Absolutely summing polynomials on Banach spaces with unconditional basis

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Abstract

If $X$ is a Banach space with a normalized unconditional Schauder basis $(x_n)$, we define $\mu_{X,(x_n)} = \inf \{ t; (a_j) \in \ell_t \text{ whenever } x = \sum_{j=1}^{\infty} a_j x_j \in X \}$ and obtain estimates for $\mu_{X,(x_n)}$ when every continuous $m$-homogeneous polynomial from $X$ into $Y$ is absolutely $(q,1)$ summing. Our results provide new information on coincidence situations between the space of absolutely summing $m$-homogeneous polynomials and the whole space of continuous $m$-homogeneous polynomials. In particular, when $m = 1$, we obtain new contributions to the linear theory of absolutely summing operators.

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1. Introduction

Since the famous paper “Absolutely summing operators in $\mathcal{L}_p$ spaces and their applications” by J. Lindenstrauss and A. Pełczyński [5], the theory of absolutely summing linear operators has been broadly studied and coincidence situations are a central topic of investigation. By co-
incidence situation we mean a case in which for particular Banach spaces $X$ and $Y$ and some special real numbers $p$ and $q$ we have that every continuous linear operator from $X$ to $Y$ is absolutely $(p; q)$-summing. The recent development of nonlinear theory of absolutely summing mappings transported this question to polynomials and multilinear mappings. In this paper we will be mainly concerned with coincidence situations for polynomials. Our results generalize estimates obtained by the second author in [8] and [9]. When restricted to the linear case, our results provide some new contributions to the linear theory of absolutely summing operators on $\ell_r$ spaces.

Throughout this paper $X$, $Y$ will stand for Banach spaces, $X'$ denotes the topological dual of $X$ and the scalar field $\mathbb{K}$ can be either $\mathbb{R}$ or $\mathbb{C}$. The Banach space of all continuous $m$-homogeneous polynomials from $X$ into $Y$ with the sup norm is denoted by $P(m)X, Y)$ ($P(m)X$ if $Y = \mathbb{K}$).

The linear space of all sequences $(x_j)_{j=1}^\infty$ in $X$ such that $\| (x_j)_{j=1}^\infty \|_p = (\sum_{j=1}^\infty \|x_j\|^p)^{1/p} < \infty$ will be denoted by $\ell_p(X)$. We will also denote by $\ell_p^w(X)$ the linear subspace of $\ell_p(X)$ composed by the sequences $(x_j)_{j=1}^\infty$ in $X$ such that $(\varphi(x_j))_{j=1}^\infty \in \ell_p$ for every continuous linear functional $\varphi : X \to \mathbb{K}$. We define $\| \cdot \|_w, p$ in $\ell_p^w(X)$ by $\| (x_j)_{j=1}^\infty \|_w, p = \sup_{\varphi \in B_X} (\sum_{j=1}^\infty | \varphi(x_j) |^p)^{1/p}$.

The case $p = \infty$ is the case of bounded sequences and in $\ell_\infty(X)$ we use the sup norm. It can be proved that $\| \cdot \|_p$ ($\| \cdot \|_w, p$) is a $p$-norm in $\ell_p(X)$ ($\ell_p^w(X)$) for $p < 1$ and a norm in $\ell_p(X)$ ($\ell_p^w(X)$) for $p > 1$.

A continuous $m$-homogeneous polynomial $P : X \to Y$ is absolutely $(p; q)$-summing (or $(p; q)$-summing) if $(P(x_j))_{j=1}^\infty \in \ell_p(Y)$ for all $(x_j)_{j=1}^\infty \in \ell_q^w(X)$. The space of all absolutely $(p; q)$-summing $m$-homogeneous polynomials from $X$ into $Y$ is denoted by $P_{as(p,q)}(m)X; Y) = P_{as(p,q)}(m)X$ if $Y = \mathbb{K}$). When $m = 1$, we have the concept of absolutely summing operators and $P_{as(p,q)}(1)X; Y)$ is represented by $L_{as(p,q)}(X; Y)$.

As in the linear case, we have useful characterizations. For example, in [6] it is shown that $P \in P_{as(p,q)}(m)X; Y)$ if and only if there exists $L > 0$ such that

$$\left( \sum_{j=1}^k \| P(x_j) \|^p \right)^{1/p} \leq L \| (x_j)_{j=1}^k \|_{w,q}^m \quad \forall k \in \mathbb{N} \text{ and } x_j \in X.$$ 

The infimum of all $L > 0$ for which the inequality always holds is a norm for the case $p \geq 1$ or a $p$-norm for the case $p < 1$ on the space of absolutely $(p; q)$-summing polynomials. In any case, we have complete topological metrizable spaces and this norm $(p$-norm) will be represented by $\| \cdot \|_{as(p,q)}$.

2. Results

If $X$ is an infinite dimensional Banach space with a normalized unconditional Schauder basis $(x_n)$, we define

$$\mu_{X,(x_n)} = \inf \left\{ t : (a_j) \in \ell_t \text{ whenever } x = \sum_{j=1}^\infty a_j x_j \in X \right\}.$$ 

For $X = c_0, \ell_1$ or $\ell_2$, we can write $\mu_X$ unambiguously because every unconditional basis is equivalent to the canonical basis. For $X = \ell_p, 1 < p < \infty, p \neq 2$, when we write $\mu_X$ it must be understood that we are considering the canonical basis.

In order to obtain new results concerning eventual coincidence between the space of all absolutely $(p, q)$ summing $m$-homogeneous polynomials and the whole space of continuous
$m$-homogeneous polynomials, the following general question will be investigated: If $X$ has unconditional basis $(x_n)$ and $P_{as(q;1)}(mX;Y) = \mathcal{P}(mX;Y)$, how is the behavior of $\mu_{X,(x_n)}$?

Some partial answers are already known as we can see in the results below:

**Theorem 2.1.** (Pellegrino [8]) Let $X$ and $Y$ be infinite dimensional Banach spaces and suppose that $X$ has an unconditional Schauder basis $(x_n)$. If $Y$ finitely factors the formal inclusion $\ell_p \rightarrow \ell_\infty$ for some $\delta$ and $P_{as(q;1)}(mX;Y) = \mathcal{P}(mX;Y)$, then

(a) $\mu_{X,(x_n)} \leq \frac{mpq}{p-q}$ if $q < p$;
(b) $\mu_{X,(x_n)} \leq mq$ if $q \leq \frac{p}{2}$.

**Theorem 2.2.** (Pellegrino [9]) Let $X$ be an infinite dimensional Banach space with an unconditional Schauder basis $(x_n)$. If $P_{as(q;1)}(mX) = \mathcal{P}(mX)$, then

(a) $\mu_{X,(x_n)} \leq \frac{mq}{1-q}$ if $q < 1$;
(b) $\mu_{X,(x_n)} \leq mq$ if $q \leq \frac{1}{2}$.

In this paper we will obtain better estimates for $\mu_{X,(x_n)}$, improving Theorems 2.1 and 2.2. These new estimates will provide several interesting consequences. Besides the contribution given to the polynomial/multilinear case, our results also provide improvements to the (linear) theory of absolutely summing operators.

The following lemma will be fundamental in order to obtain better (and optimal) estimates:

**Lemma 2.1.** Suppose that $Y$ satisfies the following condition:

There exist $C_1, C_2 > 0$ and $p \geq 1$ such that for every $n \in \mathbb{N}$, there are $y_1, \ldots, y_n$ in $Y$ with $\|y_j\| \geq C_1$ for every $j$ and

\[
\left\| \sum_{j=1}^{n} a_j y_j \right\| \leq C_2 \left( \sum_{j=1}^{n} |a_j|^p \right)^{1/p}
\]

for every $a_1, \ldots, a_n \in \mathbb{K}$.

In this case, if $X$ has a normalized unconditional Schauder basis $(x_n)$, $q < p$ and $P_{as(q;1)}(mX;Y) = \mathcal{P}(mX;Y)$, then $\mu_{X,(x_n)} \leq mq$.

**Proof.** The proof will be done by induction. When $\mathbb{K} = \mathbb{R}$ it is not hard to prove that $\|(z_j)_{j=1}^{n}\|_{w,1} = \max_{\varepsilon_j \in \{1,-1\}} \{ \| \sum_{j=1}^{n} \varepsilon_j z_j \| \}$ and when $\mathbb{K}$ denotes the complex scalar field, we have $\|(z_j)_{j=1}^{n}\|_{w,1} \leq 2 \max_{\varepsilon_j \in \{1,-1\}} \{ \| \sum_{j=1}^{n} \varepsilon_j z_j \| \}$. Hence, in order to cover both cases, it will be convenient to make use of the last inequality.

Using the Open Mapping Theorem, one can find a $K > 0$ such that $\|P\|_{as(q;1)} \leq K \|P\|$ for all continuous $m$-homogeneous polynomials $P : X \rightarrow Y$.

Let $n$ be a fixed natural number and $\{\mu_i\}_{i=1}^{n}$ be such that $\sum_{j=1}^{n} |\mu_j|^s = 1$, where $s = p/q$. Define $P : X \rightarrow Y$ by

\[
P_x = \sum_{j=1}^{n} |\mu_j|^{1/q} a_j^m y_j, \quad \text{if } x = \sum_{j=1}^{\infty} a_j x_j.
\]
Since \((x_n)\) is an unconditional basis, there exists a positive \(\rho\) such that
\[
\left\| \sum_{j=1}^{\infty} \epsilon_j a_j x_j \right\| \leq \rho \left\| \sum_{j=1}^{\infty} a_j x_j \right\| = \rho \| x \| \quad \text{for any } \epsilon_j = 1 \text{ or } \epsilon_j = -1.
\]
Hence \(\| \sum_{j=1}^{k} \epsilon_j a_j x_j \| \leq \rho \| x \|\) for all \(k\) and any \(\epsilon_j = 1\) or \(-1\). So, if \(x = \sum_{j=1}^{\infty} a_j x_j \in X\), we have \(\|a_j\| \leq \rho \|x\|\) for all \(j\) and then we get
\[
\|Px\| = \left\| \sum_{j=1}^{n} |\mu_j|^{1/q} a_j^m y_j \right\| \leq C_2 \left( \sum_{j=1}^{n} \|\mu_j\|^{1/q} a_j^m \right)^{1/p}
\leq C_2 \rho^m \|x\|^m \left( \sum_{j=1}^{n} \|\mu_j\|^{p/q} \right)^{1/p}
= C_2 \rho^m \|x\|^m \left( \sum_{j=1}^{n} \|\mu_j\|^q \right)^{1/p}.
\]
We thus have \(\|P\| \leq C_2 \rho^m\), \(\|P\|_{as(1;1)} \leq KC_2 \rho^m\) and achieve the estimate below:
\[
\left( \sum_{j=1}^{n} a_j \|\mu_j\|^{1/q} C_1^q \right)^{1/q} \leq \left( \sum_{j=1}^{n} \|a_j\| \|\mu_j\|^{1/q} y_j \| \right)^{1/q} = \left( \sum_{j=1}^{n} \|P a_j x_j\| \right)^{1/q}
\leq \|P\|_{as(1;1)} \left( a_j x_j \right)_{j=1}^{n} \left\| y \right\|^{m,1}_{w,1}
\leq \|P\|_{as(1;1)} 2^m \max_{\epsilon_j \in \{1, -1\}} \left\{ \sum_{j=1}^{n} \epsilon_j a_j x_j \right\}^{m}
\leq \|P\|_{as(1;1)} (2\rho \|x\|)^m \leq KC_2 2^m \rho^2 \|x\|^m.
\]
Note that the last inequality holds whenever \(\sum_{j=1}^{n} |\mu_j|^s = 1\). Hence, since \(1/s + 1/(s-1) = 1\), we have
\[
\left( \sum_{j=1}^{n} a_j \right)^{1/(s-1)} \leq \sup \left\{ \sum_{j=1}^{n} \mu_j |a_j|^w ; \sum_{j=1}^{n} |\mu_j|^s = 1 \right\}
\leq \sup \left\{ \sum_{j=1}^{n} |\mu_j| |a_j|^w ; \sum_{j=1}^{n} |\mu_j|^s = 1 \right\}.
\]
Again, by (2.2), it follows that
\[
\left( \sum_{j=1}^{n} (|a_j|^{1/s-1})^{mq} \right)^{1/(s-1)} \leq \left( C_1^{-1} KC_2 2^m \rho^2 \|x\|^m \right)^{q},
\]
and then
\[
\left( \sum_{j=1}^{n} |a_j|^{1/s-1} \right)^{1/(s-1)^{mq}} \leq \left( C_1^{-1} KC_2 2^m \rho^2 \|x\|^m \right)^{1/m}.
\]
Since \(s/m = mpq/p-q\) and \(n\) is arbitrary, we have \(\mu_{X,(x_n)} \leq \frac{mpq}{p-q}\).
Now, if $q \leq p/2$, define, for a fixed $n$, $S : X \to Y$ by

$$Sx = \sum_{j=1}^{n} a_j^m y_j \text{ if } x = \sum_{j=1}^{\infty} a_j x_j.$$ 

Since $mp \geq \frac{s}{s-1} mq$, combining the preceding estimates, we obtain

$$\|Sx\| = \left\| \sum_{j=1}^{n} a_j^m y_j \right\| \leq C_2 \left( \sum_{j=1}^{n} |a_j|^p \right)^{1/p} = C_2 \left( \left( \sum_{j=1}^{n} |a_j|^{mp} \right)^{1/(mp)} \right)^{m}$$

Thus $\|S\| \leq C_1^{-1} K C_2^2 m^2 \rho^{2m}$ and $\|S\|_{\text{as}(q;1)} \leq C_1^{-1} K^2 C_2^2 m^2 \rho^{2m}$. Hence

$$\sum_{j=1}^{n} \|a_j^m C_1\|^q \leq \sum_{j=1}^{n} \|a_j^m y_j\|^q = \sum_{j=1}^{n} \|S a_j x_j\|^q \leq \|S\|_{\text{as}(q;1)}^{2mq} \max_{\epsilon_j \in \{1,-1\}} \left\{ \sum_{j=1}^{n} \epsilon_j a_j x_j \right\}^{mq} \leq (C_1^{-1} K^2 C_2^4 m^3 \rho^3)^q \|x\|^{mq}.$$ 

Consequently, since $n$ is arbitrary, we have $\sum_{j=1}^{\infty} |a_j|^mq < \infty$ whenever $x = \sum_{j=1}^{\infty} a_j x_j \in X$ and $\mu_{X,(x_n)} \leq mq$ if $q \leq p/2$. 

Now we state the induction hypothesis:

Suppose that we have

(i) $\mu_{X,(x_n)} \leq \frac{mpq}{jp-jq}$ and $(\sum_{k=1}^{\infty} |a_k|^{mpq}/jp-jq)^{1/(mpq/jp-jq)} \leq A_j \|x\|$, if $\frac{jp}{j+1} < q < p$;

(ii) $\mu_{X,(x_n)} \leq mq$ and $(\sum_{k=1}^{\infty} |a_k|^mq)^{1/(mq)} \leq B_j \|x\|$, if $q \leq \frac{jp}{j+1}$,

where

- $A_1 = (C_1^{-1} K C_2^2 m^2 \rho^{2m})^{1/m}$,
- $B_1 = (C_1^{-2} K C_2^4 m^3 \rho^{3m})^{1/m}$,
- $A_j = (C_1^{-1} K C_2^2 m^2 \rho^m A_{j-1}^m)^{1/m}$ for $j \geq 2$,
- $B_j = (C_1^{-2} K C_2^2 K^2 m^3 \rho^m A_{j-1}^m)^{1/m}$ for $j \geq 2$.

Note that the case $j = 1$ is done. We assume that (i) and (ii) hold for $j$ and prove that they hold for $j+1$. To prove (i), assume $\frac{(j+1)p}{j+p} < q < p$.

Fix $n$ and let $\{\mu_i\}_{i=1}^{n}$ be such that $\sum_{i=1}^{n} |\mu_i|^{q} = 1$, where $s_j = \frac{p}{(j+1)q-jp}$. Define $P_j : X \to Y$ by

$$P_j x = \sum_{k=1}^{n} |\mu_k|^{1/q} a_k^m y_k, \text{ if } x = \sum_{k=1}^{\infty} a_k x_k.$$ 

Putting $l_j = \frac{pq}{jp-jq}$ and $t_j = \frac{pq}{(j+1)q-jp}$, we have $\frac{1}{l_j} + \frac{1}{l_j} = \frac{1}{p}, ml_j = \frac{mpq}{jp-jq}$ and so
\[ \|P_j x\| = \left( \sum_{k=1}^{n} |\mu_k|^{1/q} a_k^m y_k \right) \leq C_2 \left( \sum_{k=1}^{n} |\mu_k|^{1/q} a_k^m \right)^{1/p} \]

\[ \leq C_2 \left( \sum_{k=1}^{n} |\mu_k|^{1/q} |j^l_j| \right)^{1/t_j} \left( \sum_{k=1}^{n} |a_k^m|^{1/j} \right)^{1/j} \]

\[ \leq C_2 \left( \sum_{k=1}^{n} |\mu_k|^s_j \right)^{1/t_j} \left( \sum_{k=1}^{n} |a_k^m|^{1/j} \right)^{1/j} \leq C_2 A_j^m \|x\|^m. \]

Writing \( L = C_2 A_j^m \), we obtain \( \|P_j\| \leq L \) and \( \|P_j\|_{\text{ass}(q;1)} \leq KL \) and achieve the estimate below:

\[ \left( \sum_{k=1}^{n} |a_k^m|^{1/q} C_1^q \right)^{1/q} \leq \left( \sum_{k=1}^{n} \|a_k^m\|_{\mu_k}^{1/q} y_k \right)^{1/q} = \left( \sum_{k=1}^{n} \|P_j a_k x_k\|^q \right)^{1/q} \]

\[ \leq \|P_j\|_{\text{ass}(q;1)} \left( \frac{n}{\|x\|} \right)^m \leq \|P_j\|_{\text{ass}(q;1)} 2^m \max_{\varepsilon_k \in \{-1,1\}} \left\{ \left| \sum_{k=1}^{n} \varepsilon_k a_k x_k \right| \right\}^m \]

\[ \leq \|P_j\|_{\text{ass}(q;1)} (2\rho \|x\|)^m \leq KL 2^m \rho^m \|x\|^m. \] (2.4)

Since \( \frac{1}{s_j} + \frac{1}{(s_j - 1)} = 1 \), we have

\[ \left( \sum_{k=1}^{n} |a_k|^s_j \right)^{1/(s_j - 1)} = \sup \left\{ \left( \sum_{k=1}^{n} |\mu_k|^{s_j} \right) ; \sum_{k=1}^{n} |\mu_k|^s_j = 1 \right\} \]

\[ \leq \sup \left\{ \sum_{k=1}^{n} |\mu_k|^s_j ; \sum_{k=1}^{n} |\mu_k|^s_j = 1 \right\}. \] (2.5)

It is plain that (2.4) holds whenever \( \sum_{k=1}^{n} |\mu_k|^s_j = 1 \). Thus, by (2.4) and (2.5), it follows that

\[ \left( \sum_{k=1}^{n} |a_k|^s_j \right)^{1/(s_j - 1)} \leq \left( C_1^{-1} KL 2^m \rho^m \|x\|^m \right)^q, \]

and then

\[ \left( \sum_{k=1}^{n} |a_k|^s_j \right)^{1/(s_j - 1)} \leq \left( C_1^{-1} KL 2^m \rho^m \|x\|^m \right)^{1/m}. \]

Since \( s_j \frac{mq}{s_j - 1} = \frac{mpq}{(j+1)p - (j+1)q} \) and \( n \) is arbitrary, we have \( \mu X, (x_n) \leq \frac{mpq}{(j+1)p - (j+1)q} \) and

\[ \left( \sum_{k=1}^{n} |a_k|^s_j \right)^{1/(s_j - 1)} \leq \left( C_1^{-1} C_2 A_j^m 2^m \rho^m \|x\|^m \right)^{1/m}. \]

which proves (i) for \( j + 1 \). To prove (ii), assume \( q \leq \frac{(j+1)p}{j+2} \). Let us invoke, for a fixed \( n, S \) again. We have \( mp \geq \frac{mpq}{(j+1)p - (j+1)q} = \frac{s_j}{s_j - 1} \), so
Theorem 2.3. Let $X$ be an infinite dimensional Banach space with a normalized unconditional basis $(x_n)$ and $\mathcal{P}_{\text{as}(q; 1)}(mX; Y) = \mathcal{P}(mX; Y)$. Then $\mu_{X, (x_n)} \leq mq$ if:

(i) $q < 1$ and $\dim Y < \infty$;

(ii) $q < \cot Y$ and $\dim Y = \infty$.

Proof. (i) Since Banach spaces of the same finite dimension are isomorphic, it suffices to deal with the case $Y = \mathbb{K}^n$. Moreover, we just need to treat the case $n = 1$. In fact, if $\mathcal{P}_{\text{as}(q; 1)}(mX; \mathbb{K}^n) = \mathcal{P}(mX; \mathbb{K}^n)$, we have $\mathcal{P}_{\text{as}(q; 1)}(mX; \mathbb{K}) = \mathcal{P}(mX; \mathbb{K})$. For the case $n = 1$, it is enough to call on Lemma 2.1 with $p = C_1 = C_2 = y_1 = \cdots = y_n = 1$.

(ii) If $q < 2$, from [4, Theorem 4.2], we know that $Y$ finitely factors the inclusion $\ell_2 \to \ell_\infty$, so it suffices to choose $p = 2$ in Lemma 2.1.

For $2 \leq q < \cot Y$, since $Y$ finitely factors the inclusion $\ell_{\cot Y} \to \ell_\infty$ (see [4, p. 304]), we just have to apply Lemma 2.1 with $p = \cot Y$. □

The case $m = 1$ is useful and seems to be unknown. So, due to the relevance of the linear theory, it worths to extract the linear case (written in a more convenient form) from Theorem 2.3:

Corollary 2.1. Let $X$ be an infinite dimensional Banach space with unconditional basis $(x_n)$ and $Y$ be an infinite dimensional Banach space. If $q < \cot Y$ and $\mu_{X, (x_n)} > q$, then $\mathcal{L}_{\text{as}(q; 1)}(X; Y) \neq \mathcal{L}(X; Y)$.
A first application of Corollary 2.1 is illustrative and shows how it can be explored:

**Example 2.1.** If \( Y \) is an infinite dimensional Banach space and \( \mathcal{L}_{as(q;1)}(c_0; Y) = \mathcal{L}(c_0; Y) \), then \( q \geq \text{cot } Y \). In fact, from Corollary 2.1 it follows that

\[
\inf \{ q ; \mathcal{L}_{as(q;1)}(c_0; Y) = \mathcal{L}(c_0; Y) \} = \text{cot } Y. \tag{2.6}
\]

A well known result due to Maurey and Pisier [7] says that

\[
\inf \{ q ; \text{id}_Y \in \mathcal{L}_{as(q;1)}(Y; Y) \} = \text{cot } Y \tag{2.7}
\]

and thus, combining (2.6) and (2.7), we get

\[
\inf \{ q ; \mathcal{L}_{as(q;1)}(c_0; Y) = \mathcal{L}(c_0; Y) \} = \inf \{ q ; \text{id}_Y \in \mathcal{L}_{as(q;1)}(Y; Y) \}.
\]

More generally, if \( X \) has an unconditional Schauder basis \( (x_n) \) and \( \mu_{X,(x_n)} > \text{cot } Y \), then the estimate of Corollary 2.1 provides

\[
\inf \{ q ; \mathcal{L}_{as(q;1)}(X; Y) = \mathcal{L}(X; Y) \} = \text{cot } Y.
\]

**Remark 2.1.** Let us note that Theorem 2.3 is optimal in many directions.

- In (i) we cannot expect the result for \( q \geq 1 \). In fact, \( \mathcal{P}_{as(1;1)}(\ell_2, Y) = \mathcal{P}(\ell_2, Y) \) [6, Proposition 2.12] and \( \mu_{\ell_2} = 3 > 2 = mq \).
- In (ii) we cannot expect the result for \( q \geq \text{cot } Y \). In fact, \( \mathcal{L}_{as(2;1)}(c_0; \ell_2) = \mathcal{L}(c_0; \ell_2) \) and \( \mu_{c_0} = \infty > 2 = mq \).
- The estimate \( mq \) is also optimal. In fact, for \( q \geq \frac{2}{m} \) we have \( \mathcal{P}_{as(q;1)}(m\ell_{mq}; Y) = \mathcal{P}(m\ell_{mq}; Y) \) for every \( Y \) [2, Theorem 2.2] and \( \mu_{\ell_{mq}} = mq \), showing that the estimate \( mq \) cannot be improved.
- Lemma 2.1 and its consequences obviously have natural adaptations to multilinear mappings.

In [8, Theorem 7] it is shown that if \( m \in \mathbb{N}, r \geq 2 \) and \( Y \) is an infinite dimensional Banach space with \( \text{cot } Y = \infty \), then

\[
\mathcal{P}_{as(q;1)}(m\ell_r; Y) = \mathcal{P}(m\ell_r; Y) \quad \iff \quad q \geq \frac{r}{m}.
\]

It is also shown that if \( Y \) has cotype \( \text{cot } Y \), then \( \mathcal{P}_{as(q;1)}(m\ell_\infty; Y) = \mathcal{P}(m\ell_\infty; Y) \iff q \geq \text{cot } Y \).

If \( Y = \mathbb{K} \), in [9, Corollary 7] it is proved that for \( q \leq \frac{1}{2} \) we have \( \mathcal{P}_{as(q;1)}(m\ell_r) = \mathcal{P}(m\ell_r) \iff q \geq \frac{r}{m} \).

As a consequence of Theorem 2.3, we have a result showing all possible coincidence results for \( (q, 1) \)-summing polynomials on \( \ell_r \) spaces \( (r \geq 2) \), generalizing the aforementioned results.

**Corollary 2.2.** Let \( m \in \mathbb{N} \).

(i) If \( r \geq 1 \), \( \text{dim } Y = \infty \) and \( Y \) has cotype \( \text{cot } Y \), we have

\[
\mathcal{P}_{as(q;1)}(m\ell_r; Y) = \mathcal{P}(m\ell_r; Y) \quad \iff \quad q \geq \min \left\{ \frac{r}{m}, \text{cot } Y \right\}.
\]

(ii) If \( r \geq 2 \), \( \text{dim } Y = \infty \) and \( Y \) has cotype \( \text{cot } Y \), we have

\[
\mathcal{P}_{as(q;1)}(m\ell_r; Y) = \mathcal{P}(m\ell_r; Y) \quad \iff \quad q \geq \min \left\{ \frac{r}{m}, \text{cot } Y \right\}.
\]
(iii) For \( r \geq 2 \), we have \( \mathcal{P}_{\text{as}(q;1)}(m_\ell_r) = \mathcal{P}(m_\ell_r) \iff q \geq \min\{\frac{r}{m}, 1\} \).

**Proof.** (i) Suppose \( q < \min\{\frac{r}{m}, \cot Y\} \). Since \( q < \cot Y \), \( \ell_r \) has unconditional basis and \( \mathcal{P}_{\text{as}(q;1)}(m_\ell_r; Y) = \mathcal{P}(m_\ell_r; Y) \), Theorem 2.3 provides \( r = \mu_\ell_r \leq mq \) (contradiction).

(ii) Suppose that \( q \geq \min\{\frac{r}{m}, \cot Y\} \). Then \( q \geq \frac{r}{m} \) or \( q \geq \cot Y \). If \( q \geq \frac{r}{m} \), since \( \ell_r \) has cotype \( r \), [2, Theorem 2.2] asserts that \( \mathcal{P}_{\text{as}(q;1)}(m_\ell_r; X) = \mathcal{P}(m_\ell_r; X) \) for every \( X \). In the case that \( q \geq \cot Y \), since \( Y \) has cotype \( \cot Y \), we have \( \mathcal{P}_{\text{as}(\cot Y;1)}(m_\ell_r; Y) = \mathcal{P}(m_\ell_r; Y) \) and, a fortiori, \( \mathcal{P}_{\text{as}(q;1)}(m_\ell_r; Y) = \mathcal{P}(m_\ell_r; Y) \). The converse follows from (i).

(iii) Suppose that \( q \geq \min\{\frac{r}{m}, 1\} \). If \( q \geq 1 \), from [6, Theorem 2.2] we have that \( \mathcal{P}_{\text{as}(q;1)}(m_X) = \mathcal{P}(m_X) \) for every \( X \). If \( q \geq \frac{r}{m} \), since \( \ell_r \) has cotype \( r \), [2, Theorem 2.2] yields that \( \mathcal{P}_{\text{as}(q;1)}(m_\ell_r) = \mathcal{P}(m_\ell_r) \). Reciprocally, suppose \( \mathcal{P}_{\text{as}(q;1)}(m_\ell_r) = \mathcal{P}(m_\ell_r) \). If \( q < \min\{r/m, 1\} \), then \( \mu_\ell_r = r > mq \), but this contradicts Theorem 2.3(i). \( \Box \)

**Remark 2.2.** A well known result due (independently) to G. Bennet [1] and B. Carl [3] (see, e.g., [4, p. 209]) says that if \( 1 \leq r \leq s \leq \infty \) and \( s \geq 2 \), then the inclusion \( \ell_r \to \ell_s \) is \((r;1)\)-summing and that this result cannot be improved. We thus have the noncoincidence \( \mathcal{L}_{\text{as}(q;1)}(\ell_r; \ell_s) \neq \mathcal{L}(\ell_r; \ell_s) \) if \( 1 \leq q < r \) and \( s \geq 2 \). In the following two cases:

(i) \( 1 < r < 2 \) and

(ii) \( r, s \geq 2 \) and \( q < \min\{r, s\} \),

Corollary 2.2 (with \( m = 1 \)) extends the aforementioned noncoincidence in the sense that \( \ell_s \) can be replaced by any infinite dimensional Banach space \( Y \) having cotype \( s = \cot Y \).

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**References**