Approximation by Nörlund Means of Double Fourier Series for Lipschitz Functions*

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We study the rate of uniform approximation by Nörlund means of the rectangular partial sums of the double Fourier series of a function \( f(x, y) \) belonging to the class Lip\( \alpha \), \( 0 < \alpha \leq 1 \), on the two-dimensional torus \( -\pi < x, y \leq \pi \). As a special case we obtain the rate of uniform approximation by double Cesàro means.

1. NÖRLUND SUMMABILITY OF DOUBLE NUMERICAL SEQUENCES

Let \( P = \{ p_{jk} : j, k = 0, 1, \ldots \} \) be a double sequence of nonnegative numbers, \( p_{00} > 0 \). Set

\[
P_{mn} = \sum_{j=0}^{m} \sum_{k=0}^{n} p_{jk} \quad (m, n = 0, 1, \ldots).
\]

Given a double sequence \( \{ s_{jk} : j, k = 0, 1, \ldots \} \) of complex numbers, the Nörlund means \( t_{mn} \) are defined by

\[
t_{mn} = \frac{1}{P_{mn}} \sum_{j=0}^{m} \sum_{k=0}^{n} p_{m-j, n-k} s_{jk} \quad (m, n = 0, 1, \ldots).
\]

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We say that the Nörlund method generated by \( \mathcal{P} \), or simply the \( \mathcal{P} \)-method of summability, is regular if whenever \( s_{mn} \) tends to a finite limit \( s \) as \( m, n \to \infty \) and the \( s_{mn} \) are bounded for \( m, n = 0, 1, \ldots \), then \( t_{mn} \) also tends to the same limit \( s \) as \( m, n \to \infty \).

**Theorem A** [3, p. 39]. If \( \mathcal{P} = \{ p_{jk} \geq 0; j, k = 0, 1, \ldots; p_{00} > 0 \} \), then the necessary and sufficient conditions for the regularity of the \( \mathcal{P} \)-method of summability are

\[
\lim_{m,n \to \infty} \frac{1}{P_{m,n}} \sum_{k=0}^{n} p_{m-n,k} = 0 \quad (j = 0, 1, \ldots; m \geq j)
\]

and

\[
\lim_{m,n \to \infty} \frac{1}{P_{m,n}} \sum_{j=0}^{m} p_{j,n-k} = 0 \quad (k = 0, 1, \ldots; n \geq k).
\]

The \((C, \beta, \gamma)\)-summability, \( \beta, \gamma > -1 \), is a particular case of the Nörlund summability, where \( \mathcal{P} \) is given by

\[
p_{jk} = A_j^{\beta} A_k^{\gamma} \quad (j, k = 0, 1, \ldots)
\]

(even this is a factorable case), where

\[
A_j^{\beta} = \binom{x + 1}{l} = \frac{(x + 1)(x + 2) \cdots (x + l)}{l!}
\]

for \( l = 1, 2, \ldots \) and \( A_0^{\beta} = 1 \). Then, as is known,

\[
P_{mn} = A_m^{\beta} A_n^{\gamma} \quad (m, n = 0, 1, \ldots).
\]

Furthermore, there exist two positive constants \( C_1 \) and \( C_2 \) such that

\[
C_1 \leq \frac{A_{l}^{\beta}}{(l+1)^2} \leq C_2 \quad (l = 0, 1, \ldots; x > -1)
\]

(see, e.g., [5, p. 77]).

2. Nörlund Means for Double Fourier Series

Let \( f(x, y) \) be a complex-valued function defined on the two-dimensional real torus \( Q: -\pi < x \leq \pi, -\pi < y \leq \pi \). If \( f \in L^1(Q) \), then its double Fourier series is

\[
f(x, y) \sim \sum_{j=\tau}^{\tau} \sum_{k=-\tau}^{\tau} c_{jk} e^{i(jx + ky)} \tag{2.1}
\]
where

\[ c_{jk} = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(s, t) e^{-i(js + k_t)} ds \, dt \quad (j, k = \ldots, -1, 0, 1, \ldots). \]

We associate with (2.1) the double sequence of (symmetric) rectangular partial sums

\[ s_{mn}(x, y) = \sum_{j=-m}^{m} \sum_{k=-n}^{n} c_{jk} e^{i(jx + k_y)} \quad (m, n = 0, 1, \ldots). \]

Now, the Nörlund means for (2.1) are defined as those for the sequence \( \{s_{mn}(x, y)\} \):

\[ t_{mn}(x, y) = \frac{1}{P_{mn}} \sum_{j=-m}^{m} \sum_{k=-n}^{n} p_{m-j, n-k} s_{jk}(x, y) \quad (m, n = 0, 1, \ldots). \]

The representation

\[ t_{mn}(x, y) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x + s, y + t) K_{mn}(s, t) \, ds \, dt \quad (2.2) \]

plays a central role, where the Nörlund kernel \( K_{mn}(x, t) \) is defined by

\[ K_{mn}(s, t) = \frac{1}{P_{mn}} \sum_{j=-m}^{m} \sum_{k=-n}^{n} p_{m-j, n-k} D_j(s) D_k(t) \quad (m, n = 0, 1, \ldots), \quad (2.3) \]

and \( D_j(s) \) and \( D_k(t) \) are the Dirichlet kernels in terms of \( s \) and \( t \), respectively, e.g.,

\[ D_j(s) = \frac{1}{2} + \sum_{\nu=1}^{j} \cos \nu s = \frac{\sin(j + \frac{1}{2}) s}{2 \sin \frac{1}{2} s} \quad (j = 0, 1, \ldots). \]

From (2.2) it follows immediately that

\[ t_{mn}(x, y) - f(x, y) = \frac{4}{\pi^2} \int_{0}^{\pi} \int_{0}^{\pi} \phi_{xy}(s, t) K_{mn}(s, t) \, ds \, dt \quad (2.4) \]

where

\[ \phi_{xy}(s, t) = \frac{1}{4} \{ f(x + s, y + t) + f(x - s, y + t) + f(x + s, y - t) + f(x - s, y - t) - 4f(x, y) \}. \]
We say that the function $f$ satisfies a Lipschitz condition of order $\alpha > 0$, in symbols $f \in \text{Lip}_\alpha$, if
\[
\omega(\delta; f) = \sup_{x, \delta > 0} \sup_{\forall \|x, y\| \leq \delta} |f(x + x, y + y) - f(x, y)| \leq C\delta^\alpha \quad (\delta > 0)
\]
with a constant $C$ independent of $\delta$. The quantity $\omega(\delta; f)$ is called the (total) modulus of continuity of $f$. As usual, we consider $f$ as defined over the two-dimensional real Euclidean space $\mathbb{R}^2$ extended periodically in each variable (with period $2\pi$).

Clearly, if $f \in \text{Lip}_\alpha$ for some $\alpha > 0$, then $f$ is necessarily continuous everywhere. Only the case $0 < \alpha \leq 1$ is interesting. If $\alpha > 1$, then $\partial f/\partial x$ and $\partial f/\partial y$ exist and are zero everywhere, so $f$ must be a constant.

Condition (2.5) can be rewritten as
\[
|f(x + s, y + t) - f(x, y)| \leq C\{s^2 + t^2\}^{\alpha/2}
\]
for every real $x, y, s$, and $t$; or equivalently,
\[
|f(x + s, y + t) - f(x, y)| \leq C(|s|^\alpha + |t|^\alpha). \quad (2.6)
\]

Indeed, for every real $s, t$ and $0 < \alpha \leq 2$
\[
\{s^2 + t^2\}^{\alpha/2} \leq |s|^\alpha + |t|^\alpha \leq 2\{s^2 + t^2\}^{\alpha/2}.
\]
Here the first inequality is the Minkowski one, while the second is trivial.

Condition (2.6) obviously yields
\[
|\phi_{x,y}(s, t)| \leq C(|s|^\alpha + |t|^\alpha). \quad (2.7)
\]
During the proofs we actually use inequality (2.7) which is, in certain cases, weaker than (2.6).

3. MAIN RESULTS

We will use the notations
\[
A_{10} = p_{j,k} = p_{j+1,k}, \quad A_{01} = p_{j,k} = p_{j,k+1},
\]
and
\[
A_{11} = p_{j,k} = p_{j+1,k} - p_{j+1,k+1} + p_{j,k+1} + p_{j,k+1} \quad (j, k = 0, 1, \ldots).
\]
The double sequence \( \{p_{jk}\} \) is nondecreasing if \( \Delta_{10} p_{jk} \leq 0 \) and \( \Delta_{01} p_{jk} \leq 0 \), and is nonincreasing if \( \Delta_{10} p_{jk} \geq 0 \) and \( \Delta_{01} p_{jk} \geq 0 \) for every \( j, k = 0, 1, \ldots \). We also set

\[
q_{mn} = \frac{1}{P_{mn}} \sum_{k=0}^{n} p_{mk},
\]
\[
r_{mn} = \frac{1}{P_{mn}} \sum_{j=0}^{m} p_{jm} \quad (m, n = 0, 1, \ldots).
\]

First we consider the case where \( p_{jk} \) is nondecreasing. Then

\[
(m + 1) q_{mn} = \frac{1}{P_{mn}} \sum_{j=0}^{m} \sum_{k=0}^{n} p_{jk},
\]
\[
\geq \frac{1}{P_{mn}} \sum_{j=0}^{m} \sum_{k=0}^{n} p_{jk} = 1,
\]
\[
(3.1)
\]
and similarly,

\[
(n + 1) r_{mn} \geq 1.
\]
\[
(3.2)
\]

We also have

\[
P_{mn} \leq (m + 1)(n + 1) p_{mn} \quad (m, n = 0, 1, \ldots).
\]

In the sequel, we need the opposite inequality:

\[
\frac{(m + 1)(n + 1) p_{mn}}{P_{mn}} = O(1).
\]
\[
(3.3)
\]

This condition is satisfied, for example, if \( p_{jk} \) has a power growth both in \( j \) and in \( k \); i.e.,

\[
p_{jk} = (j + 1)^{\beta} (k + 1)^{\gamma} \quad \text{for some } \beta, \gamma \geq 0.
\]

Now, condition (3.3) implies that

\[
(m + 1) q_{mn} = \frac{m + 1}{P_{mn}} \sum_{k=0}^{n} p_{mk}
\]
\[
\leq \frac{m + 1}{P_{mn}} (n + 1) p_{mn} = O(1)
\]
\[
(3.4)
\]
and

\[
(n + 1) r_{mn} = O(1).
\]
\[
(3.5)
\]
In particular, the conditions of regularity are satisfied:

$$\lim_{m,n \to \infty} q_{mn} = \lim_{m,n \to \infty} r_{mn} = 0.$$ 

Thus, we may assume that

$$q_{mn} < \pi \quad \text{and} \quad r_{mn} < \pi \quad (m, n = 0, 1, \ldots).$$

**Theorem 1.** Let \( \{ p_{jk} > 0; j, k = 0, 1, \ldots \} \) be a nondecreasing double sequence such that \( \Delta_{11} p_{jk} \) is of fixed sign and condition (3.3) is satisfied. If \( f \in \text{Lip} \alpha, 0 < \alpha < 1 \), then

$$\sup_{(x, y) \in Q} |t_{mn}(x, y) - f(x, y)| = O(q_{mn}^\alpha + r_{mn}^\alpha) \quad \text{if} \quad 0 < \alpha < 1.$$

$$= O\left( q_{mn} \log \frac{\pi}{q_{mn}} + r_{mn} \log \frac{\pi}{r_{mn}} \right) \quad \text{if} \quad \alpha = 1$$

(3.6)

Second we treat the case where \( p_{jk} \) is nonincreasing. Then

$$(m + 1) q_{mn} \leq 1 \quad \text{and} \quad (n + 1) r_{mn} \leq 1 \quad (3.7)$$

(cf. (3.1) and (3.2)).

**Theorem 2.** Let \( \{ p_{jk} \geq 0; j, k = 0, 1, \ldots; p_{00} > 0 \} \) be a nonincreasing double sequence such that \( \Delta_{11} p_{jk} \) is of fixed sign. If \( f \in \text{Lip} \alpha, 0 < \alpha \leq 1 \), then

$$\sup_{(x, y) \in Q} |t_{mn}(x, y) - f(x, y)|$$

$$= O\left\{ \frac{1}{D_{nn}} \sum_{j=0}^{m} \sum_{k=0}^{n} \left( \frac{p_{jk}}{(j+1)^{2 \alpha + 1}(k+1)} + \frac{p_{jk}}{(j+1)(k+1)^{2 \alpha + 1}} \right) \right\}. \quad (3.8)$$

In the special case where

$$\lim_{m,n \to \infty} p_{mn} > 0, \quad (3.9)$$

we have

$$\frac{1}{(m + 1) q_{mn}} \leq \frac{p_{00}}{p_{nn}} = O(1) \quad \text{and} \quad \frac{1}{(n + 1) r_{mn}} = O(1)$$

and the right-hand side of (3.8) reduces to that of (3.6).
Corollary 1. Let \( \{ p_{jk} > 0 : j, k = 0, 1, \ldots \} \) be a nonincreasing double sequence such that \( \Delta_{11} p_{jk} \) is of fixed sign and condition (3.9) is satisfied. If \( f \in \text{Lip}_\alpha \), \( 0 < \alpha \leq 1 \), then statement (3.6) holds.

The approximation rate for \((C, \beta, \gamma)\)-summability immediately follows from Theorem 1 (for \( \beta, \gamma \geq 1 \)) and Theorem 2 (for \( \alpha < \beta, \gamma < 1 \)).

**Corollary 2.** If \( f \in \text{Lip}_\alpha \), \( 0 < \alpha < 1 \), and \( \beta, \gamma \geq \alpha \), then

\[
\sup_{(x,y) \in \mathbb{Q}} \left| \frac{1}{A_{m}^{\beta} A_{n}^{\gamma}} \sum_{j=0}^{m} \sum_{k=0}^{n} A_{m}^{\beta-1} \nu_{j-k} s_{jk}(x,y) - f(x,y) \right| = O\left( \frac{1}{(m+1)^{2}} + \frac{1}{(n+1)^{2}} \right)
\]

if \( \beta > \alpha \) and \( \gamma > \alpha \),

\[
= O\left( \frac{\log(m+2)}{(m+1)^{2}} + \frac{1}{(n+1)^{2}} \right)
\]

if \( \beta = \alpha \) and \( \gamma > \alpha \),

\[
= O\left( \frac{\log(m+2)}{(m+1)^{2}} + \frac{\log(n+2)}{(n+1)^{2}} \right)
\]

if \( \beta = \gamma = \alpha \).

Theorem 1 is an extension of that announced by T. Singh (see [2, p. 364]) from the one-dimensional case to the two-dimensional case, while Theorem 2 is an extension of that in [1]. Our method clearly applies to higher-dimensional Fourier series as well. The extensions of our results to \( d \)-dimensional cases, where \( d \) is an integer greater than 2, are straightforward.

4. ESTIMATION OF THE NÖRLUND KERNEL

We will use some well-known estimates. For \( j = 0, 1, \ldots \)

\[
|D_{j}(s)| < j + 1 \quad \text{for every } s. \quad (4.1)
\]

For \( a, b = 0, 1, \ldots ; a \leq b \),

\[
\sum_{j=a}^{b} \sin \left( j + \frac{1}{2} \right) s = \frac{\cos as - \cos (b + 1) s}{2 \sin \frac{1}{2} s},
\]

whence, on account of the inequality

\[
\frac{\sin s}{s} \geq \frac{2}{\pi} \quad \text{for } 0 < s \leq \frac{\pi}{2}, \quad (4.2)
\]
we obtain
\[
\left| \sum_{j=0}^{b} \sin \left( j + \frac{1}{2} \right) \frac{s}{\pi} \right| \leq \frac{\pi}{s} \quad \text{for } 0 < s \leq \pi. \tag{4.3}
\]

Similarly,
\[
\left| \sum_{j=0}^{b} D_j(s) \right| = \left| \frac{\cos as - \cos (b + 1) s}{(2 \sin \frac{1}{2} s)^2} \right| \leq \frac{\pi^2}{2s^2} \quad \text{for } 0 < s \leq \pi. \tag{4.4}
\]

We note that \(1/(b + 1) \sum_{j=0}^{b} D_j(s)\) is the Fejér kernel (cf. [5, pp. 49, 88]). The Nörlund kernel \(K_{mn}(s, t)\) is defined by (2.3).

\[\text{LEMMA 1.} \quad \text{Let } \{p_{jk} > 0; j, k = 0, 1, \ldots \} \text{ be a nondecreasing double sequence such that } A_{11} p_{jk} \text{ is of fixed sign. Then}
\]
\[
|K_{mn}(s, t)| \leq (m + 1)(n + 1) \quad \text{for every } s \text{ and } t,
\]
\[
\leq \frac{\pi^2}{2} \frac{1}{P_{mn} s^2} \sum_{k=0}^{m} (k + 1) p_{m,n,k} \quad \text{for every } t \text{ and } 0 < s \leq \pi,
\]
\[
\leq \frac{\pi^2}{2} \frac{1}{P_{mn} t^2} \sum_{j=0}^{n} (j + 1) p_{m,j,n} \quad \text{for every } s \text{ and } 0 < t \leq \pi,
\]
\[
\leq \frac{3\pi^4}{4} \frac{p_{mn}}{P_{mn} s^2 t^2} \quad \text{for every } 0 < s, t \leq \pi. \tag{4.5}
\]

\[\text{Proof.} \quad \text{By (4.1),}
\]
\[
|K_{mn}(s, t)| \leq \frac{1}{P_{mn}} \sum_{j=0}^{m} \sum_{k=0}^{n} p_{m,j,n-k} |D_j(s)| |D_k(t)|
\]
\[
\leq \frac{1}{P_{mn}} \sum_{j=0}^{m} \sum_{k=0}^{n} (j + 1)(k + 1) p_{m-j,n-k}
\]
\[
= \frac{1}{P_{mn}} \sum_{j=0}^{m} \sum_{k=0}^{n} p_{jk} \leq (m + 1)(n + 1),
\]

which is (4.5i). The monotonicity of the \(p_{jk}\) is not used here.

Again from (4.1),
\[
P_{mn} |K_{mn}(s, t)| \leq \sum_{k=0}^{n} \left| \sum_{j=0}^{m} p_{m-j,n-k} D_j(s) \right| |D_k(t)|
\]
\[
\leq \sum_{k=0}^{n} (k + 1) \left| \sum_{j=0}^{m} p_{m-j,n-k} D_j(s) \right|. \tag{4.6}
\]
For each $k$, we rewrite the inner sum by an Abel’s transformation (see, e.g., [5, p. 3]) as

\[
\sum_{j=0}^{m} \sum_{t=0}^{j} \Delta_{10} p_{m-j,n-k} D_{j}(s) = - \sum_{j=1}^{n} \sum_{t=0}^{j} \Delta_{10} p_{m-j,n-k} D_{j}(s)
\]

\[
+ p_{0,n-k} \sum_{t=0}^{m} D_{t}(s),
\]

whence, by (4.4) and the assumption that $p_{jk}$ is nondecreasing in $j$, we get

\[
\left| \sum_{j=0}^{m} p_{m-j,n-k} D_{j}(s) \right| \leq \frac{\pi^2}{2s^2} \left( \sum_{j=1}^{m} \Delta_{10} p_{m-j,n-k} + p_{0,n-k} \right)
\]

\[
= \frac{\pi^2}{2s^2} p_{m,n-k}.
\]  

(4.7)

Combining (4.6) and (4.7) yields (5.4ii).

Equation (4.5iii) can be shown in a similar way.

To prove (4.5iv), we first perform a double Abel’s transformation (see, e.g., [4]):

\[
P_{mn} K_{mn}(s, t)
\]

\[
= \sum_{j=0}^{m} \sum_{k=0}^{n} \Delta_{11} p_{m-j,n-k} \sum_{a=0}^{j-1} D_{a}(s) \sum_{b=0}^{k-1} D_{b}(t)
\]

\[
- \sum_{j=0}^{n} \sum_{k=0}^{n} \Delta_{10} p_{m-j,n-k} \sum_{a=0}^{j} D_{a}(s) \sum_{b=0}^{k} D_{b}(t)
\]

\[
- \sum_{j=0}^{n} \sum_{k=0}^{n} \Delta_{00} p_{m-j,n-k} \sum_{a=0}^{j} D_{a}(s) \sum_{b=0}^{k} D_{b}(t)
\]

\[
+ p_{00} \sum_{a=0}^{m} D_{a}(s) \sum_{b=0}^{n} D_{b}(t),
\]

(4.8)

whence, by (4.4),

\[
P_{mn} |K_{mn}(s, t)|
\]

\[
\leq \frac{\pi^4}{4s^2t^2} \left( \sum_{j=1}^{m} \sum_{k=1}^{n} |\Delta_{11} p_{m-j,n-k}| + \sum_{j=1}^{m} \Delta_{10} p_{m-j,0} + \sum_{k=1}^{n} \Delta_{01} p_{0,n-k} + p_{00} \right).
\]

(4.9)
Since $A_{11}p_{jk}$ is of fixed sign,

$$
\sum_{j=1}^{m} \sum_{k=1}^{n} |A_{11}p_{m-j,n-k}| = \left| \sum_{j=1}^{m} \sum_{k=1}^{n} A_{11}p_{m-j,n-k} \right| = |p_{mn} - p_{0n} - p_{00}|.
$$

Returning to (4.9), if $A_{11}p_{jk} \geq 0$,

$$
P_{mn}|K_{mn}(s,t)| \leq \frac{\pi^4}{4s^2t^2} \left[ (p_{mn} - p_{m0} - p_{0n} + p_{00}) + (p_{m0} - p_{00}) + (p_{0n} - p_{00}) + p_{00} \right]
$$

$$
= \frac{\pi^4}{4s^2t^2} p_{mn},
$$

while if $A_{11}p_{jk} \leq 0$,

$$
P_{mn}|K_{mn}(s,t)| \leq \frac{\pi^4}{4s^2t^2} \left[ (-p_{mn} + p_{m0} + p_{0n} - p_{00}) + (p_{m0} - p_{00}) + (p_{0n} - p_{00}) + p_{00} \right]
$$

$$
= \frac{\pi^4}{4s^2t^2} (-p_{mn} + 2p_{m0} + 2p_{0n} - 2p_{00})
$$

$$
\leq \frac{3\pi^4}{4s^2t^2} p_{mn}.
$$

**Lemma 2.** Let $\{p_{jk} \geq 0: j, k = 0, 1, \ldots; p_{00} > 0\}$ be a nonincreasing double sequence such that $A_{11}p_{jk}$ is of fixed sign, and let $\sigma = \lfloor 1/s \rfloor$, $\tau = \lfloor 1/t \rfloor$ where $[\cdot]$ means the integral part. Then

$$
|K_{mn}(s,t)| \leq (m+1)(n+1)
$$

for every $s$ and $t$,

$$
\leq \frac{\pi(\pi+1)}{2} \frac{1}{P_{mn}s} \sum_{k=0}^{n} (k+1) \sum_{j=0}^{\sigma} p_{j,n-k} \quad \text{for every } t \text{ and } 0 < s \leq \pi,
$$

$$
\leq \frac{\pi(\pi+1)}{2} \frac{1}{P_{mn}t} \sum_{j=0}^{m} (j+1) \sum_{k=0}^{\tau} p_{m-j,k} \quad \text{for every } s \text{ and } 0 < t \leq \pi,
$$

$$
\leq \frac{\pi^2(1 + 2\pi + 3\pi^2)}{4} \frac{P_{\sigma\tau}}{P_{mn}st} \quad \text{for every } 0 < s, t \leq \pi.
$$

(4.10)
Proof. Equation (4.10i) coincides with (4.5i), which holds without any monotonicity condition, as we remarked in the proof of Lemma 1.

By (4.6) and (4.2),

\[ P_{mn} |K_{mn}(s, t)| \leq \sum_{k=0}^n (k+1) \left| \sum_{j=0}^m p_{m-j,n-k} D_j(s) \right| \]
\[ \leq \frac{\pi}{2s} \sum_{k=0}^n (k+1) \left| \sum_{j=0}^m p_{j,n-k} \sin \left( m-j+\frac{1}{2} \right) s \right|. \quad (4.11) \]

A simple estimate shows that, for each \( k \),

\[ \left| \sum_{j=0}^m p_{j,n-k} \sin \left( m-j+\frac{1}{2} \right) s \right| \leq \sum_{j=0}^m p_{j,n-k} \left| \sum_{j=0}^m p_{j,n-k} \sin \left( m-j+\frac{1}{2} \right) s \right|. \quad (4.12) \]

Using an Abel's transformation,

\[ \sum_{j=0}^m p_{j,n-k} \sin \left( m-j+\frac{1}{2} \right) s \]
\[ = \sum_{j=0}^m A_{10} p_{j,n-k} \sum_{l=\sigma+1}^m \sin \left( m-l+\frac{1}{2} \right) s \]
\[ + p_{m,n-k} \sum_{l=\sigma+1}^m \sin \left( m-l+\frac{1}{2} \right) s. \quad (4.13) \]

From (4.3), the fact that \( p_{jk} \) is nonincreasing in \( j \), and that \( 1/s < \sigma + 1 \), we can conclude that

\[ \left| \sum_{j=\sigma+1}^m p_{j,n-k} \sin \left( m-j+\frac{1}{2} \right) s \right| \leq \frac{\pi}{s} p_{\sigma+1,n-k} \]
\[ \leq \pi(\sigma + 1) p_{\sigma+1,n-k} \leq \pi \sum_{j=0}^\sigma p_{j,n-k}. \quad (4.14) \]

Now, the combination of (4.11), (4.12), and (4.14) provides (4.10ii). Equation (4.10iii) can be deduced similarly.

To prove (4.10iv), by (4.2) we begin with the inequality

\[ P_{mn} |K_{mn}(s, t)| \]
\[ = \left| \sum_{j=0}^m \sum_{k=0}^n p_{jk} D_{m-j}(s) D_{n-k}(t) \right| \]
\[ \leq \frac{\pi^2}{4st} \left| \sum_{j=0}^m \sum_{k=0}^n p_{jk} \sin \left( m-j+\frac{1}{2} \right) s \sin \left( n-k+\frac{1}{2} \right) t \right|. \quad (4.15) \]
We divide the double sum into four parts:

\[
\left| \sum_{j=0}^{m} \sum_{k=0}^{n} p_{jk} \sin \left( m - j + \frac{1}{2} \right) s \sin \left( n - k + \frac{1}{2} \right) t \right| 
\leq \sum_{j=0}^{m} \sum_{k=0}^{n} p_{jk} + \sum_{j=0}^{m} \sum_{k=\sigma+1}^{n} p_{jk} \sin \left( m - j + \frac{1}{2} \right) s 
+ \sum_{j=0}^{m} \sum_{k=\tau+1}^{n} p_{jk} \sin \left( n - k + \frac{1}{2} \right) t 
+ \sum_{j=\sigma+1}^{m} \sum_{k=\tau+1}^{n} p_{jk} \sin \left( m - j + \frac{1}{2} \right) s \sin \left( n - k + \frac{1}{2} \right) t 
= P_\sigma + A_1 + A_2 + A_3, \quad \text{say.} \tag{4.16}
\]

For \( A_1 \), we can perform an Abel’s transformation similar to (4.13) and conclude that

\[
\left| \sum_{j=\sigma+1}^{m} p_{jk} \sin \left( m - j + \frac{1}{2} \right) s \right| 
\leq \sum_{j=\sigma+1}^{m} A_{10} p_{jk} \left| \sum_{l=\sigma+1}^{j} \sin \left( m - l + \frac{1}{2} \right) s \right| 
+ p_{m\tau} \left| \sum_{l=\sigma+1}^{m} \sin \left( m - l + \frac{1}{2} \right) s \right| 
\leq \frac{\pi}{\alpha} p_{\sigma+1,\tau} \leq \pi (\sigma + 1) P_{\sigma+1,\tau} \leq \pi \sum_{j=0}^{\sigma} p_{jk}
\]

(cf. (4.14)), which results in

\[
A_1 \leq \pi P_{\sigma+1}, \tag{4.17}
\]

Analogously,

\[
A_2 \leq \pi P_{\sigma+1}. \tag{4.18}
\]

For \( A_3 \), we perform a double Abel’s transformation (cf. (4.8)):

\[
\sum_{j=\sigma+1}^{m} \sum_{k=\tau+1}^{n} p_{jk} \sin \left( m - j + \frac{1}{2} \right) s \sin \left( n - k + \frac{1}{2} \right) t 
= \sum_{j=\sigma+1}^{m-1} \sum_{k=\tau+1}^{n-1} A_{11} p_{jk} \sum_{a=\sigma+1}^{j} \sin \left( m - a + \frac{1}{2} \right) s \sum_{b=\tau+1}^{k} \sin \left( n - b + \frac{1}{2} \right) t
\]
\[ + \sum_{j=0}^{m-1} A_{10} P_{jm} \sum_{a=\sigma+1}^{n} \sin \left( m - a + \frac{1}{2} \right) s \sum_{b=\tau+1}^{n} \sin \left( n - b + \frac{1}{2} \right) t \]
\[ + \sum_{k=\tau+1}^{n-1} A_{01} P_{mk} \sum_{a=\sigma+1}^{m} \sin \left( m - a + \frac{1}{2} \right) s \sum_{b=\tau+1}^{k} \sin \left( n - b + \frac{1}{2} \right) t \]
\[ + P_{mn} \sum_{a=\sigma+1}^{m} \sin \left( m - a + \frac{1}{2} \right) s \sum_{b=\tau+1}^{n} \sin \left( n - b + \frac{1}{2} \right) t. \]

whence, by (4.3),

\[ \left| \sum_{j=0}^{m} \sum_{k=0}^{n} P_{jk} \sin \left( m - j + \frac{1}{2} \right) s \sin \left( n - k + \frac{1}{2} \right) t \right| \]
\[ \leq \frac{\pi^2}{s^2} \left\{ \sum_{j=\sigma+1}^{m} \sum_{k=\tau+1}^{n} A_{11} P_{jk} + \sum_{j=\sigma+1}^{m} A_{10} P_{jn} + \sum_{k=\tau+1}^{n-1} A_{01} P_{mk} + P_{mn} \right\} \]
\[ = \frac{3\pi^2}{s^2} p_{\sigma+1, \tau+1} \quad \text{if} \quad A_{11} P_{jk} > 0, \]
\[ = \frac{\pi^2}{s^2} (-2P_{mn} + 2P_{\sigma+1, n} + 2P_{0\tau+1} - P_{\sigma+1, \tau+1}) \]
\[ \leq \frac{3\pi^2}{s^2} p_{\sigma+1, \tau+1} \quad \text{if} \quad A_{11} P_{jk} \leq 0. \]

Thus, in any case,

\[ A_3 \leq \frac{3\pi^2}{s^2} p_{\sigma+1, \tau+1} \leq 3\pi^2(\sigma + 1)(\tau + 1) p_{\sigma+1, \tau+1} \]
\[ \leq 3\pi^2 \sum_{j=0}^{\sigma} \sum_{k=0}^{\tau} P_{jk} = 3\pi^2 P_{\sigma \tau}. \]

Putting (4.16)–(4.19) together yields

\[ \left| \sum_{j=0}^{m} \sum_{k=0}^{n} P_{jk} \sin \left( m - j + \frac{1}{2} \right) s \sin \left( n - k + \frac{1}{2} \right) t \right| \]
\[ \leq (1 + 2\pi + 3\pi^2) P_{\sigma \tau}. \]

Hence (4.15) immediately implies (4.10iv).
5. PROOFS OF THE THEOREMS

Proof of Theorem 1. We start with representation (2.4), decomposing the integral as follows:

\[
\frac{\pi^2}{4} |t_{mn}(x, y) - f(x, y)| \\
\leq \left\{ \int_0^\pi \int_0^{\pi} \int_0^\pi \int_0^{\pi} (s^2 + t^2) \, ds \, dz \right\}
\]

\[+ \left\{ \int_0^\pi \int_0^{\pi} |\phi_{xy}(s, t)| |K_{mn}(s, t)| \, ds \, dt \right\}
\]

\[= I_1 + I_2 + I_3 + I_4, \text{ say.} \quad (5.1)\]

Each time \(\phi_{xy}(s, t)\) is estimated by (2.7) and the appropriate estimate of Lemma 1 is substituted for the kernel \(K_{mn}(s, t)\).

By (4.5), for \(\alpha > 0\)

\[I_1 \leq (m + 1)(n + 1) \int_0^\pi \int_0^{\pi} (s^2 + t^2) \, ds \, dt \]

\[= \frac{1}{\alpha + 1} (m + 1)(n + 1) q_{mn} r_{mn} (q_{mn}^2 + r_{mn}^2).\]

By (3.4) and (3.5),

\[I_1 = O(q_{mn}^2 + r_{mn}^2). \quad (5.2)\]

By (4.5ii),

\[I_2 \leq \frac{\pi^2}{2} \frac{1}{P_{mn}} \sum_{k=0}^{n} (k + 1) p_{m,n-k} \int_0^\pi \int_0^{\pi} (s^2 + t^2) \, ds \, dt,\]

whence for \(0 < \alpha < 1\),

\[I_2 \leq \frac{\pi^2}{2} \frac{r_{mn}}{q_{mn} P_{mn}} \sum_{k=0}^{n} (k + 1) p_{m,n-k} \left( \frac{q_{mn}^2}{1-\alpha} + \frac{r_{mn}^2}{\alpha + 1} \right),\]

while for \(\alpha = 1\),

\[I_2 \leq \frac{\pi^2}{2} \frac{r_{mn}}{q_{mn} P_{mn}} \sum_{k=0}^{n} (k + 1) p_{m,n-k} \left( q_{mn} \log \frac{\pi}{q_{mn}} + \frac{1}{2} r_{mn} \right).\]
Using (3.5),
\[
\frac{r_{mn}}{q_{mn} P_{mn}} \sum_{k=0}^{n} (k+1) p_{m,n-k} \leq \frac{r_{mn}}{q_{mn} P_{mn}} (n+1) \sum_{k=0}^{n} p_{mk} = (n+1) r_{mn} = O(1).
\]
So,
\[
I_2 = O(q_{mn}^2 + r_{mn}^2) \quad \text{if} \quad 0 < \alpha < 1,
\]
\[
= O \left( q_{mn} \log \frac{\pi}{q_{mn}} + r_{mn} \right) \quad \text{if} \quad \alpha = 1.
\]
(5.3)

Similarly, this time using (4.5iii),
\[
I_3 = O(q_{mn}^2 + r_{mn}^2) \quad \text{if} \quad 0 < \alpha < 1,
\]
\[
= O \left( q_{mn} + r_{mn} \log \frac{\pi r_{mn}}{r_{mn}} \right) \quad \text{if} \quad \alpha = 1.
\]
(5.4)

By (4.5iv),
\[
I_4 \leq \frac{3 \pi^4}{4} \frac{p_{mn}}{P_{mn}} \int_{q_{mn}}^{\pi} \int_{r_{mn}}^{1} \frac{s^2 + t^2}{s \cos \theta} ds dt,
\]
whence for $0 < \alpha < 1$,
\[
I_4 \leq \frac{3 \pi^4}{4(1-\alpha)} \frac{p_{mn}}{q_{mn} r_{mn} P_{mn}} (q_{mn}^2 + r_{mn}^2),
\]
while for $\alpha = 1$,
\[
I_4 \leq \frac{3 \pi^4}{4} \frac{p_{mn}}{q_{mn} r_{mn} P_{mn}} \left( q_{mn} \log \frac{\pi}{q_{mn}} + r_{mn} \log \frac{\pi}{r_{mn}} \right).
\]

By (3.1), (3.2), and (3.3),
\[
\frac{p_{mn}}{q_{mn} r_{mn} P_{mn}} = \frac{(m+1)(n+1) p_{mn}}{(m+1) q_{mn} (n+1) r_{mn} P_{mn}} = O(1).
\]
Consequently,
\[
I_4 = O(q_{mn}^2 + r_{mn}^2) \quad \text{if} \quad 0 < \alpha < 1,
\]
\[
= O \left( q_{mn} \log \frac{\pi}{q_{mn}} + r_{mn} \log \frac{\pi}{r_{mn}} \right) \quad \text{if} \quad \alpha = 1.
\]
(5.5)

Collecting (5.1)–(5.5) together yields (3.6).
Proof of Theorem 2. We use decomposition (5.1) with $q_{mn}$ and $r_{mn}$ replaced by $\pi/(m+1)$ and $\pi/(n+1)$, respectively. For brevity, we denote by $Q_{mn}$ the quantity in braces on the right-hand side of (3.8).

By (4.10i), for $\alpha > 0$

$$I_1 \leq (m+1)(n+1) \int_0^\pi \int_0^\pi (s^2 + t^2) \, ds \, dt$$

$$\leq \frac{\pi^{x+2}}{\alpha + 1} \left( \frac{1}{(m+1)^x} + \frac{1}{(n+1)^x} \right). \quad (5.6)$$

Since $p_{jk}$ is nonincreasing, we trivially have

$$P_{jk} \geq (j+1)(k+1) p_{jk} \quad (j, k = 0, 1, \ldots).$$

Therefore,

$$\frac{1}{(m+1)^x} = \frac{1}{(m+1)^x} \sum_{j=0}^m \sum_{k=0}^n p_{jk}$$

$$\leq \frac{1}{(m+1)^x} \sum_{j=0}^m \sum_{k=0}^n \frac{P_{jk}}{(j+1)(k+1)}$$

$$\leq \frac{1}{P_{mn}} \sum_{j=0}^m \sum_{k=0}^n \frac{P_{jk}}{(j+1)(k+1)^{x+1}}.$$

and similarly,

$$\frac{1}{(n+1)^x} \leq \frac{1}{P_{mn}} \sum_{j=0}^m \sum_{k=0}^n \frac{P_{jk}}{(j+1)(k+1)^{x+1}}.$$

Combining (5.6) with the last two inequalities results in

$$I_1 = O(Q_{mn}). \quad (5.7)$$

By (4.10ii),

$$I_2 \leq \frac{\pi(\pi+1)}{2P_{mn}} \sum_{k=0}^n (k+1)$$

$$\times \left\{ \frac{\pi}{\pi(m+1)} \int_0^\pi \int_0^\pi (s^2 + t^2) \frac{\sum_{j=0}^\alpha}{s} s \, p_{j,n-k} \, dt \, ds \right\}$$

$$= \frac{\pi(\pi+1)}{2P_{mn}} \sum_{k=0}^n (k+1)$$

$$\times \left\{ \frac{\pi}{n+1} \int_0^\pi \int_0^\pi s^2 \frac{1}{\pi(n+1)} \sum_{j=0}^\alpha p_{j,n-k} \, ds \right\}$$

$$+ \frac{\pi^{x+1}}{(\alpha + 1)(n+1)^{x+1}} \left\{ \frac{\pi}{\pi(m+1)} \frac{1}{\alpha} \sum_{j=0}^\alpha p_{j,n-k} \, ds \right\}.$$
In each integration replace $s$ by $1/u$ (remembering that $\sigma = [1/s]$) to get

$$I_2 = \frac{O(1)}{P_{mn}} \sum_{k=0}^{n} (k + 1) \left\{ \frac{1}{n+1} \int_{1/\pi}^{(m+1)/\pi} \frac{1}{u^{1+1}} \sum_{j=1}^{[u]} p_{j,n-k} du \right. \\
\left. + \frac{1}{(n+1)^{2+1}} \int_{1/\pi}^{(m+1)/\pi} \frac{1}{u} \sum_{j=1}^{[u]} p_{j,n-k} du \right\}.$$

Then making a simple approximation to the integrals involved yields

$$I_2 = \frac{O(1)}{P_{mn}^{n}} \sum_{k=0}^{n} (k + 1) \left\{ \frac{1}{n+1} \sum_{l=0}^{m} \frac{1}{(l+1)^{2+1}} \sum_{j=0}^{l} p_{j,n-k} \right. \\
\left. + \frac{1}{(n+1)^{2+1}} \sum_{l=0}^{m} \frac{1}{l+1} \sum_{j=0}^{l} p_{j,n-k} \right\} \quad (5.8)$$

The first sum on the right is equal to

$$A = \frac{1}{(n+1)} P_{mn} \sum_{k=0}^{n} (k + 1) \sum_{l=0}^{m} \frac{1}{(l+1)^{2+1}} \sum_{j=0}^{l} p_{j,n-k} \\
= \frac{1}{(n+1)} P_{mn} \sum_{l=0}^{m} \sum_{k=0}^{n} (k + 1) p_{j,n-k}.$$

Using the identity

$$\sum_{k=0}^{n} (k + 1) p_{j,n-k} = \sum_{k=0}^{n} \sum_{r=0}^{n-k} p_{j,r},$$

we can write

$$A = \frac{1}{(n+1)} P_{mn} \sum_{l=0}^{m} \sum_{k=0}^{n} \sum_{j=0}^{l} p_{j,n-k} \\
= \frac{1}{(n+1)} P_{mn} \sum_{l=0}^{m} \sum_{k=0}^{n} \frac{P_{l,n-k}}{(l+1)^{2+1}} \\
= \frac{1}{(n+1)} P_{mn} \sum_{l=0}^{m} \sum_{k=0}^{n} \frac{P_{lk}}{(l+1)^{2+1}} \\
\leq \frac{1}{P_{mn}} \sum_{l=0}^{m} \sum_{k=0}^{n} \frac{P_{lk}}{(l+1)^{2+1}(k+1)}. \quad (5.9)$$

The second sum in the right-hand side of (5.8) can be dominated in a similar manner:
From (5.8)–(5.10) it follows that
\[ I_2 = O(Q_{mn}). \] (5.11)

In an analogous way, by (4.10iii),
\[ I_3 = O(Q_{mn}). \] (5.12)

Using (4.10iv),
\[ I_4 = \frac{O(1)}{P_{mn}} \int_0^\pi \int_0^\pi \frac{s^2 + t^2}{st} P_{\sigma, \tau} ds \, dt. \]

We replace \( s \) by \( 1/u \) and \( t \) by \( 1/v \), keeping in mind that \( \sigma = [1/s] \) and \( \tau = [1/t] \). As a result we obtain
\[ I_4 = \frac{O(1)}{P_{mn}} \int_1^\pi \int_1^\pi \left( \frac{1}{u^{\sigma+1}v^{\tau+1}} + \frac{1}{uv^{\sigma+1}} \right) P_{[u],[v]} du \, dv. \]

A natural evaluation of this double integral shows that
\[ I_4 = \frac{O(1)}{P_{mn}} \sum_{j=0}^m \sum_{k=0}^n \frac{1}{(j+1)^{\sigma+1}(k+1)} \left( 1 + \frac{1}{(j+1)(k+1)^{\tau+1}} \right) P_{jk} = O(Q_{mn}). \] (5.13)

Combining (5.1), (5.7), (5.11)–(5.13) results in (3.8).

REFERENCES