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Circuit and fractional circuit covers of matroids

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ABSTRACT

Let M be a connected matroid having a ground set E. Lemos and Oxley proved that $|E(M)| \leq \frac{1}{2}c(M)c^*(M)$ where $c(M)$ (resp. $c^*(M)$) is the circumference (resp. cocircumference) of M. In addition, they conjectured that one can find a collection of at most $c^*(M)$ circuits which cover the elements of M at least twice. In this paper, we verify this conjecture for regular matroids. Moreover, we show that a version of this conjecture is true for fractional circuit covers.

1. Introduction

For all notation, terminology and concepts used for matroids, we refer the reader to [7]. For a matroid M we define the **circumference** (resp. **cocircumference**) to be the size of the largest circuit (resp. cocircuit) and denote it by $c(M)$ (resp. $c^*(M)$). In [2], Lemos and Oxley established the following bound for the size of a connected matroid:

Theorem 1.1 (*Lemos, Oxley*)**.** *Let M be a connected matroid. Then $|E(M)| \leq \frac{1}{2}c(M)c^*(M)$.*

Later, Oxley [6] conjectured that a stronger result holds:

Conjecture 1.2 (*Oxley*)**.** *For any connected matroid M with at least two elements, one can find a collection of at most $c^*(M)$ circuits which cover each element of M at least twice.*

Up until now, this conjecture has been verified for graphic and cographic matroids (see [3,5]). In the next section, we shall show that Conjecture 1.2 is true for regular matroids. In the last section, we shall show that this conjecture is true for fractional circuit covers.

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2. Circuit covers of regular matroids

Our goal in this section is to show that Conjecture 1.2 holds for regular matroids. To start with, we shall use a well-known result of Tutte [8].

Lemma 2.1 (*Tutte*). *Let M be a connected matroid and let $e \in E(M)$. Then either $M\backslash\{e\}$ or $M/\{e\}$ is connected.*

In [3], the following result was proven.

Theorem 2.2. *Let M be a k-connected regular matroid where $c(M) \geq 2k$. If C_1 and C_2 are disjoint circuits satisfying $r(C_1) + r(C_2) = r(C_1 \cup C_2)$, then $|C_1| + |C_2| \leq 2(c(M) - k + 1)$.*

Remark. In the case where M is 3-connected, the proof of Theorem 2.2 given for binary matroids in [3] (Theorem 5.2) shows that even if M is vertically 3-connected, the theorem is still true (for $k = 3$).

In [4], it was shown:

Theorem 2.3. *Let M be a connected binary matroid having no F_7^*-minor. Let $\mathcal{A}^*$ be a collection of cocircuits of M. Then there is a circuit intersecting all cocircuits of $\mathcal{A}^*$ if either one of two things hold:*

(i) *For any two disjoint cocircuits A_1^* and A_2^* in $\mathcal{A}^*$ it holds that $r^*(A_1^*) + r^*(A_2^*) > r^*(A_1^* \cup A_2^*)$.*
(ii) *For any two disjoint cocircuits A_1^* and A_2^* in $\mathcal{A}^*$ it holds that $r(A_1^*) + r(A_2^*) > r(M)$.*

For a matroid M, let $\mathcal{S}^*(M)$ be the set of cocircuits of size at least $c^*(M) - 1$. A collection of circuits $\mathcal{K}$ of M is said to be a **covering set** if $|\mathcal{K}| \leq c^*(M)$, and every element of M belongs to at least two circuits of $\mathcal{K}$. We shall say that M is **coverable** if it has a covering set.

Theorem 2.4. *Any connected regular matroid is coverable.*

Proof. Let M be a connected regular matroid. We shall show that M is coverable by induction on $r(M)$. If $r(M) \leq 3$, then M is either graphic or cographic and there is a circuit intersecting every cocircuit of M. Using the arguments in Case 1, one can show that M is coverable. We therefore assume that $r(M) \geq 4$, and the theorem holds for all connected regular matroids M' where $r(M') < r(M)$, or $r(M') = r(M)$ and $|E(M')| < |E(M)|$.

Suppose that M contains a 2-cocircuit $\{e, f\}$. Let $M' = M/f$. Then M' is connected, $r(M') = r(M) - 1$, and $c^*(M') \leq c^*(M)$. By assumption, M' has a covering set $\mathcal{K}'$. Let

$$\mathcal{K} = \{C \mid C \in \mathcal{K}', e \notin C\} \cup \{C \cup \{f\} \mid C \in \mathcal{K}', e \in C\}.$$

Then $\mathcal{K}$ is seen to be a covering set for M. We shall henceforth assume that M contains no 2-cocircuits.

Case 1. Suppose M is vertically 3-connected.

Proof. We have that M^* is vertically 3-connected, and following the remark after Theorem 2.2, it holds that for any two disjoint cocircuits $C_1^*, C_2^* \in \mathcal{S}^*(M)$, $r^*(C_1^*) + r^*(C_2^*) \geq r^*(C_1^* \cup C_2^*) + 1$. It follows from Theorem 2.3 that there is a circuit C which intersects each cocircuit of $\mathcal{S}^*$. According to Lemma 2.1, we can successively delete or contract each element of C to yield a connected matroid M'. Since C intersects each cocircuit of $\mathcal{S}^*$, it follows that $c^*(M') \leq c^*(M) - 2$. Furthermore, $r(M') \leq r(M)$, and $|E(M')| < |E(M)|$. By assumption, M' has a covering set $\mathcal{K}'$ where $|\mathcal{K}'| \leq c^*(M') \leq c^*(M) - 2$. Let $\mathcal{K}$ be a corresponding collection of circuits of M. Then $\mathcal{K} \cup \{C, C\}$ is seen to be a covering set of M. □

Case 2. Suppose M is not vertically 3-connected.

Proof. We can express M as a non-trivial 2-sum. We shall consider two subcases:

Case 2.1: $M = M_1 \oplus_2 M_2$, where $r(M_i) \geq 3,\ i = 1, 2$ and $E(M_1) \cap E(M_2) = \{e\}$.

For $i = 1, 2$ let $\beta_i = |B_i^* \backslash \{e\}|$ where B_i^* is a largest cocircuit in M_i containing e. Let $\beta = \min\{\beta_1, \beta_2\}$. We may assume that $\beta_1 = \beta$. Then for all $C^* \in \mathcal{C}^*(M_i),\ e \notin C^*$, it holds that $|C^*| \leq 2\beta_i$. To see this, let $C^* \in \mathcal{C}^*(M_i)$, where $e \notin C^*$. Since M_i is connected, there is a cocircuit containing e which also intersects C^*. Among such cocircuits, choose a cocircuit B^* such that $|B^* \backslash C^*|$ is minimum. Then it is seen that $B'^* = B^* \triangle C^*$ is also a cocircuit of M_i where $e \in B'^*$. Now

$$2(\beta_i + 1) \geq |B^*| + |B'^*| = |C^*| + 2|B^* \backslash C^*| \geq |C^*| + 2.$$

Thus $|C^*| \leq 2\beta_i$.

It is also seen that $\beta_1 + \beta_2 \leq c^*(M)$, and consequently $2\beta \leq c^*(M)$. Let $M_i' = M_i \oplus_2 N_i,\ i = 1, 2$ where N_i is the matroid defined by taking a 3-circuit $\{e, f, g\}$ and replacing f by β parallel elements, and doing the same for g. Then $r(M_i') < r(M),\ i = 1, 2$. It is seen that $c^*(M_1') = 2\beta \leq c^*(M)$ and $c^*(M_2') \leq c^*(M)$. Thus by assumption, M_i' has a covering set $\mathcal{K}_1$ with $q_1 \leq c(M_1^*) = 2\beta$ circuits. Since $E(N_1) \backslash \{e\}$ is a cocircuit with 2β elements, and each circuit intersecting $E(N_1) \backslash \{e\}$ does so in exactly 2 elements, it holds that $q_1 = 2\beta$. We also have that M_2' has a covering set $\mathcal{K}_2$ with $q_2 \leq c^*(M_2') \leq c^*(M)$ circuits. We have $q_1 = 2\beta \leq \beta_1 + \beta_2 \leq q_2$. Following very similar arguments to those used in [3, Theorem 1.3] for graphs, one can 'splice together' covering sets K_1 and K_2 to obtain a covering set for M.

Case 2.2: For every non-trivial 2-sum $M = M_1 \oplus_2 M_2$, either $r(M_1) = 2$, or $r(M_2) = 2$.

If M is a non-trivial 2-sum where $r(M_1) = 2$ and $E(M_1) \cap E(M_2) = \{e\}$, then it is seen that $B^* = E(M_1) \backslash \{e\}$ is a cocircuit of M where $r(B^*) = 2$. Let $\mathcal{C}_2^*$ be the set of all such cocircuits of M. If for some $B^*, B'^* \in \mathcal{C}_2^*$ it holds that $B^* \cap B'^* \neq \emptyset$, then we can express M as a non-trivial 2-sum $M = M_1' \oplus_2 M_2'$ where $E(M_1') \cap E(M_2') = \{e'\}$ and $E(M_1') = B^* \cup B'^* \cup \{e'\}$. It would then hold that $r(M_1') = 3$, and hence $r(M_2') = 2$. In this case, M is graphic and thus has a covering set. We may therefore assume that $\mathcal{C}_2^*$ is a disjoint collection of cocircuits.

We shall create a matroid M' from M in the following way: let $B^* \in \mathcal{C}_2^*$ and let $f, g \in B^*$ be non-parallel elements. Then $B^* = \mathrm{cl}(f) \cup \mathrm{cl}(g)$. If $|\mathrm{cl}(f)| \geq |\mathrm{cl}(g)|$, then contract the elements of $\mathrm{cl}(g)$; otherwise, contract the elements of $\mathrm{cl}(f)$. After performing this operation on each $B^* \in \mathcal{C}_2^*$ we obtain a vertically 3-connected matroid M'. By Theorem 2.2 (and the remark after it) and Theorem 2.3, there is a circuit $C_{M'}$ of M' which intersects every cocircuit of $\mathcal{S}^*(M')$. Let C_M be a corresponding circuit in M. If C_M intersects every cocircuit of $\mathcal{S}^*(M)$, then we can argue as in Case 1. We assume therefore that C_M does not. Since $C_{M'}$ intersects every cocircuit of $\mathcal{S}^*(M')$, it is seen that C_M intersects every cocircuit of $\mathcal{S}^*(M) \backslash \mathcal{C}_2^*(M)$, and thus for some $B_1^* \in \mathcal{C}_2^*(M)$ it holds that $C_M \cap B_1^* = \emptyset$ and $|B_1^*| \geq c^* - 1$.

Let $e_1, e_2 \in B_1^*$ be non-parallel elements and let $E_i = \mathrm{cl}(e_i),\ i = 1, 2$. Among the circuits of M containing e_1 and e_2, let D be a circuit having maximum length. If D intersects all cocircuits of $\mathcal{S}^*(M)$, then we can proceed as in Case 1. We may therefore assume that for some $B_2^* \in \mathcal{S}^*(M)$ it holds that $D \cap B_2^* = \emptyset$. Since M is connected there is a cocircuit containing e_1 and elements of B_2^*. Among all such cocircuits choose C_1^* so that $|C_1^* \backslash B_2^*|$ is minimum. Then by minimality, $C_2^* = C_1^* \triangle (B_1^* \cup B_2^*)$ is seen to be a cocircuit. Thus

$$\begin{aligned} 2c^* \geq |C_1^*| + |C_2^*| &= |B_1^*| + |B_2^*| + 2|C_1^* \backslash (B_1^* \cup B_2^*)| \\ &\geq 2(c^* - 1) + 2|C_1^* \backslash (B_1^* \cup B_2^*)|. \end{aligned} \tag{1}$$

We have that $D \cap C_1^* \neq \emptyset$ and hence $|D \cap C_1^*| \geq 2$. Thus $(D \backslash \{e_1, e_2\}) \cap C_1^* \neq \emptyset$ and hence $C_1^* \backslash (B_1^* \cup B_2^*) \neq \emptyset$. It follows from (1) that

$$|(D \backslash \{e_1, e_2\}) \cap C_1^*| = |C_1^* \backslash (B_1^* \cup B_2^*)| = |C_2^* \backslash (B_1^* \cup B_2^*)| = 1,$$

and equality holds throughout in (1). Consequently,

$$|B_1^*| = |B_2^*| = c^* - 1, \quad |C_1^*| = |C_2^*| = c^*, \quad \text{and} \quad |E_1| = |E_2| = \frac{c^* - 1}{2}.$$

Let

$$\{d_1\} = (D \backslash \{e_1, e_2\}) \cap C_1^* = C_1^* \backslash (B_1^* \cup B_2^*) = C_2^* \backslash (B_1^* \cup B_2^*).$$

It is seen that any circuit of $E(M)\backslash B_2^*$ containing d_1 must also intersect B_1^*, and as such $E(M)\backslash(B_1^* \cup B_2^*)$ has no circuit containing d_1. Since M contains no 2-cocircuits, it holds that $c^* - 1 \geq 3$, and thus $|E_i| \geq 2,\ i = 1, 2$. Let

$$F_i = C_i^* \cap B_2^*, \quad f_i \in F_i,\ i = 1, 2.$$

By the choice of C_1^*, it follows that $C_1^* \triangle B_2^* = E_1 \cup F_2 \cup \{d_1\}$ is a cocircuit, as is $C_1^* \triangle B_1^* = E_2 \cup F_1 \cup \{d_1\}$. In particular, this implies that $|F_1| = |F_2| = \frac{c^*-1}{2}$. In the remainder of the proof, we aim to show that, assuming D does not intersect all cocircuits of $\mathcal{S}^*(M)$, then either M has an F_7-minor, or there is a 2-cocircuit. In either case, we reach a contradiction.

Let

$$T_1 \in \mathcal{B}(M\backslash B_2^*), \qquad D\backslash\{d_1\} \subset T_1.$$

Since B_2^* is a cocircuit, $T_1 \cup \{f_1, f_2\}$ has a unique circuit which contains f_1 and f_2. Let

$$C \in \mathcal{C}(M), \quad \text{where } C \subset T_1 \cup \{f_1, f_2\},\ f_1, f_2 \in C.$$

Then $|C_i^* \cap C| \geq 2, i = 1, 2$, and consequently, $e_1, e_2 \in C$. Suppose d_1 is a chord of C; that is, for two circuits C', C'' it holds $C \cup \{d_1\} = C' \cup C''$, and $C' \cap C'' = \{d_1\}$. Assuming $f_1 \in C''$, it holds that $f_2 \in C''$ since $|B_2^* \cap C''| \geq 2$. We also have that $e_1, e_2 \in C'$, since $d_1 \in C'$ and $C' \subseteq E(M)\backslash B_2^*$. This implies that $C' \subseteq T_1 \cup \{d_1\}$, and given that D is the unique circuit of $T_1 \cup \{d_1\}$, it must hold that $C' = D$. However,

$$|C| = |C'| + |C''| - 2 \geq |D| + 1,$$

contradicting the maximality of D. We conclude that d_1 is not a chord of C, and $D\backslash(C \cup \{d_1\}) \neq \emptyset$.

Let

$$d_2 \in D\backslash(C \cup \{d_1\}), \quad T_2 = (T_1\backslash\{d_2\}) \cup \{d_1\}.$$

Then $d_2 \in T_1$. Moreover, $T_2 \in \mathcal{B}(M\backslash B_2^*)$ where $D\backslash d_2 \subseteq T_2$. Let H_1, H_2, H_1', H_2' be hyperplanes defined such that

$$H_1 = \mathrm{cl}((T_1\backslash\{e_1\}) \cup \{f_2\}) \qquad H_2 = \mathrm{cl}((T_1\backslash\{e_2\}) \cup \{f_1\})$$
$$H_1' = \mathrm{cl}((T_2\backslash\{e_1\}) \cup \{f_2\}) \qquad H_2' = \mathrm{cl}((T_2\backslash\{e_2\}) \cup \{f_1\}).$$

It is seen that

$$C_i^* = E(M)\backslash H_i, \quad i = 1, 2.$$

Let

$$C_i'^* = E(M)\backslash H_i', \quad i = 1, 2.$$

Then $C_i'^*,\ i = 1, 2$ are cocircuits where $f_i, d_2 \in C_i'^*$. Given that $|F_1| = |F_2| = \frac{c^*-1}{2} \geq 2$, there are elements

$$f_i' \in F_i\backslash\{f_i\}, \quad i = 1, 2.$$

For $f, g \in \{f_1, f_1', f_2, f_2'\}, f \neq g$, there is a unique circuit in $(T_1 \cup \{d_1\}\backslash\{e_1, e_2\}) \cup \{f, g\} = (T_2 \cup \{d_2\}\backslash\{e_1, e_2\}) \cup \{f, g\}$ which contains f and g. We shall denote such a circuit by $C(f, g)$. We first note that since $C \triangle D \subset (T_1 \cup \{d_1\}\backslash\{e_1, e_2\}) \cup \{f_1, f_2\}$, it holds that

$$C(f_1, f_2) = C \triangle D, \quad \text{and} \quad d_1, d_2 \in C(f_1, f_2).$$

We also observe that for any $f \in F_1$, and $g \in F_2$, it holds that $f \in C(f, g) \cap C_1^*$, and thus $|C(f, g) \cap C_1^*| \geq 2$. It follows that $\{f, d_1\} = C(f, g) \cap C_1^*$. Hence

$$d_1 \in C(f, g), \quad \forall f \in F_1,\ \forall g \in F_2.$$

Suppose $f_1' \in H_1'$. Since $H_1' = \mathrm{cl}((T_2\backslash\{e_1\}) \cup \{f_2\})$, there is a circuit K in $T_2\backslash\{e_1\} \cup \{f_1', f_2\}$ containing f_1', and such a circuit must also contain f_2. It follows that $K = C(f_1', f_2)$, and $C(f_1', f_2) \subset H_1'$. Thus $d_1 \in C(f_1', f_2)$ and $d_2 \notin C(f_1', f_2)$ since $d_2 \notin H_1'$. Since $d_1, d_2 \in C(f_1, f_2)$, and $C(f_1, f_1') = C(f_1, f_2) \triangle C(f_1', f_2)$, it holds that $d_2 \in C(f_1, f_1')$ (since $d_2 \notin C(f_1', f_2)$) and $d_1 \notin C(f_1, f_1')$ (since $d_1 \in C(f_1', f_2)$). Summarizing, we have

$$f_1' \in H_1' \Rightarrow d_1 \notin C(f_1, f_1'), \quad d_2 \in C(f_1, f_1'). \tag{2}$$

Similarly,

$$f_2' \in H_2' \Rightarrow d_1 \notin C(f_2, f_2'), \quad d_2 \in C(f_2, f_2'). \tag{3}$$

Suppose $f_i' \in H_i'$, $i = 1, 2$. From (2) and (3) we have

$$d_2 \in C(f_i, f_i'), \quad \text{and} \quad d_1, e_1, e_2 \notin C(f_i, f_i'),\ i = 1, 2.$$

We have

$$C(f_i, f_i') = C(f_1, f_2) \triangle C(f_i', f_{3-i}), \quad i = 1, 2.$$

Given that $d_1, d_2 \in C(f_1, f_2)$ and $d_1 \notin C(f_i, f_i')$, $d_2 \in C(f_i, f_i')$, $i = 1, 2$, it follows that

$$d_1 \in C(f_i', f_{3-i})\ d_2 \notin C(f_i', f_{3-i}), \quad i = 1, 2.$$

Also, since

$$C(f_1', f_2') = C(f_1', f_2) \triangle C(f_2, f_2'),$$

it holds that $d_1, d_2 \in C(f_1', f_2')$. Let $N = M|T_1 \cup \{d_1, f_1, f_2, f_1', f_2'\}$. Then

$$C(f_1, f_2) \triangle D, \qquad C(f_1', f_2') \triangle D, \qquad C(f_1', f_2), \qquad C(f_1, f_2'), \qquad C(f_1, f_1'), \qquad C(f_2, f_2'), \qquad D$$

correspond to the 3-circuits of an F_7-minor of N. This contradicts the regularity of M. This shows that if $f \in H_1'$ for some $f \in F_1 \backslash \{f_1\}$, then $F_2 \subseteq C_2'^*$. Similarly, if $f \in H_2'$ for some $f \in F_2 \backslash \{f_2\}$, then $F_1 \subseteq C_1'^*$. Thus either $F_1 \subseteq C_1'^*$, or $F_2 \subseteq C_2'^*$. Assume without loss of generality that $F_1 \subseteq C_1'^*$. Then $\{d_2\} \cup F_1 \cup E_1 \subseteq C_1'^*$. Since $|\{d_2\} \cup F_1 \cup E_1| = c^*$, it holds that $C_1'^* = \{d_2\} \cup F_1 \cup E_1$. Thus $C_1^* \triangle C_1'^* = \{d_1, d_2\}$, implying that $\{d_1, d_2\}$ is a 2-cocircuit. This contradicts our assumptions about M. Thus D must intersect all cocircuits of $\mathcal{S}^*(M)$. □

The proof of the theorem now follows Cases 1 and 2. ■

3. Fractional circuit covers

In this section, we shall prove that Conjecture 1.2 is true for fractional circuit covers. As a matter of notation, we shall view functions $\boldsymbol{\phi} : E \to \mathbb{R}_+$ interchangeably as vectors $\boldsymbol{\phi} \in \mathbb{R}_+^{|E|}$. For any subset $X \subseteq E$, we let $\boldsymbol{\phi}(X) = \sum_{e \in X} \boldsymbol{\phi}(e)$.

Lemma 3.1. *Let M be a connected matroid having a ground set E where $|E| \geq 2$. Let $\mathbf{l}, \mathbf{w} : E \to \mathbb{R}_+$ where either*

$$\mathbf{l}(e) \leq \frac{1}{2} \max\{\mathbf{l}(C) : C \in \mathcal{C}(M)\} \quad \forall e, \quad \textit{or} \quad \mathbf{w}(e) \leq \frac{1}{2} \max\{\mathbf{w}(C^*) : C^* \in \mathcal{C}^*(M)\} \quad \forall e.$$

Then

$$\mathbf{l} \cdot \mathbf{w} \leq \frac{1}{2} \max\{\mathbf{l}(C) : C \in \mathcal{C}(M)\} \times \max\{\mathbf{w}(C^*) : C^* \in \mathcal{C}^*(M)\}.$$

Proof. The proof is divided into two parts: in the first part, we shall assume that $\mathbf{l}(e) > 0$, $\mathbf{w}(e) > 0\ \forall e$. Using rational approximation, the theorem is seen to hold if it holds for $\mathbf{l}, \mathbf{w} : E \to \mathbb{Q}_+$. If $\mathbf{l}, \mathbf{w} : E \to \mathbb{Q}_+$, then one can scale $\mathbf{l}$ and $\mathbf{w}$ by multiplying each by appropriate factors to obtain integer-valued functions. Thus it suffices to prove the theorem for integral $\mathbf{l}, \mathbf{w}$, and we shall assume that $\mathbf{l}, \mathbf{w} \in \mathbb{Z}_+^{|E|}$. Taking the dual, when necessary, we may assume that $\mathbf{l}(e) \leq \frac{1}{2} \max\{\mathbf{l}(C) : C \in \mathcal{C}(M)\}\ \forall e$. We form a new matroid M' from M in the following way: for each $e \in E$ replace e by $\mathbf{w}(e)$ elements $e_1, \ldots, e_{\mathbf{w}(e)}$

in parallel. Then replace each e_i with $\mathbf{l}(e)$ elements $e_{i1}, \ldots e_{i\mathbf{l}(e)}$ in series. Then $|E(M')| = \mathbf{l}\cdot\mathbf{w}$. Let $M'_e = M'|\{e_{ij} : i = 1, \ldots, \mathbf{w}(e),\ j = 1, \ldots, \mathbf{l}(e)\}$. Clearly any cocircuit $C^* \in \mathcal{C}^*(M'),\ C^* \not\subseteq E(M'_e)$ which intersects $E(M'_e)$ does so in exactly $\mathbf{w}(e)$ elements. Furthermore, any cocircuit of $\mathcal{C}^*(M')$ contained in $E(M'_e)$ has exactly two elements. From this it is seen that $c^*(M') = \max\{\mathbf{w}(C^*) : C^* \in \mathcal{C}^*(M)\}$. Similarly, any circuit $C \in \mathcal{C}(M'),\ C \not\subseteq E(M')$ which intersects M'_e does so in exactly $\mathbf{l}(e)$ elements. Any circuit of $\mathcal{C}(M')$ contained in $E(M'_e)$ contains at most $2\mathbf{l}(e) \le \max\{\mathbf{l}(C) : C \in \mathcal{C}(M)\}$ elements. Thus $c(M') = \max\{\mathbf{l}(C) : C \in \mathcal{C}(M)\}$. Given that $\mathbf{l}(e) > 0$, $\mathbf{w}(e) > 0\ \forall e$, M' is connected, and hence Theorem 1.1 implies that $|E(M')| \le \frac{1}{2}c(M')c^*(M')$. This in turn implies that

$$\mathbf{l}\cdot\mathbf{w} \le \frac{1}{2}\max\{\mathbf{l}(C) : C \in \mathcal{C}(M)\} \times \max\{\mathbf{w}(C^*) : C^* \in \mathcal{C}^*(M)\}.$$

In the case that $\mathbf{l}(e) = 0$ or $\mathbf{w}(e) = 0$ for some elements $e \in E(M)$, we let ϵ be a small positive number and define new vectors $\mathbf{l}', \mathbf{w}' : E \to \mathbb{R}_+$ where

$$\mathbf{l}'(e) = \begin{cases} \mathbf{l}(e), & \text{if } \mathbf{l}(e) \neq 0; \\ \epsilon, & \text{if } \mathbf{l}(e) = 0. \end{cases}$$

$$\mathbf{w}'(e) = \begin{cases} \mathbf{w}(e), & \text{if } \mathbf{w}(e) \neq 0; \\ \epsilon, & \text{if } \mathbf{w}(e) = 0. \end{cases}$$

By the first part, we have that

$$\mathbf{l}'\cdot\mathbf{w}' \le \frac{1}{2}\max\{\mathbf{l}'(C) : C \in \mathcal{C}(M)\} \times \max\{\mathbf{w}'(C^*) : C^* \in \mathcal{C}^*(M)\}$$

and taking limits

$$\lim_{\epsilon\to 0} \mathbf{l}'\cdot\mathbf{w}' \le \lim_{\epsilon\to 0} \frac{1}{2}\max\{\mathbf{l}'(C) : C \in \mathcal{C}(M)\} \times \max\{\mathbf{w}'(C^*) : C^* \in \mathcal{C}^*(M)\}.$$

From this it follows that

$$\mathbf{l}\cdot\mathbf{w} \le \frac{1}{2}\max\{\mathbf{l}(C) : C \in \mathcal{C}(M)\} \times \max\{\mathbf{w}(C^*) : C^* \in \mathcal{C}^*(M)\}. \quad \blacksquare$$

We note that the inequality in the above lemma is very similar to the so-called *width–length inequality* introduced by Lehman [1] who used it to characterize ideal clutters. We shall now show that a fractional version of Conjecture 1.2 holds.

Theorem 3.2. *Let M be a connected matroid having ground set E where $|E| \ge 2$. Then there exist constants $\alpha_C \in \mathbb{R}_+,\ C \in \mathcal{C}(M)$ such that $\sum_C \alpha_C \mathbf{i}_C \ge 2\mathbf{i}_E$, and $\sum_C \alpha_C \le c^*(M)$.*

Proof. Let $\mathbf{l}, \mathbf{w} : E \to \mathbb{R}_+$ where $\mathbf{w} \equiv \mathbf{1}$ and $\mathbf{l}(C) \le 1$, $\forall C \in \mathcal{C}(M)$. Since M is connected, it holds that $c^*(M) \ge 2$, and hence

$$\mathbf{w}(e) = 1 \le \frac{1}{2}\max\{\mathbf{w}(C^*) : C^* \in \mathcal{C}^*(M)\} = \frac{c^*(M)}{2} \quad \forall\, e.$$

Lemma 3.1 now implies that

$$\mathbf{l}(E) = \mathbf{l}\cdot\mathbf{w} \le \frac{1}{2}\max\{\mathbf{l}(C) : C \in \mathcal{C}\} \times \max\{\mathbf{w}(C^*) : C^* \in \mathcal{C}^*\} \le \frac{c^*(M)}{2}.$$

Thus

$$\mathbf{l}(E) \le \frac{c^*(M)}{2},\ \forall \mathbf{l} : E \to \mathbb{R}_+ \quad \text{where } \mathbf{l}(C) \le 1, \quad \forall C \in \mathcal{C}(M). \tag{4}$$

Let $A = A(\mathcal{C}(M))$ be the circuit matrix of M. Consider the LP

$$\max\ \mathbf{x}\cdot\mathbf{2}, \qquad A\mathbf{x} \le \mathbf{1},\ \mathbf{x} \ge \mathbf{0}. \tag{5}$$

and the dual LP

$$\min\ \mathbf{y}\cdot\mathbf{1},\qquad \mathbf{y}^tA \geq \mathbf{2},\ \mathbf{y} \geq 0. \tag{6}$$

By (4) it follows that any optimal solution $\mathbf{x}^*$ to (5) satisfies $\mathbf{x}^*\cdot\mathbf{2} \leq c^*(M)$. Consequently any optimal solution $\mathbf{y}^*$ to (6) satisfies $\mathbf{y}^*\cdot\mathbf{1} \leq c^*(M)$. Indexing $\mathbf{y}^*$ by the circuits of M, letting $\mathbf{y}^* = (\mathbf{y}^*_C)_{C\in\mathcal{C}(M)}$, we achieve the desired constants by taking $\alpha_C = \mathbf{y}^*_C,\ C \in \mathcal{C}(M)$. ■

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