l^\perp}' + \widehat{\psi^M}'\|_{C(\mathbb{R})}}{4}(I_{5,l} + I_{6,l} + I_{7,l} + I_{8,l}).$$

We claim that

$$\left\|\widehat{\psi_l^\perp}' + \widehat{\psi^M}'\right\|_{C(\mathbb{R})}, \qquad \left\|m_l^{\perp\prime} - im_l^\perp\right\|_C, \qquad \int_{\mathbb{R}} \left|\widehat{\varphi^M}\right|, \qquad \left\|m_l^\perp\right\|_C, \qquad \int_{\mathbb{R}} \left|\widehat{\varphi^M}' - i\widehat{\varphi^M}\right|$$

are bounded and

$$\int_{\mathbb{R}} \left|\widehat{\varphi_l^\perp} - \widehat{\varphi^M}\right|, \qquad \left\|m_l^{\perp\prime} - m^{M\prime}\right\|_C, \qquad \int_{\mathbb{R}} \left|\widehat{\varphi_l^\perp}' - \widehat{\varphi^M}'\right|, \qquad \left\|m_l^\perp - m^M\right\|_C$$

tend to 0 as $l \to \infty$. Indeed, the scheme of the proof is the same for these cases, and we already used it in Theorems 3 and 6. Let us give a brief explanation. First, we note that multiplication and division by nonvanishing functions (in our case, Φ_l are these functions) are continuous operations with respect to the supremum norm. Then we use lemmas stating the convergence of the new functions (such as m_l, m_l', Φ_l, Φ_l', $\widehat{\varphi}_l$, $\widehat{\varphi}_l'$) to the corresponding Meyer functions (m^M, $m^{M\prime}$, 1, 0, $\widehat{\varphi^M}$, $\widehat{\varphi^M}'$, respectively). For example, Lemmas 2, 5, 11 are employed to estimate $\|m_l^{\perp\prime} - m^{M\prime}\|_C$. In the case of the integrals $\int_{\mathbb{R}} |\widehat{\varphi_l^\perp} - \widehat{\varphi^M}|$ and $\int_{\mathbb{R}} |\widehat{\varphi_l^\perp}' - \widehat{\varphi^M}'|$, we additionally apply (17) and (18), respectively. By the definition of the Meyer scaling function, $\int_{\mathbb{R}} |\widehat{\varphi^M}|$ and $\int_{\mathbb{R}} |\widehat{\varphi^M}' - i\widehat{\varphi^M}|$ are bounded.

Therefore the application of Lemmas 3, 5, the estimate (17), and the proof of Theorem 3 yields

$$I_{5,l} = O\left(\|\Phi_l - 1\|_C\right) + O\left(\left(4e^{2\omega_0}\right)^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}\right) = O\left(\max\left\{\mu(l), \left(4e^{2\omega_0}\right)^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}\right\}\right).$$

Using the definition of $m_l^\perp$, Lemmas 11, 5, and 2, we get

$$I_{6,l} = O\left(\left\|\Phi_l'\right\|_C\right) + O\left(\|\Phi_l - 1\|_C\right) + O\left(\left\|m_l' - m^{M\prime}\right\|_C\right) = O\left(\max\left\{\mu(l), \left(4e^{2\omega_0}\right)^{-2l+4\log_2\frac{1+\varepsilon(l)}{c}}\right\}\right).$$

From the proof of Theorem 6 it follows that

$$I_{7,l} = O\left(\max\left\{\mu(l), lC_0^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}\right\}\right).$$

Finally, using Lemmas 5 and 1, we get

$$I_{8,l} = O\left(\|\Phi_l - 1\|_C\right) + O\left(\left\|m_l - m^M\right\|_C\right) = O\left(\max\left\{\mu(l), \left(4e^{2\omega_0}\right)^{-2l+4\log_2\frac{1+\varepsilon(l)}{c}}\right\}\right).$$

Thus collecting the estimates we obtain

$$\left|\Delta^2_{\psi_l^\perp} - \Delta^2_{\psi^M}\right| = O\left(\max\left\{\mu(l), \left(4e^{2\omega_0}\right)^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}\right\}\right). \qquad \square$$

Acknowledgments

The author thanks Professor I.Ya. Novikov for setting the problem and Professor M.A. Skopina for valuable discussions.

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