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ON FEIGENBAUM'S FUNCTIONAL EQUATION $g \circ g(\lambda x) + \lambda g(x) = 0$

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NUMERICAL STUDIES by M. Feigenbaum have exhibited what appears to be a new codimension 1 bifurcation for maps $f: [-1, 1] \mapsto [-1, 1]$. Feigenbaum's heuristic approach (see [4, 5]) is in the process of being rigorized (see [1, 3, 7]) and extended to diffeomorphisms and flows in several dimensions (see [2, 6]). We refer to [3] for a lucid introduction to the problem. We shall here be concerned only with Feigenbaum's first step, which was to solve the equation

$$\left.\begin{array}{c} g \circ g(\lambda x) + \lambda g(x) = 0 \\ g: [-1,1] \mapsto [-1,1] \text{ even, } g(0) = 1. \end{array}\right\} \tag{1}$$

Feigenbaum showed numerically that there is $\lambda = 0.39953528\ldots$ such that eqn (1) has a solution g which behaves like 1-const. x^2 at the origin. This has been made rigorous by Lanford[7] who found that eqn (1) has an analytic solution. Lanford first guesses (numerically) a good approximation to g by a polynomial of order 40. Then he *proves* by Newton's method that eqn (1) has a solution close to the guessed approximation. This is simple and perfectly rigorous, but involves calculations beyond human ability (they are done by computer). In the present note a method for solving eqn (1) is outlined, which does not involve superhuman calculations (although a small computer was used in fact to do them). The details are in [1]. The solution which we discuss is Feigenbaum's solution, shown in Fig. 1. If numerical computations are to be trusted, Fig. 2 presents another solution h behaving like 1-const. x^4 at the origin. Figure 3 shows $x \mapsto h(\sqrt{x})^2$ which is again a solution, but corresponding to negative λ.

We look for a solution g of eqn (1) satisfying also

$$g \text{ smooth‡ and } g''(0) < 0. \tag{2}$$

Our basic idea is that the functional equation for f_2

$$f_2(x) = \varphi \circ f_2(\lambda x) \tag{3}$$

(where λ, φ are given) is relatively easy to analyze. [This equation just says that the graph of f_2 is invariant under $(x, y) \mapsto (\lambda^{-1}x, \varphi(y))$.] We replace therefore eqns (1), (2) by the problem

$$f_1 \circ f_2(\lambda x) + \lambda f_2(x) = 0 \tag{4}$$

$$f_1 = f_2 \tag{5}$$

$$f_2: [-1,1] \mapsto [-1,1] \text{ smooth, even, } f_2(0) = 1,\ f_2''(0) < 0. \tag{6}$$

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‡We shall later take $g(x)$ of class C^3 as a function of x^2. There exist many C^1 solutions. In particular, the existence of a solution which behaves like 1-const $|x|^{1+\epsilon}$ is established in [3] for small ϵ, and suitable $\lambda(\epsilon)$.

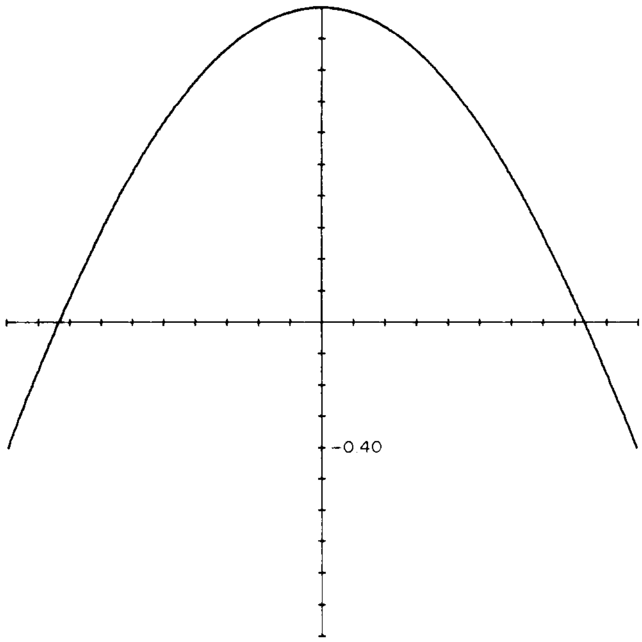

Fig. 1.

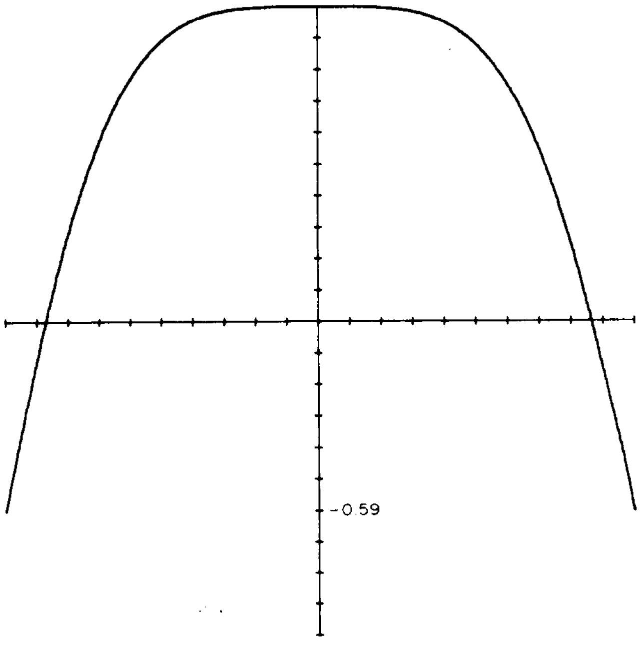

Fig. 2.

The solvability of eqn (4) with respect to f_2 (with $f_2(0) = 1$, $f_2''(0) \neq 0$) requires

$$f_1(1) + \lambda = 0 \tag{7}$$

$$\lambda f_1'(1) + 1 = 0. \tag{8}$$

Modulo eqn (7) we may rewrite eqn (4) as

$$f_1 \circ f_2(\lambda x) + \lambda f_2(x) = f_1(1) + \lambda \tag{4a}$$

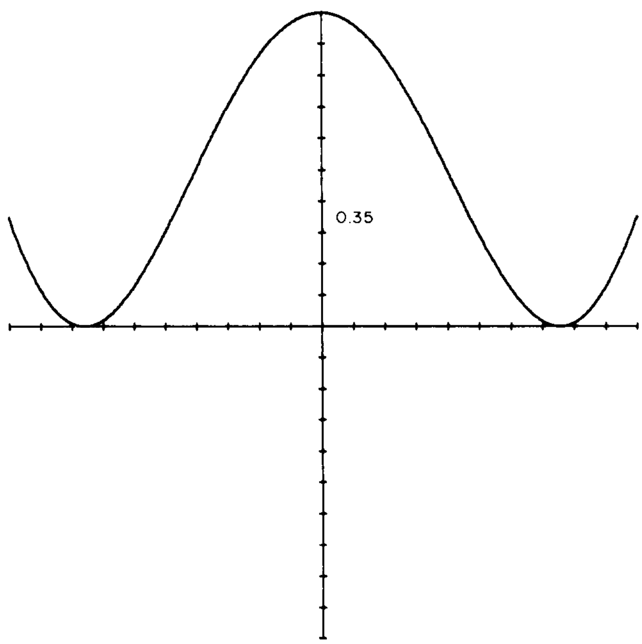

Fig. 3.

(which is again of the form of eqn 3). We shall try to solve the system, eqns (4a), (5) and (6), adjust λ such that $f_1(1)+\lambda=0$, and take $g=f_1=f_2$. The condition $f_1(1)+\lambda=0$ shows that λ is not arbitrary: our problem is a non linear eigenvalue problem. Let f_2 be a solution of eqn (4a) for given f_1, λ. Then $x\mapsto f_2(kx)$ is again a solution. In view of eqns (5) and (8) we shall lift this ambiguity by choosing the solution f_2 such that $\lambda f_2'(1)+1=0$.

Notice that eqn (4a) determines $f_2(x)$ for x near 0 in terms of $f_1(y)$ for y near 1†. In view of these dissimilar roles of f_1 and f_2 it is convenient to introduce new variables. Let us write

$$F(x)=\lambda^{-1}[f_1(1-x)-f_1(1)]$$

$$f_2(x)=1-\psi(x^2).$$

Then eqns (4a) and (5) become

$$\psi(t)=F\circ\psi(\lambda^2 t) \tag{4b}$$

$$\left.\begin{aligned} G(x)\equiv\lambda^{-1}[-\psi((1-x)^2)+\psi(1)] \\ F=G \end{aligned}\right\} \tag{5b}$$

where it is assumed that $F(0)=0$, $F'(0)=\lambda^{-2}$. One looks for a solution ψ of eqn (4b) satisfying

$$2\lambda\psi'(1)=1 \tag{8b}$$

and imposes eqn (5b). If λ is such that

$$\psi(1)=1+\lambda \tag{7b}$$

we have a solution of the original problem.

†In particular one cannot hope to determine simply from eqn (1) the coefficients of the power series expansion of g at the origin.

We may reformulate the problem as that of finding a fixed point F of the map Φ_λ: $F \mapsto \Psi \mapsto G$ where Ψ is defined by

$$\Psi(t) = F(\Psi(\lambda^2 t)),\ \Psi(0) = 0,\ \Psi'(0) = 1 \tag{9}$$

and

$$G(x) = \lambda^{-1}[\Psi(\alpha) - \Psi(\alpha(1-x)^2)] \tag{10}$$

where α is determined by

$$2\alpha\lambda\Psi'(\alpha) = 1$$

[in this notation $\psi(t) = \Psi(\alpha t)$]. Finally determine λ such that $\Psi(\alpha) = 1 + \lambda$.

From eqn (9) and the assumed smoothness one gets formulae such as

$$\Psi'(t) = \prod_{n=1}^{\infty} (\lambda^2 F'(\Psi(\lambda^{2n}t))$$

$$\frac{\Psi''\cdot(t)}{\Psi'(t)} = \sum_{n=1}^{\infty} \lambda^{2n}\Psi'(\lambda^{2n}t)\cdot\frac{F''(\Psi(\lambda^{2n}t))}{F'(\Psi(\lambda^{2n}t))}$$

$$(S\Psi)(t) = \sum_{n=1}^{\infty} \lambda^{4n}[\Psi'(\lambda^{2n}t)]^2(SF)(\Psi(\lambda^{2n}t))$$

where $Sf = (f''/f')' - 1/2(f''/f')^2$ is the Schwarzian derivative. These formulae give a good control on Ψ. Notice that these formulae require the knowledge of F only on the range of $t \mapsto \Psi(\lambda^2 t)$, $t \in [0, \alpha]$. For the purpose of finding fixed points of Φ_λ, it will thus be possible to consider functions F on $[0, A]$ with A smaller than 1.

The strategy will now be the following. We choose an interval J of values of λ and for each $\lambda \in J$ define a nonempty set $\mathcal{M}_\lambda$ of functions f on some interval $[0, A]$ such that $\Phi_\lambda\mathcal{M}_\lambda \subset \mathcal{M}_\lambda$ and Φ_λ is a contraction on $\mathcal{M}_\lambda$ with respect to some metric d. The map Φ_λ has thus a unique fixed point F_λ in the closure of $\mathcal{M}_\lambda$. Uniqueness implies continuity of $\lambda \mapsto F_\lambda$ and thus of $\lambda \mapsto \Psi(\alpha)-1-\lambda$. Finally one checks that $\Psi(\alpha)-1-\lambda$ has different signs at both ends of the interval J. Therefore there is at least one $\lambda \in J$ for which $\Psi(\alpha) = 1 + \lambda$, and this yields a solution of our original problem, eqn (1). *A priori*, F_λ is only in the closure of $\mathcal{M}_\lambda$, there may thus be an annoying loss of differentiability. A little miracle occurs however which saves the situation: $\mathcal{M}_\lambda$ contains analytic functions, and Φ_λ is analyticity improving. The fixed point F_λ is thus real analytic, and the same is true of the solution g of eqn (1).

Implementing the details of the above program is real work (see Ref. [1]), and involves in particular numerical computations. Here we give only general indication. The interval J is chosen as $[\sqrt{(0.152)}, \sqrt{(0.165)}]$. Then A is chosen as a function of λ (piecewise constant and $\leqq 0.261$). The set $\mathcal{M}_\lambda$ is convex and defined in terms of a set $\mathcal{M}'_\lambda$ such that $F \in \mathcal{M}_\lambda \Leftrightarrow (F''/F') \in \mathcal{M}'_\lambda$ (notice that if $s = (F''/F')$, then $F(x) = \int_0^x \mathrm{d}y\lambda^{-2}\exp\int_0^y s(z)\mathrm{d}z$). The convex set $\mathcal{M}'_\lambda$ consists of the C^1 functions on $[0, A]$ such that

$$\frac{1}{1-x} - l_1(1-x) - l_3(1-x)^3 \leqq -s(x) \leqq \frac{1}{1-x} - c_1(1-x) - c_3(1-x)^3$$

$$s'(x) + s(x)^2 \leqq 0 \tag{11}$$

$$-s'(x) \leqq L \tag{12}$$

where l_1, l_3, c_1, c_3, L are given as piecewise constant functions of λ, $0 \leqq c_1 \leqq l_1$, $0 \leqq c_3 \leqq l_3$, $l_1 + l_3 < 1$. It turns out that if $F \in \mathcal{M}_\lambda$, then G''/G' satisfies eqn (11) on [0,1] (not just [0, A]). In particular, $G''/G' \leqq 0$ and $G'''/G' \leqq 0$. Since $G'(0) = \lambda^{-2}$, we have $G' \geqq 0$, $G'' \leqq 0$, $G''' \leqq 0$ on [0,1]. The metric d on $\mathcal{M}_\lambda$ is given by the following norm on $\mathcal{M}'_\lambda$:

$$\|s\| = \sup_{0 \leqq x \leqq A} |(1-x)^{-1} s(x)|.$$

As to the analyticity character of Φ_λ, one shows that if $F \in \mathcal{M}_\lambda$ and

$$\left|\frac{1}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n F(x)\right| \leqq \lambda^{-2} B^{n-1} \text{ for } x \in [0,1],\ n \geqq 1$$

with $B \geqq 1.8$, then

$$\left|\frac{1}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n G(x)\right| \leqq \lambda^{-2} \tilde{B}^{n-1} \text{ for } x \in [0,1],\ n \geqq 1$$

with $\tilde{B} < B$.

THEOREM. *There is at least one number* $\lambda \in [\sqrt{(0.152)}, \sqrt{(0.165)}]$ *for which the functional equation*

$$g \circ g(\lambda x) + \lambda g(x) = 0,\ g(0) = 1$$

has an even smooth solution on $[-1,1]$. *The solution found has the following further properties*

$$g''(0) < 0$$

$$g(1) + \lambda = 0,\ \lambda g'(1) + 1 = 0$$

$$g'(x) \leqq 0,\ g''(x) \leqq 0,\ g'''(x) \geqq 0 \text{ on } [0,1]$$

$$\left|\frac{1}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n g(x)\right| \leqq \lambda^{-1}(1.8)^{n-1} \text{ for } x \in [-1,1],\ n \geqq 1.$$

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