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Toeplitz matrix method and nonlinear integral equation of Hammerstein type

M.A. Abdou[a,*], M.M. El-Borai[b], M.M. El-Kojok[b]

[a] Alexandria University, Faculty of Education, Department of Mathematics, Alexandria, Egypt
[b] Alexandria University, Faculty of Science, Department of Mathematics, Alexandria, Egypt

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Abstract

The existence and uniqueness solution of the nonlinear integral equation of Hammerstein type with discontinuous kernel are discussed. The normality and continuity of the integral operator are proved. Toeplitz matrix method is used, as a numerical method, to obtain a nonlinear system of algebraic equations. Also, many important theorems related to the existence and uniqueness of the produced algebraic system are derived. Finally, numerical examples, when the kernel takes a logarithmic and Carleman forms, are discussed and the estimate error, in each case, is calculated.

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1. Introduction

Integral equations play an important role in many branches of linear and nonlinear functional analysis and their applications in the theory of elasticity, engineering, mathematical physics and contact mixed problems (see [1–4]). Therefore, many different methods are used to obtain the solution of the nonlinear integral equation. In [5], Brunner et al., introduced a class of methods depending on some parameters to obtain the numerical solution of Abel integral equation of the second kind. In [6], Kaneko and Xu used degenerate kernel method to obtain the solution of Hammerstein integral equation. The linear multistep methods were applied in [7], to obtain the numerical solution of a singular nonlinear Volterra integral equation. Also, in [8], Kilbas and Saigo used an asymptotic method to obtain numerically the solution of nonlinear Abel–Volterra integral equation. In [9], Orsi used a Product Nyström method, as a numerical method, to obtain the solution of nonlinear Volterra integral equation, when its kernel takes a logarithmic and Carleman forms. Moreover, some methods can be found in Refs. [10–12] to discuss and obtain the solution of Hammerstein integral equation.

* Corresponding author. Tel.: +20 105446099; fax: +20 35534455.
E-mail addresses: abdella_77@yahoo.com (M.A. Abdou), m_m_elborai@yahoo.com (M.M. El-Borai), maha_mko@hotmail.com (M.M. El-Kojok).

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In this work, we will consider the following nonlinear integral equation of Hammerstein type:

$$\mu\phi(x) - \lambda \int_{-b}^{b} k(x,y)\gamma(y,\phi(y))\mathrm{d}y = f(x), \quad (|x| \le b). \tag{1.1}$$

Here, $f(x)$ with its derivatives and $\gamma(x,\phi(x))$ are given functions in the Banach space $L_2[-b,b]$. The known function $k(x,y)$ is called the kernel of the integral equation which has a weak singularity, while the unknown function $\phi(x)$ represents the solution of the nonlinear integral Eq. (1.1). The constant μ defines the kind of the integral equation, where $\mu = 0$, for the first kind, and $\mu =$ constant $\neq 0$, for the second kind. Also, λ is a constant, may be complex, that has a physical meaning, which is explained in [11,12].

Differentiating Eq. (1.1) with respect to the variable x, we have

$$\mu\phi'(x) - \lambda \int_{-b}^{b} \frac{\partial k(x,y)}{\partial x}\gamma(y,\phi(y))\mathrm{d}y = g(x), \quad (g(x) = f'(x), |x| \le b). \tag{1.2}$$

The integro differential Eq. (1.2) is equivalent to the integral Eq. (1.1). Therefore, the same solution will satisfy both of the two equivalent equations, after neglecting the constant term.

The existence and uniqueness solution of the nonlinear integral Eq. (1.1) are discussed and proved. Moreover, the normality and continuity of the integral operator are obtained. The Toeplitz matrix method is used, as a numerical method, to obtain a nonlinear system of algebraic equations. Also, we derive many important theorems related to the existence and uniqueness solution of the integral equation and its algebraic system. Finally, we obtain the solution of the produced algebraic system when the kernel takes a logarithmic and Carleman forms. Also, the estimate error, in each case, is calculated.

2. The existence and uniqueness solution

In this section, Banach fixed point theorem will be used as a source of existence and uniqueness solution of Eq. (1.1). For this, we write it in the integral operator form

$$\bar{W}\phi(x) = \frac{1}{\mu} f(x) + W\phi(x), \quad (\mu \neq 0) \tag{2.1}$$

where,

$$W\phi(x) = \frac{\lambda}{\mu} \int_{-b}^{b} k(x,y)\gamma(y,\phi(y))\mathrm{d}y. \tag{2.2}$$

Also, we assume the following conditions:

(i) The kernel $k(x,y)$ satisfies the discontinuity condition

$$\left\{\int_{-b}^{b}\int_{-b}^{b} |k(x,y)|^2 \mathrm{d}x\mathrm{d}y\right\}^{\frac{1}{2}} = c, \quad (c \text{ is a constant}).$$

(ii) The given function $f(x)$ is continuous in the space $L_2[-b,b]$, and its norm is defined as

$$\|f(x)\|_{L_2[-b,b]} = \left\{\int_{-b}^{b} |f(x)|^2 \mathrm{d}x\right\}^{\frac{1}{2}} = \zeta, \quad (\zeta \text{ is a constant}).$$

(iii) The known continuous function $\gamma(x,\phi(x))$ satisfies, for the constants $A > A_1, A > P$, the following conditions:

(a) $\left\{\int_{-b}^{b} |\gamma(x,\phi(x))|^2 \mathrm{d}x\right\}^{\frac{1}{2}} \le A_1 \|\phi(x)\|_{L_2[-b,b]}$,

(b) $|\gamma(x,\phi_1(x)) - \gamma(x,\phi_2(x))| \le M(x)|\phi_1(x) - \phi_2(x)|$,
where $\|M(x)\|_{L_2[-b,b]} = P$.

Theorem 1. *If the conditions* (i)–(iii) *are verified, then Eq.* (1.1) *has a unique solution* $\phi(x) \in L_2[-b,b]$.

The proof of this theorem depends on the following two lemmas:

Lemma 1. *Under the conditions* (i)–(iii-a) *, the operator* $\bar{W}$ *defined by* (2.1)*, maps the space* $L_2[-b,b]$ *into itself.*

Proof. In view of the formulas (2.1) and (2.2), we get

$$\left\|\bar{W}\phi(x)\right\|_{L_2[-b,b]} \leq \frac{1}{|\mu|}\left\|f(x)\right\|_{L_2[-b,b]} + \left|\frac{\lambda}{\mu}\right| \left\|\int_{-b}^{b}|k(x,y)||\gamma(y,\phi(y))|\mathrm{d}y\right\|_{L_2[-b,b]}.$$

Applying Cauchy–Schwarz inequality, then using the conditions (i)–(iii-a), the above inequality can be adapted to

$$\left\|\bar{W}\phi(x)\right\|_{L_2[-b,b]} \leq \frac{\zeta}{|\mu|} + \sigma\left\|\phi(x)\right\|_{L_{2[-b,b]}}, \qquad \left(\sigma = \left|\frac{\lambda}{\mu}\right| cA\right). \tag{2.3}$$

The last inequality (2.3) shows that, the operator $\bar{W}$ maps the ball S_ρ into itself, where

$$\rho = \frac{\zeta}{(|\mu| - |\lambda|cA)}. \tag{2.4}$$

Since $\rho > 0$, $\zeta > 0$, therefore we have $\sigma < 1$. Moreover, the inequality (2.3) involves the boundedness of the operator W of Eq. (2.1), where

$$\left\|W\phi(x)\right\|_{L_{2\{-b,b\}}} \leq \sigma\left\|\phi(x)\right\|_{L_{2\{-b,b\}}}. \tag{2.5}$$

Also, the inequalities (2.3) and (2.5) define the boundedness of the operator $\bar{W}$. □

Lemma 2. *If the two conditions* (i) *and* (iii-b) *are satisfied, then the operator* $\bar{W}$ *is contractive in the Banach space* $L_2[-b,b]$.

Proof. For the two functions $\phi_1(x)$ and $\phi_2(x)$ in the space $L_2[-b,b]$, the formulas (2.1) and (2.2) lead to

$$\left\|(\bar{W}\phi_1 - \bar{W}\phi_2)(x)\right\|_{L_2[-b,b]} \leq \left|\frac{\lambda}{\mu}\right| \left\|\int_{-b}^{b}|k(x,y)||\gamma(y,\phi_1(y)) - \gamma(y,\phi_2(y))|\mathrm{d}y\right\|_{L_2[-b,b]}.$$

Using condition (iii-b), then applying Cauchy–Schwarz inequality, and with the aid of condition (i), we obtain

$$\left\|(\bar{W}\phi_1 - \bar{W}\phi_2)(x)\right\|_{L_2[-b,b]} \leq \sigma\left\|\phi_1(x) - \phi_2(x)\right\|_{L_2[-b,b]}. \tag{2.6}$$

Inequality (2.6) shows that, the operator $\bar{W}$ is continuous in the space $L_2[-b,b]$, then $\bar{W}$ is a contraction operator under the condition $\sigma < 1$. □

The Proof of Theorem 2 is directly obtained after the following discussion:

Since the previous two lemmas showed that, the operator $\bar{W}$ is contractive in the Banach space $L_2[-b,b]$, then by Banach fixed point theorem, the operator $\bar{W}$ has a unique fixed point which is, of course, the unique solution of Eq. (1.1). □

3. The Toeplitz matrix method, (see [13])

Here, we will discuss the solution of Eq. (1.1) numerically using the Toeplitz matrix method. For this, the integral term in Eq. (1.1) can be written as

$$\int_{-b}^{b} k(x,y)\gamma(y,\phi(y))\mathrm{d}y = \sum_{n=-N}^{N-1}\int_{nh}^{(n+1)h} k(x,y)\gamma(y,\phi(y))\mathrm{d}y; \quad \left(h = \frac{b}{N}\right). \tag{3.1}$$

We approximate the integral in the right-hand side of Eq. (3.1) by

$$\int_{a}^{a+h} k(x,y)\gamma(y,\phi(y))\mathrm{d}y = A_n(x)\gamma(a,\phi(a)) + B_n(x)\gamma(a+h,\phi(a+h)) + R, \quad (a = nh), \tag{3.2}$$

where $A_n(x)$ and $B_n(x)$ are arbitrary functions to be determined, and R is the estimate error.

To determine $A_n(x)$ and $B_n(x)$, in the light of the Toeplitz matrix method, we put $\phi(x) = 1$ and $\phi(x) = x$, respectively, in Eq. (3.2). In this case, the error R will be neglected, then we obtain

$$\int_a^{a+h} k(x, y)\gamma(y, 1)\mathrm{d}y = A_n(x)\gamma(a, 1) + B_n(x)\gamma(a + h, 1), \tag{3.3}$$

and

$$\int_a^{a+h} k(x, y)\gamma(y, y)\mathrm{d}y = A_n(x)\gamma(a, a) + B_n(x)\gamma(a + h, a + h). \tag{3.4}$$

Solving the two Eqs. (3.3) and (3.4), we obtain

$$A_n(x) = \frac{1}{h_1}[\gamma(a + h, a + h)I(x) - \gamma(a + h, 1)J(x)], \tag{3.5}$$

$$B_n(x) = \frac{1}{h_1}[\gamma(a, 1)J(x) - \gamma(a, a)I(x)], \tag{3.6}$$

where

$$I(x) = \int_a^{a+h} k(x, y)\gamma(y, 1)\mathrm{d}y, \qquad J(x) = \int_a^{a+h} k(x, y)\gamma(y, y)\mathrm{d}y,$$

and

$$h_1 = \gamma(a, 1)\gamma(a + h, a + h) - \gamma(a, a)\gamma(a + h, 1), \quad (h_1 \neq 0).$$

In view of Eqs. (3.3)–(3.6), the formula (3.1) becomes

$$\int_{-b}^{b} k(x, y)\gamma(y, \phi(y))\mathrm{d}y = \sum_{n=-N}^{N} D_n(x)\gamma(nh, \phi(nh)), \tag{3.7}$$

where,

$$D_n(x) = \begin{cases} A_{-N}(x), & n = -N \\ A_n(x) + B_{n-1}(x), & -N < n < N \\ B_{N-1}(x), & n = N. \end{cases} \tag{3.8}$$

Thus, the integral Eq. (1.1) takes the form

$$\mu\phi(x) - \lambda \sum_{n=-N}^{N} D_n(x)\gamma(nh, \phi(nh)) = f(x). \tag{3.9}$$

Putting $x = mh$ in (3.9), and using the following notations:

$$\phi(\ell h) = \phi_\ell, \quad D_n(mh) = D_{mn}, \quad f(mh) = f_m, \quad \gamma(nh, \phi(nh)) = \gamma_n(\phi_n), \tag{3.10}$$

we obtain the following nonlinear algebraic system:

$$\mu\phi_m - \lambda \sum_{n=-N}^{N} D_{mn}\gamma_n(\phi_n) = f_m, \quad -N \leq m \leq N, \tag{3.11}$$

where, D_{mn} is defined by Eq. (3.8), after putting $x = mh$ and using the notations of (3.10).

The matrix D_{mn} can be written in the Toeplitz matrix form

$$D_{mn} = G_{mn} - E_{mn}.$$

Here, the matrix

$$G_{mn} = A_n(mh) + B_{n-1}(mh), \quad -N \leq m, n \leq N, \tag{3.12}$$

is called a Toeplitz matrix of order $(2N+1)$, and

$$E_{mn} = \begin{cases} B_{-N-1}(mh), & n = -N \\ 0, & -N < n < N \\ A_N(mh), & n = N, \end{cases} \tag{3.13}$$

represents a matrix of order $(2N+1)$ whose elements are zeros except for the first and the last rows (columns).

4. The nonlinear algebraic system

Now, our aim is to prove the existence and uniqueness solution of the nonlinear algebraic system (3.11) in Banach space ℓ^∞. For this, we write it in the operator form

$$\bar{T}\phi_m = T\phi_m + \frac{1}{\mu} f_m, \tag{4.1}$$

where,

$$T\phi_m = \frac{\lambda}{\mu}\sum_{n=-N}^{N} D_{mn}\gamma_n(\phi_n); \quad (\mu \neq 0, -N \le m \le N). \tag{4.2}$$

Then, we consider the following:

Lemma 3. *If the kernel of Eq.* (1.1) *satisfies the following conditions:*

$$k(x,y) \in L_q; \quad q > 1, \tag{4.3}$$

$$\lim_{x'\to x} \left\| k(x',y) - k(x,y) \right\|_{L_q} = 0; \quad x, x' \in [-b, b], \tag{4.4}$$

then, $\sup_N \sum_{n=-N}^{N} |D_{mn}|$ *exists, and*

$$\lim_{m'\to m} \sup_N \sum_{n=-N}^{N} |D_{m'n} - D_{mn}| = 0.$$

Proof. From the formula (3.5), we have

$$|A_n(x)| \le \frac{1}{|h_1|}\left[|\gamma(a+h,a+h)| \int_a^{a+h} |k(x,y)||\gamma(y,1)|\mathrm{d}y + |\gamma(a+h,1)| \int_a^{a+h} |k(x,y)||\gamma(y,y)|\mathrm{d}y \right].$$

Applying Hölder inequality for $p > 1$, and $q > 1$; $\frac{1}{p} + \frac{1}{q} = 1$, then summing from $n = -N$ to $n = N$, we get

$$\sum_{n=-n}^{N} |A_n(x)| \le \frac{1}{|h_1|} \|k(x,y)\|_{L_q} \left[\sum_{n=-N}^{N} |\gamma(a+h,a+h)| \, \|\gamma(y,1)\|_{L_p} + |\gamma(a+h,1)| \, \|\gamma(y,y)\|_{L_p} \right].$$

In view of the condition (4.3), and the continuity of the function γ in the interval $[-b, b]$, there exists a small constant E_1, such that

$$\sum_{n=-N}^{N} |A_n(x)| \le E_1, \quad \forall N.$$

Since, each term of $\sum_{n=-N}^{N} A_n(x)$ is bounded above, hence for $x = mh$, we deduce

$$\sup_N \sum_{n=-N}^{N} |A_n(mh)| \le E_1. \tag{4.5}$$

Similarly, from the formula (3.6), we can find a small constant E_2, such that

$$\sup_N \sum_{n=-N}^{N} |B_n(mh)| \le E_2. \tag{4.6}$$

In the light of (3.8), and with the help of (4.5) and (4.6), there exists a small constant E, such that

$$\sup_N \sum_{n=-N}^{N} |D_{mn}| \le E; (E = E_1 + E_2),$$

hence, $\sup_N \sum_{n=-N}^{N} |D_{mn}|$ exists.

By virtue of the formula (3.5), we get for $x, x' \in [-b, b]$

$$|A_n(x') - A_n(x)| \le \frac{1}{|h_1|}\left\{|\gamma(a+h, a+h)| \int_a^{a+h} |k(x', y) - k(x, y)||\gamma(y, 1)|\mathrm{d}y + |\gamma(a+h, 1)| \int_a^{a+h} |k(x', y) - k(x, y)||\gamma(y, y)|\mathrm{d}y\right\}.$$

Applying Hölder inequality, then summing from $n = -N$ to $n = N$, and taking in account the continuity of the function γ, the above inequality can be adapted in the form

$$\sup_N \sum_{n=-N}^{N} |A_n(x') - A_n(x)| \le \frac{1}{|h_1|} \left\| k(x', y) - k(x, y)\right\|_{L_q} \left\{\sup_N \sum_{n=-N}^{N} |\gamma(a+h, a+h)| \, \|\gamma(y, 1)\|_{L_p} + \sup_N \sum_{n=-N}^{N} |\gamma(a+h, 1)| \, \|\gamma(y, y)\|_{L_p}\right\}.$$

Putting $x = mh$, $x' = m'h$, then using the condition (4.4), we get as $x' \to x$

$$\lim_{m' \to m} \sup_N \sum_{n=-N}^{N} |A_n(m'h) - A_n(mh)| = 0. \tag{4.7}$$

Similarly, in view of the formula (3.6), we can prove

$$\lim_{m' \to m} \sup_N \sum_{n=-N}^{N} |B_n(m'h) - B_n(mh)| = 0. \tag{4.8}$$

Finally, with the aid of (3.8), (4.7) and (4.8), we have

$$\lim_{m' \to m} \sup_N \sum_{n=-N}^{N} |D_{m'n} - D_{mn}| = 0. \quad \square$$

Theorem 2. *The algebraic system (3.11), in Banach space ℓ^∞, has a unique solution under the following conditions:*

(1) $\sup_m |f_m| \le H < \infty$, ($H$ *is a constant*)
(2) $\sup_N \sum_{n=-N}^{N} |D_{mn}| \le E$, ($E$ *is a constant*)
(3) *The known functions* $\gamma(nh, \phi(nh))$, *for the constants* $Q > Q_1$, $Q > P_1$ *satisfy*
  (a) $\sup_n |\gamma(nh, \phi(nh))| \le Q_1 \, \|\Phi\|_{\ell^\infty}$,
  (b) $\sup_n |\gamma(nh, \phi(nh)) - \gamma(nh, \psi(nh))| \le P_1 \, \|\Phi - \Psi\|_{\ell^\infty}$,

where $\|\Phi\|_{\ell^\infty} = \sup_n |\phi_n|$, *for each integer* n.

To prove this theorem, we must consider the following lemmas:

Lemma 4. *If the conditions* (1)–(3-a) *are verified, then the operator* $\bar{T}$ *defined by Eq.* (4.1) *maps the space* ℓ^∞ *into itself.*

Proof. Let U be the set of all functions $\Phi = \{\phi_m\}$ in ℓ^∞ such that $\|\Phi\|_{\ell^\infty} \le \beta$, β is a constant. Define the norm of the operator $\bar{T}\Phi$ in Banach space ℓ^∞ by

$$\left\|\bar{T}\Phi\right\|_{\ell^\infty} = \sup_m |\bar{T}\phi_m|, \quad \text{for each integer } m. \tag{4.9}$$

From the formulas (4.1) and (4.2), we obtain

$$|\bar{T}\phi_m| \leq \left|\frac{\lambda}{\mu}\right| \sum_{n=-N}^{N} |D_{mn}| \sup_n |\gamma(nh, \phi(nh))| + \frac{1}{|\mu|} \sup_m |f_m|.$$

In view of the conditions (1)–(3-a), the above inequality can be adapted to

$$\sup_m |\bar{T}\phi_m| \leq \sigma_1 \, \|\Phi\|_{\ell^\infty} + \frac{1}{|\mu|} H, \qquad \left(\sigma_1 = \left|\frac{\lambda}{\mu}\right| QE\right).$$

Since, the above inequality is true for each integer m, then with the aid of (4.9), we deduce

$$\left\|\bar{T}\Phi\right\|_{\ell^\infty} \leq \sigma_1 \, \|\Phi\|_{\ell^\infty} + \frac{1}{|\mu|} H. \tag{4.10}$$

The inequality (4.10) shows that, the operator $\bar{T}$ maps the set U into itself, where

$$\beta = \frac{H}{(|\mu| - |\lambda| QE)}. \tag{4.11}$$

Since $\beta > 0$, $H > 0$, therefore we have $\sigma_1 < 1$. Also, the inequality (4.10) involves the boundedness of the operator T, where

$$\|T\Phi\|_{\ell^\infty} \leq \sigma_1 \, \|\Phi\|_{\ell^\infty}. \tag{4.12}$$

Furthermore, the inequalities (4.10) and (4.12) define the boundedness of the operator $\bar{T}$. □

Lemma 5. *Under the two conditions* (2) *and*(3-b), $\bar{T}$ *is a contraction operator in Banach space* ℓ^∞.

Proof. The formulas (4.1) and (4.2) lead to,

$$|\bar{T}\phi_m - \bar{T}\psi_m| \leq \left|\frac{\lambda}{\mu}\right| \sum_{n=-N}^{N} |D_{mn}| \sup_n |\gamma(nh, \phi(nh)) - \gamma(nh, \psi(nh))|.$$

Using the conditions (2) and (3-b), we obtain

$$|\bar{T}\phi_m - \bar{T}\psi_m| \leq \sigma_1 \, \|\Phi - \Psi\|_{\ell^\infty}.$$

The above inequality is true for each integer m, hence in view of (4.9) we have

$$\left\|\bar{T}\Phi - \bar{T}\Psi\right\|_{\ell^\infty} \leq \sigma_1 \, \|\Phi - \Psi\|_{\ell^\infty}. \tag{4.13}$$

Then, $\bar{T}$ is a contraction operator in Banach space ℓ^∞ under the condition $\sigma_1 < 1$. □

Proof of Theorem 2. In the light of the Lemmas 4 and 5, the operator $\bar{T}$ defined by (4.1) is contractive in Banach space ℓ^∞. Hence, by Banach fixed point theorem, $\bar{T}$ has a unique fixed point which is, of course, represents the unique solution of the nonlinear algebraic system in Banach space ℓ^∞. □

Definition 1. The estimate local error R_j is determined by the following equation:

$$\phi(x) - \phi_j(x) = \sum_{n=-N}^{N} D_{mn}[\gamma(nh, \phi(nh)) - \gamma(nh, \phi_j(nh))] + R_j, \tag{4.14}$$

where $\phi_j(x)$ is the approximate solution of Eq. (1.1).

Also, Eq. (4.14) gives

$$R_j = \left|\int_{-b}^{b} k(x, y)\gamma(y, \phi(y))\mathrm{d}y - \sum_{n=-N}^{N} D_{mn}\gamma(nh, \phi(nh))\right|. \tag{4.15}$$

Definition 2. The Toeplitz matrix method is said to be convergent of order r in the interval $[-b, b]$, if and only if for sufficiently large N, there exists a constant $D > 0$ independent on N such that

$$\|\phi(x) - \phi_N(x)\| \le DN^{-r}. \tag{4.16}$$

Theorem 3. *If the conditions* (2) *and* (3-b) *of Theorem* 2 *are satisfied, and the sequence of functions* $\{f_j(x)\} = \{(f_m)_j\}$ *converges uniformly to the function* $f(x) = \{f_m\}$ *in the Banach space* ℓ^∞. *Then the sequence of functions* $\{\Phi_j\} = \{(\phi_m)_j\}$ *converges uniformly to the solution* $\Phi = \{\phi_m\}$ *of Eq.* (4.1) *in the Banach space* ℓ^∞.

Proof. By virtue of Eq. (3.11), we have

$$|\phi_m - (\phi_m)_j| \le \left|\frac{\lambda}{\mu}\right| \sum_{n=-N}^{N} |D_{mn}| \sup_n |\gamma(nh, \phi(nh)) - \gamma(nh, \phi_j(nh))| + \frac{1}{|\mu|}|f_m - (f_m)_j|.$$

The above inequality, after using condition (3-b), holds for each integer m, hence from condition (2), we find

$$\sup_m |\phi_m - (\phi_m)_j| \le \sigma_1 \left\| \Phi - \Phi_j \right\|_{\ell^\infty} + \frac{1}{|\mu|} \left\| f - f_j \right\|_{\ell^\infty}.$$

Finally, the previous inequality takes the form,

$$\left\| \Phi - \Phi_j \right\|_{\ell^\infty} \le \frac{1}{[|\mu| - |\lambda| EQ]} \left\| f - f_j \right\|_{\ell^\infty}; \quad (\sigma_1 < 1). \tag{4.17}$$

Since $\left\| f - f_j \right\|_{\ell^\infty} \to 0$ as $j \to \infty$, so that $\left\| \Phi - \Phi_j \right\|_{\ell^\infty} \to 0$ as $j \to \infty$. □

Corollary 1. *Assume that, the hypothesis of Theorem* 3 *are verified, then*

$$\lim_{j\to\infty} R_j = 0. \tag{4.18}$$

Proof. In view of the formula (4.14), we have

$$|R_j| \le |\phi_m - (\phi_m)_j| + \sum_{n=-N}^{N} |D_{mn}| \sup_n |\gamma(nh, \phi(nh)) - \gamma(nh, \phi_j(nh))|.$$

Using condition (3-b), we obtain

$$|R_j| \le \sup_m |\phi_m - (\phi_m)_j| + Q \left\| \Phi - \Phi_j \right\|_{\ell^\infty} \sup_N \sum_{n=-N}^{N} |D_{mn}|.$$

The above inequality is true for each integer j, hence from condition (2), we obtain

$$\left\| R_j \right\|_{\ell^\infty} \le (1 + EQ) \left\| \Phi - \Phi_j \right\|_{\ell^\infty}, \quad \text{for each } j. \tag{4.19}$$

Since $\left\| \Phi - \Phi_j \right\|_{\ell^\infty} \to 0$ as $j \to \infty$, then $\left\| R_j \right\|_{\ell^\infty} \to 0$ and consequently $R_j \to 0$ as $j \to \infty$. □

Finally, it is convenient to consider the following theorem, which proves the convergence of the sequence of approximate solution $\{\phi_j(x)\}$ to the exact solution of Eq. (1.1) in the Banach space $L_2[-b, b]$.

Theorem 4. *If the sequence of continuous functions* $\{f_j(x)\}$ *converges uniformly to the function* $f(x)$ *in Banach space* $L_2[-b, b]$, *then under the conditions of Theorem* 1, *the sequence of approximate solution* $\{\phi_j(x)\}$ *converges uniformly to the exact solution of Eq.* (1.1) *in Banach space* $L_2[-b, b]$.

Proof. The formula (1.1) with its approximate solution give

$$\begin{aligned}\left\| \phi(x) - \phi_j(x) \right\|_{L_2[-b,b]} \le \frac{|\lambda|}{|\mu|} &\left\| \int_{-b}^{b} |k(x, y)|.|\gamma(y, \phi(y)) - \gamma(y, \phi_j(y))| \mathrm{d}y \right\|_{L_2[-b,b]} \\ &+ \frac{1}{|\mu|} \left\| f(x) - f_j(x) \right\|_{L_2[-b,b]}.\end{aligned}$$

Using condition (iii-b), and applying Cauchy–Schwarz inequality, we obtain

$$\left\|\phi(x)-\phi_j(x)\right\|_{L_2[-b,b]} \leq \left|\frac{\lambda}{\mu}\right| \left\{\int_{-b}^{b}\int_{-b}^{b}|k(x,y)|^2 \mathrm{d}x\mathrm{d}y\right\}^{\frac{1}{2}} \left\{\int_{-b}^{b}|M(y)|^2\mathrm{d}y\right\}^{\frac{1}{2}} \left\|\phi(x)-\phi_j(x)\right\|_{L_2[-b,b]} + \frac{1}{|\mu|}\left\|f(x)-f_j(x)\right\|_{L_2[-b,b]}.$$

In the light of conditions (i) and (iii-b), the above inequality takes the form

$$\left\|\phi(x)-\phi_j(x)\right\|_{L_2[-b,b]} \leq \left(\frac{1}{|\mu|-|\lambda|cA}\right)\left\|f(x)-f_j(x)\right\|_{L_2[-b,b]}.$$

Finally, we have

$$\left\|\phi(x)-\phi_j(x)\right\|_{L_2[-b,b]} \to 0 \text{ as } j\to\infty, \quad \text{since } \left\|f(x)-f_j(x)\right\|_{L_2[-b,b]} \to 0 \text{ as } j\to\infty. \quad \square$$

5. Applications

Here, we will consider the nonlinear integral Eq. (1.1) when the weak discontinuous kernel takes a logarithmic and Carleman forms, and the given function $\gamma(x,\phi(x)) = \phi^i(x)$, $i \geq 1$ is a positive integer.

Application (I):

Consider the nonlinear integral equation

$$\mu\phi(x) - \lambda\int_{-1}^{1}\ln|x-y|\phi^i(y)\mathrm{d}y = f(x), \quad (|x|\leq 1, i\geq 1). \tag{5.1}$$

When $i = 1$, we have a Fredholm integral equation of the second kind

$$\mu\phi(x) - \lambda\int_{-1}^{1}\ln|x-y|\phi(y)\mathrm{d}y = f(x), \quad (|x|\leq 1), \tag{5.2}$$

with logarithmic kernel. The formula (5.2) appears in variety of applications concerning the problems in the theory of elasticity [14], wave scattering in quantum mechanics [15], contact problems [16], and diffraction problems of aero-hydroacoustis [17]. The common approach to the solution of this type of integral equation involves its reduction to an equation approaches to the solution of this type of integral equation involves its reduction to an equation with Cauchy type singularity.

Differentiating the integral Eq. (5.2) with respect to x, we obtain

$$\mu\frac{\mathrm{d}\phi(x)}{\mathrm{d}x} - \lambda\int_{-1}^{1}\frac{1}{(x-y)}\phi(y)\mathrm{d}y = g(x), \quad (g(x) = f'(x)). \tag{5.3}$$

The formula (5.3) represents an integro-differential equation with Cauchy kernel.

Taking the transformations $x = 2z-1$ and $y = 2\eta-1$, in (5.3), we have

$$\mu\frac{\mathrm{d}\psi(z)}{\mathrm{d}z} - \lambda\int_{0}^{1}\frac{\psi(\eta)}{z-\eta}\mathrm{d}\eta = g(z). \tag{5.4}$$

The formula (5.4) under the conditions $\psi(0) = \psi(1) = 0$, has appeared in both combined infrared gaseous radiation and molecular conduction and elastic contact studies. When $\mu = 1$ and $g(z) = z$, Eq. (5.4) is solved and discussed numerically in [18]. Also, when $\mu = 0$, we have an integral equation of the first kind with Cauchy kernel which appears in airfoil theory and combined infrared radiation, in molecular conditions and in contact problems (see [19–21]).

Applying the Toeplitz matrix method to Eq. (5.1), we get Toeplitz matrix G_{mn} in the form

$$G_{mn} = \frac{h}{[n^i - (n+1)^i]}\left\{\frac{1}{(i+1)}[[(n+1)^{i+1} - m^{i+1}]\ln|(m-n-1)h| - [n^{i+1} - m^{i+1}]\ln|(m-n)h|]\right.$$
$$+ (n+1)^i[(m-n-1)\ln|(m-n-1)h| - (m-n)\ln|(m-n-1)h| + 1]$$
$$\left.- \frac{1}{(i+1)}\sum_{k=1}^{i+1}\frac{[(n+1)^{i-k+2} - n^{i-k+2}]m^{k-1}}{(i-k+2)}\right\}$$
$$+ \frac{h}{[n^i - (n-1)^i]}\left\{\frac{1}{(i+1)}[[n^{i+1} - m^{i+1}]\ln|(m-n)h| - [(n-1)^{i+1} - m^{i+1}]\ln|(m-n+1)h|]\right.$$
$$+ (n-1)^i[(m-n)\ln|(m-n)h| - (m-n+1)\ln|(m-n+1)h| + 1]$$
$$\left.- \frac{1}{(i+1)}\sum_{k=1}^{i+1}\frac{[n^{i-k+2} - (n-1)^{i-k+2}]m^{k-1}}{(i-k+2)}\right\}. \quad (5.5)$$

Also, the estimate error takes the form

$$|R| \le Ch^{2i+1},$$

where,

$$C = \left|\frac{-i}{(2i+1)(i+1)}\ln|mh| + \frac{i}{(2i+1)(i+1)}\sum_{k=1}^{2i+1}\frac{1}{km^k}\right|. \quad (5.6)$$

Using Maple 8, we obtain Table 1.

Table 1 lists the values of the exact and approximate solution of Eq. (5.1) together with various values of x in the linear and nonlinear case for $n = 3, 5$. From this table, we observe that:

1. The minimum value of the error in the linear case is 1.17×10^{-7} at $x = 0$ for $n = 5$, while the minimum value of the error in the nonlinear case is 5.1×10^{-6}at $x = \pm 0.2$ for $n = 5$.
2. The maximum value of the error in the linear case is 7.92×10^{-5} at $x = \pm 0.8$ for $n = 5$, while the maximum value of the error in the nonlinear case is 0.0001 at $x = \pm 0.667$ for $n = 3$.
3. The error in the linear case is less than the error in the nonlinear case.

Table 1

n	x	Exact	Nonlinear		Linear	
			Approx.	Error	Approx.	Error
3	−1	1	0.999988067	1.2E−05	1.0000064	6.36E−06
	−0.667	0.4444	0.444339863	0.0001	0.4443766	6.78E−05
	−0.333	0.1111	0.111093577	1.8E−05	0.1110967	1.44E−05
	0	0	−1.95184E−05	2E−05	−5.14E−07	5.14E−07
	0.3333	0.1111	0.111093577	1.8E−05	0.1110967	1.44E−05
	0.6667	0.4444	0.444339863	0.0001	0.4443766	6.78E−05
	1	1	0.999988067	1.2E−05	1.0000064	6.36E−06
5	−1	1	1.000027389	2.7E−05	1.0000511	5.11E−05
	−0.8	0.64	0.639904645	9.5E−05	0.6399208	7.92E−05
	−0.6	0.36	0.3599531	4.7E−05	0.3599641	3.59E−05
	−0.4	0.16	0.159982501	1.7E−05	0.1599857	1.43E−05
	−0.2	0.04	0.040005096	5.1E−06	4.00E−02	3.45E−06
	0	0	−2.2716E−05	2.3E−05	−1.17E−07	1.17E−07
	0.2	0.04	0.040005096	5.1E−06	4.00E−02	3.45E−06
	0.4	0.16	0.159982501	1.7E−05	0.1599857	1.43E−05
	0.6	0.36	0.3599531	4.7E−05	0.3599641	3.59E−05
	0.8	0.64	0.639904645	9.5E−04	0.6399208	7.92E−05
	1	1	1.000027389	2.7E−05	1.0000511	5.11E−05

Application (II):

Consider the nonlinear integral equation

$$\mu\phi(x) - \lambda \int_{-1}^{1} |x-y|^{-\alpha} \phi^i(y)\mathrm{d}y = f(x), \quad (0 \le \alpha < 1), \tag{5.7}$$

with Carelman kernel. The importance of Carleman kernel came from the work of Arutiunian [22], who has shown that the plane contact problem in the nonlinear theory of plasticity, in its first approximation can be reduced to Fredholm integral equation of the first kind with Carleman kernel.

Applying the Toeplitz matrix method to Eq. (5.7), the Toeplitz matrix G_{mn} takes the form

$$\begin{aligned} G_{mn} = h^{1-\alpha} \Bigg\{ & \frac{1}{[n^i - (n+1)^i]} \left[\sum_{k=0}^{i} \frac{i![n^{i-k}|m-n|^{k+1-\alpha} - (n+1)^{i-k}|m-n-1|^{k+1-\alpha}]}{(i-k)!(1-\alpha)(2-\alpha)\ldots(k+1-\alpha)} \right. \\ & \left. + \frac{(n+1)^i}{(1-\alpha)}[|m-n-1|^{1-\alpha} - |m-n|^{1-\alpha}] \right] \\ & + \frac{1}{[n^i - (n-1)^i]} \left[\sum_{k=0}^{i} \frac{i![(n-1)^{i-k}|m-n+1|^{k+1-\alpha} - n^{i-k}|m-n|^{k+1-\alpha}]}{(i-k)!(1-\alpha)(2-\alpha)\ldots(k+1-\alpha)} \right. \\ & \left. + \frac{(n-1)^i}{(1-\alpha)}[|m-n|^{1-\alpha} - |m-n+1|^{1-\alpha}] \right] \Bigg\}. \end{aligned} \tag{5.8}$$

Also, the estimate error can be determined as

$$|R| \le C h^{2i+1-\alpha},$$

where,

$$C = \left| \sum_{k=0}^{i} \frac{i! m^{k+1-\alpha} |1-\frac{1}{m}|^{k+1-\alpha}}{(i-k)!(1-\alpha)(2-\alpha)\ldots(k+1-\alpha)} - \sum_{k=0}^{2i} \frac{(2i)! m^{k+1-\alpha} |1-\frac{1}{m}|^{k+1-\alpha}}{(2i-k)!(1-\alpha)(2-\alpha)\ldots(k+1-\alpha)} \right|. \tag{5.9}$$

Using Maple 8, we get Table 2.

Table 2 shows the values of the exact and approximate solution of Eq. (5.7) together with different values of x in the linear and nonlinear case for $n = 3, 5$ and $\nu = 0.2, 0.4$. From this table, we notice that:

1. For $\nu = 0.2$,
   (a) The minimum value of the error in linear and nonlinear case are 1.5×10^{-4}, 2.02×10^{-5} respectively, and each of which occurs at $x = \pm 1$ for $n = 3$.
   (b) The maximum value of the error in the linear case is 3.89×10^{-4} at $x = \pm 0.7$ for $n = 3$, while the maximum value of the error in the nonlinear case is 6.81×10^{-4} at $x = 0$ for $n = 3$.
   (c) The maximum value of the error in the linear case is less than the maximum value of the error in the nonlinear case, while the minimum value of the error in the linear case is greater than the minimum value of the error in the nonlinear case.
2. For $\nu = 0.4$,
   (d) The minimum value of the error in the linear case is 0 at $x = \pm 1$ for $n = 3$, and at every value of x except at $x = 0$ for $n = 5$. Also, the minimum value of the error in the nonlinear case is 1.4×10^{-5} at $x = \pm 1$ for $n = 3$.
   (e) The maximum values of the error in the linear and nonlinear case are 4×10^{-4}, 0.00069 respectively, and each of them occurs at $x = 0$ for $n = 3$.
   (f) The minimum and maximum values of the error in the linear case are less than the minimum and maximum values of the error in the nonlinear case.

Table 2

n	x	Exact	Nonlinear				Linear			
			$\nu = 0.4$		$\nu = 0.2$		$\nu = 0.4$		$\nu = 0.2$	
			App.	Error	App.	Error	App.	Error	App.	Error
3	−1	1	1	1.4E−05	0.99998	2.02E−05	1	0	0.99985	1.50E−04
	−0.7	0.44	0.445	0.00047	0.444862	4.18E−04	0.445	∼0	0.444834	3.89E−04
	−0.3	0.11	0.111	0.00033	0.111434	3.23E−04	0.111	∼0	0.111487	3.76E−04
	0	0	7E−04	0.00069	6.81E−04	6.81E−04	4E−04	∼0	3.73E−04	3.73E−04
	0.3	0.11	0.111	0.00033	0.111434	3.23E−04	0.111	∼0	0.111487	3.76E−04
	0.7	0.44	0.445	0.00047	0.444862	4.18E−04	0.445	∼0	0.444834	3.89E−04
	1	1	1	1.4E−05	0.99998	2.02E−05	1	0	0.99985	1.50E−04
5	−1	1	1	0.00012	0.999894	1.06E−04	1	0	0.99976	2.40E−04
	−0.8	0.64	0.64	0.00029	0.640249	2.49E−04	0.64	0	0.640235	2.35E−04
	−0.6	0.36	0.36	0.00025	0.360233	2.33E−04	0.36	0	0.360223	2.23E−04
	−0.4	0.16	0.16	0.00023	0.160218	2.18E−04	0.16	0	0.160218	2.18E−04
	−0.2	0.04	0.04	0.00016	4.02E−02	1.57E−04	0.04	0	4.02E−02	2.15E−04
	0	0	4E−04	0.00045	4.41E−04	4.41E−04	2E−04	∼0	2.14E−04	2.14E−04
	0.2	0.04	0.04	0.00016	4.02E−02	1.57E−04	0.04	0	4.02E−02	2.15E−04
	0.4	0.16	0.16	0.00023	0.160218	2.18E−04	0.16	0	0.160218	2.18E−04
	0.6	0.36	0.36	0.00025	0.360233	2.33E−04	0.36	0	0.360223	2.23E−04
	0.8	0.64	0.64	0.00029	0.640249	2.49E−04	0.64	0	0.640235	2.35E−04
	1	1	1	0.00012	0.999894	1.06E−04	1	0	0.99976	2.40E−04

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