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# Note

# Lower bounds on the stability number of graphs computed in terms of degrees

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*Abstract*

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Wei discovered that the stability number, $\alpha(G)$, of a graph, $G$, with degree sequence $d_1, d_2, \ldots, d_n$ is at least

$$w(G) = \sum_{i=1}^{n} \frac{1}{d_i + 1}.$$

It is shown that this bound can be replaced by a function $b(G)$, computable from $d_1, d_2, \ldots, d_n$ using only $O(n)$ additions and comparisons. For all graphs, $b(G) \geqslant \lceil w(G) \rceil$, the inequality sometimes holding as strict. In addition, it is shown that Wei's bound can be increased by $(w(G)-1)/\Delta(\Delta+1)$ when $G$ is connected, by $w(G)k/2\Delta(\Delta+1)$ when $G$ is $k$-connected but not complete, and by $(w(G)+m-n)/\Delta(\Delta+1)$ when $G$ is triangle-free; in each case, $\Delta$, $n$, and $m$ denote the largest degree of a vertex in $G$, the number of vertices of $G$, and the number of edges of $G$, respectively.

## 1. Introduction

The stability number, $\alpha(G)$, of a graph, $G$, is the largest number of pairwise non-adjacent vertices in $G$. The celebrated theorem of Turán [7] states that every graph, $G$, has at least as many edges as some graph, $H$, where $H$ consists of $\alpha(G)$ disjoint cliques of the same size to within one on the same vertex set as $G$. Erdős [3] refined this theorem by proving that $d_G(v) \geqslant d_H(v)$ for all $v$ where $d_X(v)$ denotes the degree of vertex $v$ in graph $X$. (See [1, p. 294–296].) Implicit in his proof is an algorithm that computes a collection of pairwise non-adjacent vertices $w_1, w_2, \ldots, w_s$ in $G$ along with the graph $H$ consisting of at most $\alpha(G)$ disjoint cliques and such that $d_G(v) \geq d_H(v)$ for all $v$. Let $s$ be the number of disjoint cliques. A description of this algorithm, which will be referred to as GREEDY, is given below.

*Input*: $G$
*Output*: $H$ consisting of $s$ disjoint cliques
$j \leftarrow 0$
**while** $G \neq \emptyset$ **do**
$j \leftarrow j + 1$
$w_j \leftarrow$ any vertex of smallest degree in $G$
$C_j \leftarrow \{w_j\} \cup \{v\colon v$ is adjacent to $w_j$ in $G\}$
$G \leftarrow G - C_j$
**endwhile**
$s \leftarrow j$
$H \leftarrow$ the graph in which two vertices are adjacent if and only if they belong to the same $C_j$.

To see that $d_G(v) \geq d_H(v)$ for each $v$, let

$$F = G - \{C_1 \cup C_2 \cup \cdots \cup C_{j-1}\}, \tag{1}$$

where $v \in C_j$, and observe that

$$d_G(v) \geq d_F(v) \geq d_F(w_j) = |C_j| - 1 = d_H(v).$$

Letting $V$ denote the set of vertices of $G$ (and $H$) gives

$$s = \sum_{v \in V} \frac{1}{1 + d_H(v)}, \tag{2}$$

and since $\alpha(G) \geq s$ and $d_G(v) \geq d_H(v)$ for all $v$, it follows that

$$\alpha(G) \geq w(G) = \sum_{v \in V} \frac{1}{d_G(v) + 1}.$$

This inequality was discovered independently by Caro [2] and Wei [8]. Wei also showed that

$$w(G) \geq \frac{n}{1 + \bar{d}} \tag{3}$$

where $n$ and $\bar{d}$ denote the number of vertices and the average degree, respectively.

Johnson [6] studied the application of algorithm GREEDY to the analogous maximum clique problem and showed that the worst case ratio of the optimal to the solution computed by GREEDY grows as fast as $O(n)$. This is not surprising since the problem of determining $\alpha(G)$ for most graphs is NP-complete. The proof of NP-completeness along with other theorems which indicate the difficulty of this problem can be found in Garey and Johnson [4].

## 2. New bounds on the stability number

Wei's bound can be improved in two different ways. First, enumerate the vertices of $G$ as $v_1, v_2, \ldots, v_n$ in such a way that

$$d_G(v_1) \leqslant d_G(v_2) \leqslant \cdots \leqslant d_G(v_n)$$

and let $d_k = d_G(v_k)$. The following algorithm computes a lower bound, $b(G)$, on $\alpha(G)$.

*Input*: $d_1, d_2, \ldots, d_n$
*Output*: $b(G)$
$j \leftarrow 0$
$a_0 \leftarrow 0$
**while** $a_j < n$ **do**
$i \leftarrow a_j$
$a_{j+1} \leftarrow a_j + d_{i+1} + 1$
$j \leftarrow j + 1$
**endwhile**
$a_j \leftarrow n$
$b(G) \leftarrow j$.

**Theorem 1.** $\alpha(G) \geqslant b(G)$.

**Proof.** It suffices to show that if $j \leqslant b(G)$, then GREEDY computes the sets $C_1$, $C_2, \ldots, C_j$ such that

$$|C_1 \cup C_2 \cup \cdots \cup C_j| \leqslant a_j. \tag{4}$$

This will be done by induction on $j$. Trivially (4) holds for $j = 0$. Assume that (4) holds for some $j < b(G)$. If $j < b(G)$, then $a_j < n$. Since (4) holds, and since $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$, it means that $d_G(w_{j+1}) \leqslant d_{1+a_j}$. Therefore, since $|C_{j+1}| \leqslant 1 + d_G(w_{j+1})$, it follows that

$$|C_1 \cup C_2 \cup \cdots \cup C_{j+1}| \leqslant a_{j+1}. \qquad \square$$

To see that $w(G) \leqslant b(G)$ for all $G$, observe that

$$w(G) = \sum_{j=0}^{b(G)-1} \sum_{k=1+a_j}^{a_{j+1}} \frac{1}{1+d_k} \leqslant \sum_{j=0}^{b(G)-1} \frac{a_{j+1} - a_j}{1 + d_{1+a_j}} \leqslant b(G).$$

For some graphs, Theorem 1 gives a genuine improvement over Wei's bound. For instance, if $(d_1, d_2, \ldots, d_7) = (1, 3, 3, 4, 5, 5, 5)$, then $w(G) = 1.7$ and $b(G) = 3$.

Wei's bound can be improved in a second way. Consider the application of the algorithm GREEDY to the graph, $G$, and the resulting partition $\{C_1, C_2, \ldots, C_s\}$ of the vertices in $G$. An edge that is incident on two vertices in the same set of the partition is called an interior edge. An edge that is not an interior edge is called an exterior edge. Let $\Delta(G)$ denote the largest degree of a vertex in $G$.

**Lemma.** *If $t$ is the number of exterior edges determined by an execution of algorithm* GREEDY *then*

$$s \geq w(G) + \frac{t}{\Delta(G)(\Delta(G)+1)}.$$

**Proof.** For each vertex, $v$, of $G$, let $j$ be defined by $v \in C_j$, and let $e(v)$ denote the number of edges $(u, v)$ with $u \in C_i$, $i < j$. If $F$ is defined by (1), observe that

$$d_G(v) = e(v) + d_F(v) \geq e(v) + d_F(w_j) = e(v) + d_H(v),$$

and so

$$\frac{1}{1+d_H(v)} \geq \frac{1}{1+d_G(v)-e(v)} \geq \frac{1}{1+d_G(v)} + \frac{e(v)}{d_G(v)(d_G(v)+1)}.$$

The lemma then follows from (2). □

**Theorem 2.** *If $G$ is a connected graph, then*

$$\alpha(G) \geq w(G) + \frac{(w(G)-1)}{\Delta(G)(\Delta(G)+1)}. \tag{5}$$

*If $G$ is $k$-connected but not complete, then*

$$\alpha(G) \geq w(G) + \frac{k}{2}\frac{w(G)}{\Delta(G)(\Delta(G)+1)}. \tag{6}$$

*If $G$ is triangle-free, then*

$$\alpha(G) \geq w(G) + \frac{w(G)+m-n}{\Delta(G)(\Delta(G)+1)}, \tag{7}$$

*where $m$ is the number of edges in $G$.*

**Proof.** In all three cases, the lemma will be used to give lower bounds on the value $s$ computed by algorithm GREEDY. If $G$ is connected, then clearly $t \geq s-1$. If $G$ is $k$-connected, then the vertices of each $C_j$ must be incident with at least $k$ exterior edges, and so $t \geq ks/2$. If $G$ is triangle-free, then its interior edges are precisely those joining each $w_j$ to another vertex in $C_j$, and so $t = m-(n-s)$. The theorem then follows directly from the lemma. □

## 3. Conclusions

The bounds (5) and (6) are the best known for connected graphs. A result similar to (7) is due to Griggs [5]: if $G$ is triangle-free then

$$\alpha(G) \geqslant w(G) + \frac{n}{\Delta(G)(\Delta(G)+1)}, \tag{8}$$

unless $G$ is a circuit with $n$ odd or a path with $n$ even. It is easy to see using (3) that the difference between (7) and (8) is no less than

$$\frac{n}{\Delta(G)(\Delta(G)+1)} \frac{\bar{d}^2 - 3\bar{d} - 2}{2(\bar{d}+1)},$$

where (7) is the superior bound whenever $\bar{d} \geqslant 3.6$.

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## References

[1] B. Bollobás, Extremal Graph Theory (Academic Press, New York, 1978).
[2] Y. Caro, unpublished.
[3] P. Erdős, On the graph theorem of Turán, Mat. Lapok 21 (1970) 249–251.
[4] M.R. Garey and D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness (Freeman, New York, 1979).
[5] J.R. Griggs, Lower bounds on the independence number in terms of degrees, J. Combin. Theory Ser. B 34 (1983) 22–39.
[6] D.S. Johnson, Approximation algorithms for combinatorial problems, J. Comput. System Sci. 9 (1974) 256–278.
[7] P. Turán, An extremal problem in graph theory, Mat. Fiz. Lapok 48 (1941) 436–452.
[8] V.K. Wei, A lower bound on the stability number of a simple graph, Bell Laboratories Technical Memorandum, 81-11217-9, Murray Hill, NJ, 1981.